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CONSTRUCTION OF MEASURE BY MASS DISTRIBUTION

Abstract

In this note we show that a measure can be constructed on an arbitrary set by iterated arbitrary mass distribution over arbitrary subsets of the set.

A Borel outer measure on $\mathbb{R}^n$ is an outer measure μ on $\mathbb{R}^n$ whose σ-algebra $\mathcal{M}$ of μ-measurable subsets of $\mathbb{R}^n$ contains the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$. The restriction of a Borel outer measure μ on $\mathbb{R}^n$ to the σ-algebra $\mathcal{M}$ is then a measure on this σ-algebra which contains all the Borel sets in $\mathbb{R}^n$. K. Falconer [3] describes a method of constructing a Borel outer measure on $\mathbb{R}^n$ by repeated mass distributions. (What is called a measure in [3] is an outer measure in our terminology.) The process of mass distribution may be described as follows.

Let E be a bounded Borel set in $\mathbb{R}^n$. Let $\mathcal{E}_0 = \{E\}$. For $k = 1, 2, \ldots$, let $\mathcal{E}_k$ be a collection of disjoint Borel subsets of E such that each member U of $\mathcal{E}_k$ is contained in one of the members of $\mathcal{E}_{k-1}$ and contains a finite number of members of $\mathcal{E}_{k+1}$ and the maximum diameter of the members of $\mathcal{E}_k$ tends to 0 as $k \to \infty$. It is not required that a member of $\mathcal{E}_k$ is equal to the union of the members of $\mathcal{E}_{k+1}$ it contains. Let us assign a mass $\mu(E) \in (0, \infty)$ to the set E. Subdivide this mass between the members $U_1, \ldots, U_m$ of $\mathcal{E}_1$ in such a way that $\sum_{i=1}^m \mu(U_i) = \mu(E)$. For each set U in $\mathcal{E}_1$, subdivide the mass $\mu(U)$ between the members $U_1, \ldots, U_n$ of $\mathcal{E}_2$ contained in U in such a way that $\sum_{j=1}^n \mu(U_j) = \mu(U)$. We repeat this subdivision of mass indefinitely. Consider the sequence $(\mathcal{E}_k : k \in \mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, of collections of Borel subsets of E. For each $k \in \mathbb{Z}_+$, let E_k be the union of the members of $\mathcal{E}_k$. Then $(E_k : k \in \mathbb{Z}_+)$ is a decreasing sequence and $(\mathbb{R}^n \setminus E_k : k \in \mathbb{Z}_+)$ is an increasing sequence. We set $\mu(\mathbb{R}^n \setminus E_k) = 0$ for every $k \in \mathbb{Z}_+$. The collection

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$\mathcal{E}$ of all sets in $\mathcal{E}_k$ and $\mathbb{R}^n \setminus E_k$ for all $k \in \mathbb{Z}_+$ is a covering class for $\mathbb{R}^n$ in the sense that there exists a sequence in $\mathcal{E}$ the union of whose members is equal to $\mathbb{R}^n$. Let $\mathcal{P}(\mathbb{R}^n)$ be the collection of all subsets of $\mathbb{R}^n$. If we define a set function μ on $\mathcal{P}(\mathbb{R}^n)$ by setting for every $A \in \mathcal{P}(\mathbb{R}^n)$

$$\mu(A) = \inf \left\{ \textstyle\sum_{i\in\mathbb{N}} \mu(U_i) : A \subset \bigcup_{i\in\mathbb{N}} U_i \text{ and } U_i \in \mathcal{E} \right\},$$

then μ is a Borel outer measure on $\mathbb{R}^n$ and the support of the measure μ on the σ-algebra $\mathcal{M}$ is contained in $\bigcap_{k\in\mathbb{Z}_+} \overline{E_k}$.

A net $\mathcal{N}$ of subsets of $\mathbb{R}^n$ is a subcollection of $\mathcal{P}(\mathbb{R}^n)$ with the property that if $V_1, V_2 \in \mathcal{N}$ then either $V_1 \cap V_2 = \emptyset$ or else $V_1 \subset V_2$ or $V_2 \subset V_1$. In Falconer's construction of a measure on $\mathcal{B}_{\mathbb{R}^n}$ by mass distribution described above, the mass is distributed over a net of subsets of $\mathbb{R}^n$.

A covering class $\mathcal{V}$ of subsets of $\mathbb{R}^n$ is a subcollection of $\mathcal{P}(\mathbb{R}^n)$ such that $\emptyset \in \mathcal{V}$ and there exists $\{V_i : i \in \mathbb{N}\} \subset \mathcal{V}$ such that $\bigcup_{n\in\mathbb{N}} V_n = \mathbb{R}^n$.

A premeasure γ on $\mathbb{R}^n$ is a nonnegative extended real-valued set function on a covering class $\mathcal{V}$ of subsets of $\mathbb{R}^n$ such that $\gamma(\emptyset) = 0$. Given a premeasure γ defined on a covering class $\mathcal{V}$ of subsets of $\mathbb{R}^n$, a set function μ on $\mathcal{P}(\mathbb{R}^n)$ defined by setting for every $E \in \mathcal{P}(\mathbb{R}^n)$

$$\mu(E) = \inf \left\{ \textstyle\sum_{i\in\mathbb{N}} \gamma(V_i) : (V_i : i \in \mathbb{N}) \subset \mathcal{V}, \bigcup_{i\in\mathbb{N}} V_i \supset E \right\},$$

is an outer measure on $\mathbb{R}^n$. Restriction of μ to the σ-algebra $\mathcal{M}$ of μ-measurable subsets of $\mathbb{R}^n$, that is, $E \in \mathcal{P}(\mathbb{R}^n)$ satisfying the Carathéodory condition

$$\mu(A) = \mu(E \cap A) = \mu(E^c \cap A) \quad \text{for every } A \in \mathcal{P}(\mathbb{R}^n),$$

is the a measure generated by the premeasure γ. In general, the σ-algebra $\mathcal{M}$ may not contain the covering class $\mathcal{V}$ on which the premeasure γ is based. If the covering class $\mathcal{V}$ is a semialgebra of subsets of $\mathbb{R}^n$ and γ is additive on $\mathcal{V}$ then $\mathcal{V} \subset \mathcal{M}$ and if γ is countably additive on the semialgebra $\mathcal{V}$ then we have $\gamma = \mu$ on $\mathcal{V}$.

A net measure is a measure generated by a premeasure whose covering class is a net. Thus Falconer's measure on $\mathcal{B}_{\mathbb{R}^n}$ constructed by mass distribution is a net measure on $\mathbb{R}^n$.

Net measures are a useful tool in the study of Hausdorff measures. A. S. Besicovitch [1] constructed net measures comparable to a Hausdorff measure $\mathcal{H}^s$ on $\mathbb{R}^n$ to show that any closed set of infinite $\mathcal{H}^s$-measure contains subsets of positive and finite $\mathcal{H}^s$-measure. For a treatise of comparable net measures we refer to K. Falconer [2].

In this article we show that a measure can be constructed on an arbitrary set X by mass distribution on a sequence $(\mathcal{D}_k : k \in \mathbb{Z}_+)$ of successive decompositions of X into arbitrary subsets. We show that the collection of subsets of X resulting from the decompositions $(\mathcal{D}_k : k \in \mathbb{Z}_+)$ together with $\emptyset$ constitute a semialgebra of subsets of X and an arbitrary mass distribution over these sets is a countably additive set function on the semialgebra. This set function is then extended to be a measure on the σ-algebra generated by the semialgebra.

A collection $\mathcal{S}$ of subsets of a set X is called a semialgebra if it satisfies the following conditions:

1° $\emptyset, X \in \mathcal{S}$,
2° if $E, F \in \mathcal{S}$, then $E \cap F \in \mathcal{S}$,
3° if $E \in \mathcal{S}$, then E^c is a finite disjoint union of members of $\mathcal{S}$.

It follows that a finite union of members of $\mathcal{S}$ is always equal to a finite disjoint union of members of $\mathcal{S}$. The collection of all finite unions of members of a semialgebra $\mathcal{S}$ is equal to the algebra $\alpha(\mathcal{S})$ generated by $\mathcal{S}$, that is, the smallest algebra of subsets of X containing $\mathcal{S}$.
Let μ be a nonnegative extended real-valued set function on a semialgebra $\mathcal{S}$ of subsets of a set X with $\mu(\emptyset) = 0$. We say that μ is finitely additive on $\mathcal{S}$ if for every finite disjoint collection $\{E_1, \dots, E_n\}$ of members of $\mathcal{S}$ such that $\bigcup_{i=1}^n E_i \in \mathcal{S}$ we have $\mu\big(\bigcup_{i=1}^n E_i\big) = \sum_{i=1}^n \mu(E_i)$. We say that μ is countably additive on $\mathcal{S}$ if for every countable disjoint collection $\{E_n : n \in \mathbb{N}\}$ of members of $\mathcal{S}$ such that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{S}$ we have $\mu\big(\bigcup_{n\in\mathbb{N}} E_n\big) = \sum_{n\in\mathbb{N}} \mu(E_n)$.

For $A \in \alpha(\mathcal{S})$ given as a finite disjoint union of member $E_1, \dots, E_n$ of $\mathcal{S}$, let us define $\mu(A) = \sum_{i=1}^n \mu(E_i)$. Then μ is finitely additive on $\alpha(\mathcal{S})$ if and only if μ is finitely additive on $\mathcal{S}$ and μ is countably additive on $\alpha(\mathcal{S})$ if and only if μ is countably additive on $\mathcal{S}$. According to the Hopf Extension Theorem, if a nonnegative extended real-valued set function μ on an algebra $\mathcal{A}$ of subsets of a set X with $\mu(\emptyset) = 0$ is countably additive on $\mathcal{A}$ then μ is extendible to a measure on the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$, that is, the smallest σ-algebra of subsets of X containing $\mathcal{A}$. Moreover the extension of μ to a measure on $\sigma(\mathcal{S}) = \sigma(\alpha(\mathcal{S}))$ is unique when μ is σ-finite on $\mathcal{S}$. For proofs of these statements regarding an extension of a set function on a semialgebra $\mathcal{S}$ to a measure on the σ-algebra $\sigma(\mathcal{S}) = \sigma(\alpha(\mathcal{S}))$ we refer to [4].

Definition 1. *Let X be a non-empty set. Let $(\mathcal{D}_k : k \in \mathbb{Z}_+)$ be a sequence of decompositions of X into arbitrary subsets of X and let γ be a set function on $\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$. We call $\big(\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k, \gamma\big)$ a mass distribution on X if $(\mathcal{D}_k : k \in \mathbb{Z}_+)$ and γ satisfy the following conditions;*

1° *For each* $k \in \mathbb{Z}_+$, $\mathcal{D}_k$ *is a finite disjoint collection of non-empty subsets of* X *whose union is equal to* X.
2° *For each* $k \in \mathbb{Z}_+$, $\mathcal{D}_{k+1}$ *is a refinement of* $\mathcal{D}_k$, *that is, every member of* $\mathcal{D}_k$ *is a union of some members of* $\mathcal{D}_{k+1}$
3° $\mathcal{D}_0 = \{X\}$.
4° $\gamma(X) \in (0, \infty)$.
5° $\gamma(E) \in [0, \infty)$ *for every* $E \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$.
6° $\gamma(E) = \sum_{i=1}^{p} \gamma(E_i)$ *for every* $E \in \mathcal{D}_k$ *such that* $E = \bigcup_{i=1}^{p} E_i$ *where* $\{E_1, \dots, E_p\} \subset \mathcal{D}_{k+1}$.

Lemma 2. *Let* $\left(\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k, \gamma\right)$ *be a mass distribution on a non-empty set* X. *Let* $\mathcal{S} = \{\emptyset\} \cup \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ *and let* $\gamma(\emptyset) = 0$. *Then we have:*
(a) $\mathcal{S}$ *is a semialgebra of subsets of* X.
(b) *If* $E, F \in \mathcal{S}$ *then either* $E \cap F = \emptyset$ *or else* $E \subset F$ *or* $F \subset E$.
(c) γ *is a countably additive set function on the semialgebra* $\mathcal{S}$.

Proof. 1. Let us show that $\mathcal{S}$ is a semialgebra of subsets of X. Clearly $\emptyset, X \in \mathcal{S}$. Let us show that $\mathcal{S}$ is closed under intersections. Let $E, F \in \mathcal{S}$. If at least one of the two sets is the empty set then their intersection is the empty set which is a member of $\mathcal{S}$. Suppose neither of the two sets E and F is the empty set. Then $E, F \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ so that $E \in \mathcal{D}_m$ and $F \in \mathcal{D}_n$ for some $m, n \in \mathbb{Z}_+$. We may assume without loss of generality that $m \le n$. If $m = n$ then $E, F \in \mathcal{D}_m$ so that either $E = F$ so that $E \cap F = E \in \mathcal{S}$ or $E \cap F = \emptyset \in \mathcal{S}$. Now consider the case $m < n$. In this case $F \in \mathcal{D}_n$ is contained in a member of $\mathcal{D}_m$. If our $E \in \mathcal{D}_m$ is the member of $\mathcal{D}_m$ containing F then $E \cap F = F \in \mathcal{S}$. If E does not contain F then F is contained in another member of $\mathcal{D}_m$ which is disjoint from E so that $E \cap F = \emptyset \in \mathcal{S}$. This shows proves that $\mathcal{S}$ is closed under intersections. We have shown also that if $E, F \in \mathcal{S}$ then either $E \cap F = \emptyset$ or else $E \subset F$ or $F \subset E$.

Let us show that for every $E \in \mathcal{S}$ the complement E^c is a finite disjoint union of members of $\mathcal{S}$. Let $E \in \mathcal{S}$. If $E = \emptyset$ then $E^c = X \in \mathcal{S}$ so that E^c is trivially a finite disjoint union of members of $\mathcal{S}$. If $E \neq \emptyset$ then $E \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ so that $E \in \mathcal{D}_k$ for some $k \in \mathbb{Z}_+$. Then since $\mathcal{D}_k$ is a finite disjoint collection of sets the union of all of which is equal to X, E^c is equal to the union of all the members of $\mathcal{D}_k$ other than E. Then E^c is a finite disjoint union of members of $\mathcal{S}$. Thus we have verified (a) and (b).

2. Let us show that γ is countably additive on $\mathcal{S}$, that is, if $(E_n : n \in \mathbb{N})$ is a disjoint sequence in $\mathcal{S}$ such that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{S}$ then $\sum_{n\in\mathbb{N}} \gamma(E_n) = \gamma\big(\bigcup_{n\in\mathbb{N}} E_n\big)$. Since $\gamma(\emptyset) = 0$, it suffices to consider the case $(E_n : n \in \mathbb{N}) \subset \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$. Now $\gamma(X) \in (0, \infty)$. Let $J = [0, \gamma(X))$, an interval in $\mathbb{R}$ with length $\ell(J) = \gamma(X)$. Let $\mathcal{J}$ be the collection of all left-closed and right-open subintervals of J. We construct below that a mapping φ of $\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ into $\mathcal{J}$ satisfying the following conditions:

1° φ is one-to-one,
2° if $E_1, E_2 \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ and $E_1 \supset E_2$ then $\varphi(E_1) \supset \varphi(E_2)$,
3° if $E_1, E_2 \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ and $E_1 \cap E_2 = \emptyset$ then $\varphi(E_1) \cap \varphi(E_2) = \emptyset$,
4° $\ell\big(\varphi(E)\big) = \gamma(E)$ for $E \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$.

Consider $\mathcal{D}_1 = \{E_{1,1}, \dots, E_{1,p_0}\}$, a finite disjoint collection of non-empty subsets of X such that $\bigcup_{j=1}^{p_0} E_{1,j} = X$ and $\sum_{j=1}^{p_0} \gamma(E_{1,j}) = \gamma(X)$. Decompose J into p_0 disjoint left-closed and right-open subintervals of J, $\{J_{1,1}, \dots, J_{1,p_0}\}$ with $\ell(J_{1,j}) = \gamma(E_{1,j})$ for $j = 1, \dots, p_0$ so that

$$\sum_{j=1}^{p_0} \ell(J_{1,j}) = \sum_{j=1}^{p_0} \gamma(E_{1,j}) = \gamma(X) = \ell(J).$$

We define a mapping φ of $\mathcal{D}_1$ into $\mathcal{J}$

$$\varphi(E_{1,j}) = J_{1,j} \quad \text{for } j = 1, \dots, p_0.$$

Next consider $\mathcal{D}_2$, a refinement of $\mathcal{D}_1$. Thus each $E_{1,i}$ in $\mathcal{D}_1$ is decomposed into a subcollection $\{E_{1,i,1}, \dots, E_{1,i,p_{1,i}}\}$ of $\mathcal{D}_2$. By the definition of γ we have $\sum_{j=1}^{p_{1,i}} \gamma(E_{1,i,j}) = \gamma(E_{1,i})$. Decompose $J_{1,i} = \varphi(E_{1,i})$ into $p_{1,i}$ disjoint left-closed and right-open intervals $\{J_{1,i,1}, \dots, J_{1,i,p_{1,i}}\}$ with $\ell(J_{1,i,j}) = \gamma(E_{1,i,j})$ for $j = 1, \dots, p_{1,i}$. Then we have

$$\sum_{j=1}^{p_{1,i}} \ell(J_{1,i,j}) = \sum_{j=1}^{p_{1,i}} \gamma(E_{1,i,j}) = \gamma(E_{1,i}) = \ell(J_{1,i}).$$

We define we extend the definition of φ to $\mathcal{D}_2$ by setting

$$\varphi(E_{1,i,j}) = J_{1,i,j} \quad \text{for } j = 1, \dots, p_{1,i}.$$

We extend the definition of φ to $\mathcal{D}_3$ in the same manner and so on for $\mathcal{D}_k$ for $k \in \mathbb{Z}_+$. The mapping φ thus defined on $\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ satisfies conditions 1° to 4°.

Let us show the countable additivity of the set function γ on the semi-algebra $\mathcal{S}$. Let $(E_n : n \in \mathbb{N})$ be a disjoint sequence in $\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ such that $\bigcup_{n\in\mathbb{N}} E_n \in \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$. Then $\big(\varphi(E_n) : n \in \mathbb{N}\big)$ is a disjoint sequence in $\mathcal{J}$ and

$\varphi\big(\bigcup_{n\in\mathbb{N}} E_n\big) \in \mathcal{J}$. Let $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ be the Lebesgue measure space on $\mathbb{R}$. By the fact that $\mathcal{J} \subset \mathcal{M}_L$ and the fact that $\mu_L(J) = \ell(J)$ for $J \in \mathcal{J}$, we have

$$\begin{aligned}\gamma\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) &= \ell\Big(\varphi\Big(\bigcup_{n\in\mathbb{N}} E_n\Big)\Big) = \mu_L\Big(\varphi\Big(\bigcup_{n\in\mathbb{N}} E_n\Big)\Big) \\ &= \mu_L\Big(\bigcup_{n\in\mathbb{N}} \varphi(E_n)\Big) = \sum_{n\in\mathbb{N}} \mu_L\big(\varphi(E_n)\big) \\ &= \sum_{n\in\mathbb{N}} \ell\big(\varphi(E_n)\big) = \sum_{n\in\mathbb{N}} \gamma(E_n).\end{aligned}$$

This proves the countable additivity of γ on the semialgebra $\mathcal{S}$. □

Theorem 3. *Let* $\big(\bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k, \gamma\big)$ *be a mass distribution on a non-empty set* X. *Let* $\mathcal{S} = \{\emptyset\} \cup \bigcup_{k\in\mathbb{Z}_+} \mathcal{D}_k$ *and let* $\gamma(\emptyset) = 0$. *Then there exists a measure space* $(X, \sigma(\mathcal{S}), \mu)$ *such that* $\mu = \gamma$ *on* $\mathcal{S}$.

Proof. By Lemma 2, γ is a nonnegative real-valued countably additive set function on the semialgebra $\mathcal{S}$ of subsets of X with $\gamma(\emptyset) = 0$. Thus γ has a unique extension to a measure μ on $\sigma(\mathcal{S})$. (See Theorem 21.10, [4] for instance.) □

As an example of constructing a measure on an arbitrary set by mass distributions let us consider the space of infinite sequences of finitely many objects. Let $A = \{a_1, \ldots, a_m\}$. Let X be the collection of all infinite sequences of elements of A given by

$$\big(a_{n_j} : n \in \mathbb{N}\big) \quad \text{where } n_j = 1, \ldots, m \text{ for } n \in \mathbb{N}. \tag{1}$$

For $k \in \mathbb{N}$ consider the k-term sequences of elements of A given by

$$\big(a_{n_1}, \ldots, a_{n_k}\big) \quad \text{where } n_j = 1, \ldots, m \text{ for } n = 1, \ldots, k. \tag{2}$$

There are m^k such sequences. Let $E\big[a_{n_1}, \ldots, a_{n_k}\big]$ be the collection of all elements of X whose first k entries are $\big(a_{n_1}, \ldots, a_{n_k}\big)$, that is,

$$E\big[a_{n_1}, \ldots, a_{n_k}\big] = \big\{(s_n : n \in \mathbb{N}) \in X : (s_1, \ldots, s_k) = \big(a_{n_1}, \ldots, a_{n_k}\big)\big\}. \tag{3}$$

For $k \in \mathbb{N}$ let

$$\mathcal{D}_k = \big\{E\big[a_{n_1}, \ldots, a_{n_k}\big]; n_j = 1, \ldots, m \quad \text{for } n = 1, \ldots, k\big\} \tag{4}$$

and let

$$\mathcal{D}_0 = \{X\}. \tag{5}$$

Then for each $k \in \mathbb{Z}_+$, $\mathcal{D}_k$ is a collection of m^k disjoint non-empty subset of X. Moreover every member $E\big[a_{n_1}, \dots, a_{n_k}\big]$ of $\mathcal{D}_k$ is the union of m members of $\mathcal{D}_{k+1}$, that is,

$$E\big[a_{n_1}, \dots, a_{n_k}\big] = E\big[a_{n_1}, \dots, a_{n_k}, a_1\big] \cup \dots \cup E\big[a_{n_1}, \dots, a_{n_k}, a_m\big]. \tag{6}$$

Thus our $(\mathcal{D}_k : k \in \mathbb{Z}_+)$ satisfies conditions $1°$, $2°$ and $3°$ of Definition 1.

Let us define a set function γ on the semialgebra $\mathcal{S} = \{\emptyset\} \cup \bigcup_{k \in \mathbb{Z}_+} \mathcal{D}_k$ as follows. Let $p_1, \dots, p_m$ be arbitrary positive numbers satisfying the condition $\sum_{i=1}^m p_i = 1$. Let $\gamma(\emptyset) = 0$, $\gamma(X) = 1$ and for $E\big[a_{n_1}, \dots, a_{n_k}\big] \in \mathcal{D}_k$ let

$$\gamma\big(E\big[a_{n_1}, \dots, a_{n_k}\big]\big) = p_{n_1} \cdots p_{n_k}.$$

Then for a decomposition of $E\big[a_{n_1}, \dots, a_{n_k}\big]$ into m members of $\mathcal{D}_{k+1}$ as given by (6) we have

$$\gamma\big(E\big[a_{n_1}, \dots, a_{n_k}\big]\big) = p_{n_1} \cdots p_{n_k} \cdot \sum_{i=1}^m p_i = \sum_{i=1}^m \gamma\big(E\big[a_{n_1}, \dots, a_{n_k}, a_i\big]\big).$$

This shows that γ satisfies conditions $4°$, $5°$ and $6°$ of Definition 1. Thus by Theorem 3 the set function γ can be extended uniquely to be a measure μ on the σ-algebra $\sigma(\mathcal{S})$ generated by the semialgebra $\mathcal{S} = \{\emptyset\} \cup \bigcup_{k \in \mathbb{Z}_+} \mathcal{D}_k$ with $\mu(X) = 1$.

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