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A CONVERGENCE THEOREM FOR THE HENSTOCK-KURZWEIL INTEGRAL

Dedicated to Professor Štefan Schwabik (1941-2009)

Abstract

We present a convergence theorem for the Henstock-Kurzweil integral of functions taking values in a locally convex topological vector space, which is sequentially complete with respect to its weak topology.

1 Introduction.

The first simple convergence theorems for HK-integral were shown by Kurzweil (see [3]) and by McLeod (see [5]). A convergence theorem for HK-integral of functions taking values in a complete locally convex space was given by Marraffa (see [4, Theorem 5]). We prove another convergence theorem, Theorem 2.4, which is similar to Marraffa's Theorem. The Banach version of [4, Theorem 5] is used in proving our Theorem 2.4. In spite of that, those theorems are independent of each other, because in general there are no relations between the completeness of a locally convex topological vector space and the sequential completeness of its weak topology. If a locally convex topological vector space is weakly sequential complete but not complete, then our Theorem has to be used instead of Marraffa's Theorem. Do locally convex topological vector spaces of this type exist? According to Kōmura (see [2, p. 153]), there exist locally convex topological vector spaces that are reflexive but not complete. Since a reflexive space is a semi-reflexive space (see [6, p.144]), by Schaefer (see [6, Th.IV.5.5, p.144]), we see that a reflexive locally convex topological

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vector space is a weakly quasi-complete space (for the quasi-complete spaces see [6, p.27]). Since a quasi-complete space is a sequentially complete or semi-complete space (for the sequentially complete or semi-complete spaces see [6, p.17]), a reflexive locally convex topological vector space is a weakly sequentially complete space. Consequently, there exist locally convex topological vector spaces that are weakly sequential complete but not complete.

These sorts of different types of convergence theorems for the HK-integral are based on the concept of HK-*equi-integrability* (see Definition 1.4).

In this paper (V, τ) is a locally convex topological vector space, which is Hausdorff (or separated) space. We set P the family of all continuous semi-norms in this space; for every $p \in P$, $\widetilde{V}^p$ denotes the quotient vector space of the vector space V with respect to the equivalence relation $x \sim^p y \Leftrightarrow p(x-y) = 0$; the map $\phi_p : V \to \widetilde{V}^p$ is the canonical quotient map, thus $\phi_p(x)$ is the equivalence class of an element $x \in V$ with respect to the relation "$\sim^p$"; the quotient normed space $(\widetilde{V}^p, \widetilde{p})$ is called the normed component of the space V, where $\widetilde{p}(\phi_p(x)) = p(x)$, for each $x \in V$; the Banach space $(\overline{V}^p, \overline{p})$, which is the completion of the space $(\widetilde{V}^p, \widetilde{p})$, is called the Banach component of the space (V, τ);V', V'_p, $\widetilde{V}'_p$ and $\overline{V}'_p$ are topological duals of (V, τ), (V, p), $(\widetilde{V}^p, \widetilde{p})$ and $(\overline{V}^p, \overline{p})$, respectively.

Assume that an interval $S = [a, b]$ and a function $f : S \to V$ are given. Let $\pi = \{(s_i, J_i);\ i = 1, 2, ..., n\}$ be a set such that $s_i \in S$ and J_i is a compact subinterval of S for $i = 1, 2, ..., n$; this set π is called an HK-partition of S if the finite sequence $(J_i)_{i=1}^n$ satisfies the following statements:

1. $\bigcup_{i=1}^n J_i = S$,
2. $s_i \in J_i$, for $i = 1, 2, ..., n$,
3. $(J_i)_{i=1}^n$ is a finite sequence of pairwise non-overlapping intervals (two intervals I and J are called non-overlapping if $I^o \cap J^o = \phi$, where I^o and J^o denote the interiors of I, J respectively).

A positive function $\delta : S \to (0, +\infty)$ is called a gauge on S; an HK-partition $\pi = \{(s_i, J_i);\ i = 1, 2, ..., n\}$ in S is called δ-fine (it is denoted by $\pi << \delta$) if we have

$$J_i \subset (s_i - \delta(s_i), s_i + \delta(s_i)),$$

for $i = 1, 2, ..., n$. We set:

$$S(f, \pi) = \sum_{i=1}^n f(s_i)\mu_L(J_i),$$

where $\pi = \{(s_i, J_i);\ i = 1, 2, ..., n\}$ is an HK-partition of S and μ_L is the Lebesgue outer measure in S.

Definition 1.1. *A function $f : S \to V$ is called HK-integrable in (V, τ) if there exists a vector $I_f \in V$ satisfying the following property: for every $p \in P$ and $\epsilon > 0$ there exists a gauge $\delta_p(\epsilon)$ on S, such that inequality*

$$p(S(f, \pi) - I_f) < \epsilon$$

holds for every HK-partition π in S, such that $\pi << \delta_p(\epsilon)$. Since the family P is separated, then the vector I_f is the only one satisfying the definition and it is called the HK-integral of the function f in (V, τ). It is denoted

$$(HK) \int_S f = I_f.$$

The following theorem guarantees a simple and important relation of HK-integral in a locally convex spaces and in its components.

Theorem 1.2. *A function $f : S \to V$ is HK-integrable in (V, τ) if and only if there exists a vector $I_f \in V$ such that for every $p \in P$ the function $\phi_p \circ f$ is HK-integrable in the normed component $(\widetilde{V}^p, \widetilde{p})$ while*

$$(HK) \int_S \phi_p \circ f = \phi_p(I_f).$$

Theorem 1.2 can be easily proved by the Definition 1.1. An application of Theorem 1.2 is the following result, which is an analogue of [1, Corollary III.3.2].

Theorem 1.3. *Let $f : S \to V$ be a function taking values in (V, τ). If $f = 0$ almost everywhere (with respect to μ_L), then the function f is HK-integrable and*

$$(HK) \int_S f = 0.$$

Definition 1.4. *A sequence of functions (f_n), $f_n : S \to V$ is called HK-*equi-integrable *in (V, τ) if every function f_n is HK-integrable in (V, τ) and for every $p \in P$ and $\epsilon > 0$ there exists a gauge $\delta_p(\epsilon)$ on S, such that inequality:*

$$p(S(f_n, \pi) - (HK) \int_S f_n) < \epsilon,$$

holds for every HK-partition $\pi << \delta_p(\epsilon)$ and for all $n \in N$.

2 A convergence theorem for Henstock-Kurzweil integral.

The main result in this paper is Theorem 2.4. The following lemmas prepare the proof of this theorem. The Lemmas 2.1 and 2.2 can be proved in a similar manner as the analogous results in Banach spaces (see [1, Theorem III.5.2] and [1, Proposition IV.1.1]).

Lemma 2.1. *If (V,τ) is the sequentially complete, the sequence (f_n),$f_n : S \to V$ is HK*-equi-integrable *in (V,τ) and converges point-wise to a function $f : S \to V$ in (V,τ), then the function f is HK-integrable and*

$$\lim_{n\to\infty}(HK)\int_S f_n = (HK)\int_S f.$$

Lemma 2.2. *If a function $f : S \to V$ is HK-integrable in (V,τ), then for every $v' \in V'$ the real function $v' \circ f$ is KH-integrable and*

$$(HK)\int_S v' \circ f = v'((HK)\int_S f).$$

Lemma 2.3. *If (V,τ) is the sequentially complete with respect to the weak topology $\sigma(V,V')$, the sequence (f_n),$f_n : S \to V$ is HK*-equi-integrable *in (V,τ) and converges point-wise to the function $f : S \to V$ in the weak topology, then there exists $I_f \in V$ such that the equality*

$$\lim_{n\to\infty} v'((HK)\int_S f_n) = v'(I_f),$$

holds for every $v' \in V'$.

PROOF. The locally convex topologically vector space $(V,\sigma(V,V'))$ is Hausdorff (see [7, Corollary IV.6.1, p.107]). Let denote by P' the family of all continuous semi-norms in $(V,\sigma(V,V'))$. Since $P' \subset P$, the sequence (f_n) is HK-*equi-integrable* in $(V,\sigma(V,V'))$ and converges to the function f in this space. Thus, we are in conditions of Lemma 2.1. Hence there exists $I_f \in V$ such that:

$$\lim_{n\to\infty} p'((HK)\int_S f_n - I_f) = 0,$$

for every $p' \in P'$. Therefore, we obtain:

$$\lim_{n\to\infty} v'((HK)\int_S f_n) = v'(I_f),$$

for every $v' \in V'$, because $|v'(.)| \in P'$. □

Now, we are ready to present the main theorem.

Theorem 2.4. *If (V,τ) is the sequentially complete with respect to the weak topology $\sigma(V,V')$, the sequence of the functions (f_n),$f_n : S \to V$ is HK*-equi-integrable *in (V,τ) and converges to $f : S \to V$ in the weak topology, then f is HK-integrable in (V,τ) and*

$$\lim_{n\to\infty}(HK)\int_S f_n = (HK)\int_S f,$$

in the weak topology.

PROOF. Let p be any continuous semi-norm in (V,τ). Since the sequence (f_n) converges to f in (V,τ) in the weak topology, then the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$ with respect to the weak topology. Consequently the sequence $(\phi_p \circ f_n)$ also converges to the functions $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$ with respect to the weak topology. Thus, we have the sequence $(\phi_p \circ f_n)$ is HK-*equi-integrable* in $(\widetilde{V}^p, \widetilde{p})$ and converges to $\phi_p \circ f$ in $(\overline{V}^p, \overline{p})$ with respect to the weak topology. According to the Banach version of [4, Theorem 5], the function $\phi_p \circ f$ is HK-integrable in $(\overline{V}^p, \overline{p})$ and the equality

$$\lim_{n\to\infty}\overline{v}'_p((HK)\int_S \phi_p \circ f_n) = \overline{v}'_p((HK)\int_S \phi_p \circ f),$$

holds for every $\overline{v}'_p \in \overline{V}'_p$, and since every $\overline{v}'_p \in \overline{V}'_p$ is the continuous extension of an element $\widetilde{v}'_p \in \widetilde{V}'_p$, it follows that the equality

$$\lim_{n\to\infty}\widetilde{v}'_p((HK)\int_S \phi_p \circ f_n) = \overline{v}'_p((HK)\int_S \phi_p \circ f) \tag{1}$$

holds for every $\widetilde{v}'_p \in \widetilde{V}'_p$, where $\overline{v}'_p$ is continuous extension of $\widetilde{v}'_p$.

By applying Lemma 2.2, for every $\phi_p \circ f_n$, we obtain

$$\begin{aligned}\widetilde{v}'_p((HK)\int_S \phi_p \circ f_n) =&(HK)\int_S (\widetilde{v}'_p \circ (\phi_p \circ f_n))\\ =&(HK)\int_S v'_p \circ f_n,\end{aligned}$$

where $v'_p = \widetilde{v}'_p \circ \phi_p$, and by another application of Lemma 2.2, we obtain that for every f_n,

$$(HK)\int_S v'_p \circ f_n = v'_p((HK)\int_S f_n)$$

and consequently we obtain

$$\widetilde{v}'_p((HK)\int_S \phi_p \circ f_n) = v'_p((HK)\int_S f_n) \tag{2}$$

Hence, inserting the right-hand-side of (2) into (1), we get

$$\lim_{n\to\infty} v'_P(((HK)\int_S f_n) = \overline{v}'_p((HK)\int_S \phi_p \circ f) \tag{3}$$

Also, according to Lemma 2.3, there exists $I_f \in V$ such that the equality

$$\lim_{n\to\infty} v'_p((HK)\int_S f_n) = v'_p(I_f) = \widetilde{v}'_p(\phi_p(I_f)) \tag{4}$$

holds for every $v'_p \in V'_p$. Hence, inserting the right-hand-side of (4) into (3), we obtain

$$\widetilde{v}'_p(\phi_p(I_f)) = \overline{v}'_p((HK)\int_S \phi_p \circ f),$$

for every $\widetilde{v}'_p \in \widetilde{V}'_p$, where $\overline{v}'_p$ is continuous extension of $\widetilde{v}'_p$. Consequently,

$$\overline{v}'_p(\phi_p(I_f)) = \overline{v}'_p((HK)\int_S \phi_p \circ f),$$

for every $\overline{v}'_p \in \overline{V}'_p$ and according to [7, Cor.IV.6.2, p.108], this means that

$$(HK)\int_S \phi_p \circ f = \phi_p(I_f) \in \widetilde{V}^p.$$

Therefore, by Theorem 1.2, the function f is HK-integrable and

$$(HK)\int_S f = I_f.$$

□

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