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Miguel Antonio Jiménez-Pozo, Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, Puebla, México. email: mjimenez@fcfm.buap.mx, mapozo@ujaen.es

ANOTHER LOOK AT THE PRODUCT MEASURE PROBLEM

Abstract

Here we give a positive answer to the so-called *Product Measure Problem* under the relatively simple hypothesis that the measure in one of the factors is inner regular and its support with the induced Hausdorff topology is locally metrizable. No special hypothesis on the other topological measure space is required. The proof is inspired by the rather imprecise conjecture that the open sets of a topological space must satisfy some restrictions in order to support any strictly positive σ-finite measure.

1 Introduction of the Problem.

There is such an extensive literature dealing with topologies and measures, that we will be only able to touch on those items directly related to the *Product Measure Problem* considered here. First we establish sufficient notation to introduce the problem.

For the time being we let $(X, \mathbb{T}_X, \mathbb{M}_X, \mu_X)$ and $(Y, \mathbb{T}_Y, \mathbb{M}_Y, \mu_Y)$ denote nontrivial complete σ-finite topological measure spaces. In particular this means there are inclusions of the corresponding topologies within the σ-algebras, i. e. $\mathbb{T}_X \subset \mathbb{M}_X$ and $\mathbb{T}_Y \subset \mathbb{M}_Y$. We shall say these measures are topological. The topology on X will be assumed to be Hausdorff. Let us denote by $\sigma(\mathbb{D})$ the $\sigma-$algebra generated by a class $\mathbb{D}$ of subsets of a given set, then $\mathbb{B}_X = \sigma(\mathbb{T}_X)$ and $\mathbb{B}_Y = \sigma(\mathbb{T}_Y)$ are the Borel $\sigma-$algebras generated by these

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topologies. Sometimes the Borel sets refer only to the elements of the $\sigma-$ring generated by the class of all compact subsets. In classical spaces such as R^n, this and other related definitions coalesce.

In the product space $X \times Y$, we always have

$$\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y = \sigma\left(\mathbb{B}_X \times \mathbb{B}_Y\right) \subset \sigma\left(\mathbb{T}_X \otimes_\tau \mathbb{T}_Y\right) = \mathbb{B}_{X \times Y}.$$

It is known (see König [10, page 132]) that for every set Z whose cardinal is strictly larger than the cardinal $\mathfrak{c}$ of the continuum and any two $\sigma-$algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ in Z, the diagonal $\Delta_{Z \times Z}$ does not belong to the product of $\sigma-$algebras $\mathbb{A}_1 \otimes_\sigma \mathbb{A}_2$. Thus, if X has a strictly larger cardinal than $\mathfrak{c}$, we have $\Delta_{X \times X} \notin \mathbb{B}_X \otimes_\sigma \mathbb{B}_X$. On the other hand, since $(X, \mathbb{T}_X)$ is a Hausdorff space, the diagonal $\Delta_{X \times X}$ is closed in the product topological space $(X \times X, \mathbb{T}_X \otimes_\tau \mathbb{T}_X)$. Thus $\Delta_{X \times X} \in \mathbb{B}_{X \times X}$. For instance, we may choose the Stone-Čech compactification $X = \beta N$ of the natural numbers N whose cardinal is $2^{\mathfrak{c}}$. This example is particularly remarkable because βN is separable.

To summarise, $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y \subset \mathbb{B}_{X \times Y}$ and the inclusion can be strict even for separable compact Hausdorff spaces. However, for different applications, the main point is to know whether the completion of the product measure is a topological one. In what follows we examine this problem in the extreme case in which $\mathbb{M}_X$ and $\mathbb{M}_Y$ are simply the completions of $\mathbb{B}_X$ and $\mathbb{B}_Y$ with respect to μ_X and μ_Y, respectively. Obviously, an answer to the problem depends on the definition of product measure and in this paper we will restrict ourselves to the standard product derived by Carathéodory's method from an outer measure generated by the set function $\mu_X \otimes \mu_Y (A \times B) = \mu_X (A) \mu_Y (B)$, where $A \in \mathbb{M}_X, B \in \mathbb{M}_Y, \mu_X (A) < \infty$ and $\mu_Y (B) < \infty$.

We assume that μ_X and μ_Y are $\sigma-$finite in order to have the product $\mu_X \otimes \mu_Y$ uniquely defined on the $\sigma-$algebra $\mathbb{A}$, the completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ or equivalently, on the completion of $\mathbb{M}_X \otimes_\sigma \mathbb{M}_Y$.

It might be expected that $(X \times Y, \mathbb{T}_{X \times Y}, \mathbb{A}, \mu_X \otimes \mu_Y)$ is again a topological measure space and if such were the case, that $\mathbb{A}$ would coincide with $\mathbb{M}_{X \times Y}$. Stated in this way, the conjecture was known as the *Product Measure Problem* and it remained open for a long time until Fremlin [3] presented a counterexample in 1976. Another, albeit simpler counterexample was presented in 1999 by Gryllakis and Grekas [6]. In an important paper [4], published in 1995, Fremlin and Grekas presented a very general sufficient condition that guarantees a positive answer to the *Product Measure Problem*:

> If X is quasi-dyadic, μ_X completion regular and μ_Y $\tau-$additive, then $(X \times Y, \mathbb{T}_{X \times Y}, \mathbb{A}, \mu_X \otimes \mu_Y)$ is a topological measure space.

They also treated the problem of whether the product of completion regular measures is completion regular and considered product measures defined by integral formulas as well. The terminology and the scope of these results are too extensive to be described here and the proofs of the main theorems are rather complicated and difficult. An evolution of the ideas of the subject may be found in Grekas' 2002 work, [5].

The main objective in this paper is to present a simple and quite general sufficient condition under which the *Product Measure Problem* has a positive answer. In such a case, $\mathbb{B}_{X\times Y}-$measurable functions are $\mu_X \otimes \mu_Y-$measurable almost everywhere and the extended version of the Fubini Theorem applies. To do this, we need to present certain beautiful links between locally metrizable Hausdorff topologies and strictly positive measures. We refer the reader to Bourbaki [1] and [2], Halmos [7] and Hewitt & Ross [8], for the foundation of the theory we are concerned with. In the main however, we will recall the concepts and results employed in the current paper.

2 Auxiliary Tools.

Given a nontrivial Hausdorff, σ-finite, complete topological measure space, to simplify notation we will eliminate subindexes and abbreviate expressions whenever no confusion should arise. A set $E \in \mathbb{M}$ is $\mu-$inner regular (or simply inner regular) if

$$\mu(E) = \sup\{\mu(K) : K \subset E,\, K \in \mathbb{K}\}$$

where $\mathbb{K}$ stands for the class of all compact subsets of X, and outer regular if

$$\mu(E) = \inf\{\mu(W) : E \subset W,\, W \in \mathbb{T}\}.$$

E is called regular whenever it is simultaneously inner and outer regular. The measure itself is called inner, outer or regular whenever all elements of $\mathbb{M}$ have the corresponding properties of regularity. A topological measure that is bounded on the compact sets is called a Borel measure.

Lemma 1. *Suppose F is a locally metrizable Hausdorff space and μ is σ-finite and inner regular in F. Then there exists a sequence of compact, metrizable and $\mu-$finite sets (K_n) in F, such that $\mu(F \setminus \cup_n K_n) = 0$.*

PROOF. Since μ is $\sigma-$finite, there exists a sequence (E_n) of disjoint measurable sets, each of finite $\mu-$measure, such that $F = \cup_n E_n$. Using that μ is inner regular, we find sequences $\left(K^*_{m,n}\right) \subset \mathbb{K}$, such that for every $n = 1, 2, ...$

$$K^*_{m,n} \subset E_n \text{ and } \mu(E_n) = \mu\left(\cup_m K^*_{m,n}\right).$$

As local metrizability is hereditary, the sets $K^*_{m,n}$ with the induced topology are locally metrizable and Hausdorff. Since they also are compact, they are metrizable. Rewrite this double sequence as the sequence (K_n) we are looking for.

$\square$

Remark 2. *In Lemma 1, μ is neither necessarily a Borel measure nor a regular one. For instance, let $X = N \cup \{\infty\}$ be the one point Alexandrov compactification of the natural numbers with the discrete topology and μ the counting measure in X. In this example the compact set X has infinite measure and $\{\infty\}$ is not an outer regular Borel set. This example is presented for other purposes in the text book of the author [9, page 184].*

The equality $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y = \mathbb{B}_{X\times Y}$ is well known whenever X and Y satisfy the second axiom of countability, but a weaker form also holds:

Lemma 3. *If one of the topological spaces X or Y satisfies the second axiom of countability, then $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y = \mathbb{B}_{X\times Y}$.*

Proof. Since $\mathbb{B}_{X\times Y} = \sigma(\mathbb{T}_{X\times Y})$, it is sufficient to prove that $\mathbb{T}_{X\times Y} \subset \mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$. Suppose that $\mathbb{U} = \{C_m : m \in \mathbb{N}\}$ is a topological basis of $\mathbb{T}_X$. Fix any open set $\cup_{i\in I}(A_i \times B_i)$ in $\mathbb{T}_{X\times Y} = \mathbb{T}_X \otimes_\tau \mathbb{T}_Y$, where $A_i \in \mathbb{T}_X$ and $B_i \in \mathbb{T}_Y$. Express A_i as $\cup_{(n,i)\in\mathbb{N}_i} C_{(n,i)}$, $\mathbb{N}_i \subset \mathbb{N}$, $C_{(n,i)} \in \mathbb{U}$. Define $J_m = \left\{i \in I : \exists\,(n,i)\,,\, C_{(n,i)} = C_m\right\}$, $m \in \mathbb{N}$ and $D_m = \cup_{i\in J_m} B_i$. Then

$$\begin{aligned}\cup_{i\in I}(A_i \times B_i) &= \cup_{i\in I}\left(\cup_{(n,i)\in\mathbb{N}_i} C_{(n,i)} \times B_i\right) = \cup_{i\in I}\cup_{(n,i)\in\mathbb{N}_i}\left(C_{(n,i)} \times B_i\right)\\ &= \cup_m (C_m \times D_m) \in \sigma(\mathbb{B}_X \times \mathbb{B}_X) = \mathbb{B}_X \otimes_\sigma \mathbb{B}_X\ .\end{aligned}$$

$\square$

To prove the existence of a maximal $\mu-$negligible open set W or equivalently, the existence of a smaller closed set F that supports the measure, it is sufficient to assume that all open sets are μ-inner regular. To see this, simply define $F= W^C$ and W to be the union of all $\mu-$negligible open sets $(U_i)_{i\in I}$. If $K \subset W$ and $K \in \mathbb{K}$, there exists a finite subfamily $(U_i)_{i\in J}$, $J \subset I$, that still covers the compact set K. Then $\mu(K) \leq \sum_{i\in J}\mu(U_i) = 0$. Since W is μ-inner regular and K was arbitrarily chosen, it follows that $\mu(W) = 0$.

We are now in position to announce and prove the contributions of this paper.

3 Results.

A topological measure in X is strictly positive whenever every non-empty open subset of X has a strictly positive measure. A topological space is said to satisfy the countable chain condition (ccc) if every pairwise disjoint collection of non-empty open subsets is countable. It is known that a topological $\sigma-$finite and strictly positive measure space X is ccc. In case X is also metrizable, it is separable. However, the first part of next theorem shows that more is true. Moreover, the second part is the main contribution of this paper.

Theorem 4. *Suppose that μ_X is a σ-finite and inner regular topological Hausdorff measure such that its support F in X with the induced topology is locally metrizable. Let $(Y, \mathbb{T}_Y, \mathbb{M}_Y, \mu_Y)$ be any other topological σ-finite measure space. Then F is separable and the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{B}_{X\times Y}$.*

PROOF. Since the compact subsets of F with the induced Hausdorff topology are just the compact subsets of X that are contained in F, the restriction of the measure μ_X to F is still inner regular. By hypothesis, F is locally metrizable so that F, with the induced topology and measure, satisfies the hypothesis of Lemma 1

Set $W = F^C$. Then W is the maximal μ_X-negligible open set of X. Let V be an open set in F. Then $V = U \cap F$, for some open set U in X. From $U = V \cup (U \setminus F)$, we obtain $\mu_X(U) = \mu_X(V)$. If $\mu_X(V) = 0$, then $U \subset W$ so that $V = \phi$. It follows that the restriction of μ_X to F is strictly positive.

The compact and metrizable sets $(K_n) \subset F$ given by Lemma 1 are separable. For each n, let A_n be a countable dense set in K_n. Thus, $A = \cup_n A_n$ is a countable set that is dense in $\cup_n K_n$. We claim A is also dense in F. To see this, let x be an element of F and V an open set that contains x. Then $\mu(V) > 0$ and, consequently, the relatively open set $V \cap (\cup_n K_n)$ in $(\cup_n K_n)$ is not empty. Thus, $V \cap A \neq \phi$ which shows F is separable.

For the second statement it is sufficient to prove the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{T}_{X\times Y}$. Once again by Lemma 1, let (K_n) be a sequence of compact sets in F such that $\mu_X(F \setminus \cup_n K_n) = 0$. Since these K_n are metrizable, each of them has a countable topological basis $\{V_{n,k} : k \in N\}$ of relatively open sets induced by the topology of F or equivalently by the topology of X. Set $V = \cup_{n,k} V_{n,k}$ and $P = F \setminus V$. As all sets under consideration are measurable, $\mu_X(P) = 0$ and V with the induced topology satisfies the second axiom of countability. Now express X as the disjoint union $V \cup P \cup W$. In this way, any open set U in $\mathbb{T}_{X\times Y}$ can be expressed as a disjoint union $A \cup B \cup C$ of relatively open sets $A \in \mathbb{T}_{V\times Y}$, $B \in \mathbb{T}_{P\times Y}$ and an open set $C \in$

$\mathbb{T}_{W\times Y}$. By Lemma 3

$$A \in \mathbb{T}_{V\times Y} = \mathbb{T}_V \otimes_\tau \mathbb{T}_Y \subset \mathbb{B}_V \otimes_\sigma \mathbb{B}_Y.$$

On the other hand $P\times Y \in \mathbb{B}_P \otimes_\sigma \mathbb{B}_Y$, $W\times Y \in \mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, $\mu_X \otimes \mu_Y (P\times Y) = 0$ and $\mu_X \otimes \mu_Y (W\times Y) = 0$. It follows that the $\mu_X \otimes \mu_Y-$ completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains U, and moreover that, $\mu_X \otimes \mu_Y (U) = \mu_X \otimes \mu_Y (A)$. $\square$

Remark 5. *Obviously, the statement of Theorem 4 could be simplified if we use the weaker hypothesis that X itself is locally metrizable, but we should loose a great deal of generality under that circumstance. For instance, consider the trivial case in which μ_X is the Dirac measure concentrated at a point x of any Hausdorff topological space X. Since in this case $F = \{x\}$, the hypothesis of the theorem holds.*

Remark 6. *The main point in Theorem 4 is the simplicity of its statement and its relatively easy proof while it keeps so many possibilities of applications. For instance, suppose X and Y are locally compact Hausdorff spaces, μ_X and μ_Y $\sigma-$finite Radon measures in the sense of Bourbaki and the support of one of these measures is locally metrizable. Then the completeness of the Radon product measure $\mu_{X\times Y}$ defined on $\mathbb{B}_{X\times Y}$ and the completeness of the product measure $\mu_X \otimes \mu_Y$ defined on $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, are equal. In fact, these measures are regular, the restriction $\mu_{X\times Y} / \mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$, coincides with $\mu_X \otimes \mu_Y$ and from Theorem 4, the $\mu_X \otimes \mu_Y-$completion of $\mathbb{B}_X \otimes_\sigma \mathbb{B}_Y$ contains $\mathbb{B}_{X\times Y}$.*

In spite its relatively easy proof, Theorem 4 is not trivial in the sense that F could be non-metrizable. This is exhibited by the following example.

Example 7. *Let $X = [0,1]$ be the closed unit interval. Partition X as $Q \cup I$, where Q stands for the rational numbers and I for the irrational numbers in X, respectively. Consider the class $\mathcal{C}$ of subsets of X defined by the set X, the entire collection of subsets of Q, and for every $x \in I$ and every natural number n, the subsets*

$$A(x,n) = \{x\} \cup \{q \in Q : |x-q| < 1/n\}.$$

Observe this class is closed under finite intersections. Thus $\mathcal{C}$ is a base of a topology $\mathbb{T}$ in X. It is easy to check that $\mathbb{T}$ is a Hausdorff topology.

Now we prove that every $x \in X$ has a metrizable neighborhood. This fact is trivial if $x \in Q$. For the other case, define a distance d in the neighborhood

$A(x,n)$ of $x \in I$, where n is chosen such that $x^{+}_{-}1/n \in X$, by the following rule: First divide $A(x,n) \setminus \{x\}$ as the disjoint union of rational intervals

$$A_m^- = (x - 1/m, x - 1/(m+1)) \cap Q, \text{ and}$$
$$A_m^+ = (x + 1/(m+1), x + 1/m) \cap Q$$

for $m = n, n+1, n+2, \ldots$. Then for every $q \in A_m^- \cup A_m^+$, set $d(q,x) = d(x,q) = 1/(m+1)$. If $p, q \in A_m^-$ or $p, q \in A_m^+$, with $p \neq q$, set $d(q,p) = d(p,q) = 1/m^3$. If $p \in A_r^+$, $q \in A_s^+$, or $p \in A_r^-$, $q \in A_s^-$, with $r \neq s$, set $d(q,p) = d(p,q) = |1/(r+1) - 1/(s+1)|$. If $p \in A_r^-$, $q \in A_s^+$, set $d(q,p) = d(p,q) = 1/(r+1) + 1/(s+1)$. Finally, for every $a \in A(x,n)$, set $d(a,a) = 0$. It is a straightforward task to prove d is a distance that defines the same topology of the restriction of $\mathbb{T}$ to $A(x,n)$.

We claim $\mathbb{T}$ is not metrizable. Suppose we have a distance d that defines the topology $\mathbb{T}$. Let $x \in I$. There exists $r > 0$, such that the open ball B centered at x with radius r, does not contain any irrational number with the exception of x. Let n be such that $A(x,n)$ is contained in the open ball centered at x with radius $r/2$. Let $p > x$, $p \in A(x,n)$. Choose an irrational number y such that $x < y < p$. Then there exists a rational number $q \in A(x,n)$, such that $d(y,q) < r/2$. We find the following contradiction. On one hand $d(x,y) \geq r$, but on the other one

$$d(x,y) \leq d(x,q) + d(q,y) < r/2 + r/2 = r.$$

Finally, define μ as the simplest topological measure that has mass 1 at each singleton $\{x\}$, $x \in Q$.

We have constructed an example in which the hypothesis of Theorem 4 are fulfilled, but $F = X$ is not metrizable.

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