RESEARCH

Real Analysis Exchange
Vol. 35(2), 2009/2010, pp. 431–444

Ricardo Estrada, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A. email: restrada@math.lsu.edu

Jasson Vindas,* Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A. email: jvindas@cage.ugent.be

ON ROMANOVSKI'S LEMMA

Abstract

Romanovski introduced a procedure, Romanovski's lemma, to construct the Denjoy integral without the use of transfinite induction. Here we give two versions of Romanovski's lemma which hold in general topological spaces. We provide several applications in various areas of mathematics.

1 Introduction.

In an article published in 1932 [20], Romanovski introduced a procedure to construct the Denjoy integral without the use of transfinite induction. The basic tool used in such a procedure is the following result, known as Romanovski's lemma.

Theorem 1.1. *(Romanovski's lemma) Let $\mathfrak{F}$ be a family of open intervals in (a, b) with the following four properties:*

I. If $(\alpha, \beta) \in \mathfrak{F}$ and $(\beta, \gamma) \in \mathfrak{F}$, then $(\alpha, \gamma) \in \mathfrak{F}$.

II. If $(\alpha, \beta) \in \mathfrak{F}$ and $(\gamma, \delta) \subset (\alpha, \beta)$ then $(\gamma, \delta) \in \mathfrak{F}$.

III. If $(\alpha, \beta) \in \mathfrak{F}$ for all $[\alpha, \beta] \subset (c, d)$ then $(c, d) \in \mathfrak{F}$.

Mathematical Reviews subject classification: Primary: 26B05, 28A33, 30E25, 46F10, 54A05
Key words: Romanovski's lemma, measures, analytic functions
Received by the editors May 24, 2009
Communicated by: Luisa Di Piazza
*Present Address: Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281, Gebouw S22, B 9000 Ghent, Belgium

IV. If all the intervals contiguous to a perfect closed set $E \subset [a,b]$ belong to $\mathfrak{F}$ then there exists an interval $I \in \mathfrak{F}$ with $I \cap E \neq \emptyset$.

Then $(a,b) \in \mathfrak{F}$.

The proof is very simple, but most importantly, the use of this lemma in several problems is also very simple, and many times provides rather short and direct proofs of hard results. The book of Gordon [11] gives several applications, such as the construction of the Denjoy integral we mentioned before or the study of the functions of the Baire classes.

It should be mentioned that the construction of the Denjoy integral by transfinite induction is also a very nice piece of mathematics. A detailed presentation can be found in the book by Hobson [13]. It is a matter of taste, of course, whether one prefers one approach or the other.

The aim of this article is to give versions of Romanovski's lemma that are useful in problems involving several variables. These versions are actually valid in general topological spaces. We present several applications to highlight the use of this rather powerful tool. The plan of the article is as follows. We start by giving two versions of the lemma that are valid in general spaces in Section 2. Our first application is a proof of the well known Cantor-Baire stationary principle by using the new versions of the lemma; this we present in Section 3. In Section 4 we prove a rather interesting characterization of positive measures in several variables in terms of the *almost everywhere* angular boundary behavior of the distributional $\phi-$transform, generalizing the corresponding results for everywhere behavior [22]. We then employ these ideas to prove that a function with a derivative that exists almost everywhere in a region $\Omega \subset \mathbb{C}$ will in some cases be analytic in Ω. In Section 5 we apply Romanovski's lemma in the study of analytic functions with known *radial* behavior at the boundary of the unit disc, a subject where many unexpected counterexamples exist.

Naturally, there are many other areas where Romanovski's lemma would prove to be very helpful, and our purpose in writing this article is to invite the reader to try it in such domains.

2 Versions of Romanovski's Lemma.

Our first version applies to any topological space.

Theorem 2.1. *Let X be a topological space. Let $\mathfrak{U}$ be a non empty family of open sets of X that satisfies the following four properties:*

I_{v_1}. $\mathfrak{U} \neq \{\varnothing\}$.

II_{v_1}. *If $U \in \mathfrak{U}$, $V \subset U$, and V is open, then $V \in \mathfrak{U}$.*

III_{v_1}. If $U_\alpha \in \mathfrak{U}$ $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathfrak{U}$.

IV_{v_1}. Whenever $U \in \mathfrak{U}$, $U \neq X$, then there exists $V \in \mathfrak{U}$ such that $V \cap (X \setminus U) \neq \varnothing$.

Then $\mathfrak{U}$ must be the class of all open subsets of X.

Proof. Let $W = \bigcup_{U \in \mathfrak{U}} U$. Using I_{v_1} it follows that $W \neq \varnothing$, employing III_{v_1} we obtain that $W \in \mathfrak{U}$, while from IV_{v_1} it follows that $W = X$. If we now use II_{v_1} we obtain that all open subsets of X belong to $\mathfrak{U}$. □

The next version deals with local basis of neighborhoods.

Theorem 2.2. *Let X be a topological space. For each $x \in X$ let $\mathfrak{C}_x$ be a local basis of non-empty open neighborhoods at x, and let $\mathfrak{C} = \bigcup_{x \in X} \mathfrak{C}_x$. Let $\mathfrak{B} \subset \mathfrak{C}$ be a family that satisfies the following properties:*

I_{v_2}. $\mathfrak{B} \neq \varnothing$.

II_{v_2}. If $U \in \mathfrak{B}$, $V \subset U$, and $V \in \mathfrak{C}$, then $V \in \mathfrak{B}$.

III_{v_2}. If $U \in \mathfrak{C}$ and if for each $x \in U$ there exists $V_x \in \mathfrak{B}$ with $x \in V_x \subset U$, then $U \in \mathfrak{B}$.

IV_{v_2}. If F is closed, $F \neq \varnothing$, and for each $x \in X \setminus F$ there exists $V_x \in \mathfrak{B}$ with $x \in V_x \subset X \setminus F$, then $\exists U_0 \in \mathfrak{B}$ with $U_0 \cap F \neq \varnothing$.

Then $\mathfrak{B} = \mathfrak{C}$.

Proof. Let $W = \bigcup_{U \in \mathfrak{B}} U$. Then $W = X$, since if not $F = X \setminus W$ would satisfy the conditions of IV_{v_2}, but this is not possible.

Let now $V \in \mathfrak{C}$. Since $X = \bigcup_{U \in \mathfrak{B}} U$, for each $x \in V$ there exists $B_x \in \mathfrak{B}$ with $x \in B_x$, and thus there is $C_x \in \mathfrak{C}$ with $x \in C_x \subset V \cap B_x$. Using II_{v_2} we obtain that $C_x \in \mathfrak{B}$, and thus we can use III_{v_2} to conclude that $V \in \mathfrak{B}$. □

This version, Theorem 2.2, applies to balls in a metric space. Indeed, one can take $\mathfrak{C}$ to be the family all of non-empty balls, or a suitable subfamily, say $\mathfrak{C}_x$ equal to the set of balls centered at x and radius smaller than some given number $r_x > 0$.

Observe that, as it is easy to see, the usual Romanovski's lemma, Theorem 1.1, follows from Theorem 2.2.

3 The Cantor-Baire Stationary Principle.

In this section we shall show how Romanovski's lemma can be used to give a proof of the well known and useful Cantor-Baire stationary principle [18]. The symbol Ω refers to the first uncountable ordinal number.

Theorem 3.1. *(Cantor-Baire Stationary Principle) Let $\{F_\alpha\}_{\alpha<\Omega}$ be a family of closed subsets of $\mathbb{R}^n$, indexed by the countable ordinal numbers. Suppose $\{F_\alpha\}_{\alpha<\Omega}$ is decreasing; i.e., $F_\alpha \subseteq F_\beta$ if $\alpha \geq \beta$. Then there exists $\alpha^* < \Omega$ such that $F_\alpha = F_{\alpha^*}$ for $\alpha \geq \alpha^*$.*

PROOF. Let $X = \mathbb{R}^n \setminus \bigcap_{\alpha<\Omega} F_\alpha$. Let $\mathfrak{B}$ be the family of non-empty balls $B \subset X$ that satisfy that there exists $\alpha < \Omega$ with $B \cap F_\alpha = \varnothing$. We shall show that $\mathfrak{B} = \mathfrak{C}$, the family of all non-empty balls of $\mathbb{R}^n$ contained in X, using Theorem 2.2. Observe that $\mathrm{I}_{\mathrm{v}_2}$ and $\mathrm{II}_{\mathrm{v}_2}$ are clear. In order to prove $\mathrm{III}_{\mathrm{v}_2}$, let $B = \bigcup_{\sigma\in\Sigma} B_\sigma$ be an element of $\mathfrak{C}$ that is an arbitrary union of elements B_σ of $\mathfrak{B}$; since B is a ball in $\mathbb{R}^n$, there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $B = \bigcup_{n\in\mathbb{N}} B_{\sigma_n}$. If $\alpha_n < \Omega$ is such that $B_{\sigma_n} \cap F_{\alpha_n} = \varnothing$, and $\widehat{\alpha} = \sup_{n\in\mathbb{N}} \alpha_n$, then $B \cap F_{\widehat{\alpha}} = \varnothing$, so that $B \in \mathfrak{B}$. Condition $\mathrm{IV}_{\mathrm{v}_2}$ is easy because if $x \in X$ then there exists $\alpha < \Omega$ such that $x \notin F_\alpha$ and thus there exists a ball B with $x \in B$ and $B \cap F_\alpha = \varnothing$, which yields that $B \in \mathfrak{B}$. Finally, since we obtain that $X = \bigcup_{B\in\mathfrak{B}} B$, then $X = \bigcup_{n\in\mathbb{N}} B_n$, where $\{B_n\}_{n\in\mathbb{N}}$ is an enumeration of the elements of $\mathfrak{B}$ with rational centers and rational radii; choosing $\alpha_n < \Omega$ with $B_n \cap F_{\alpha_n} = \varnothing$, and putting $\alpha^* = \sup_{n\in\mathbb{N}} \alpha_n$, we obtain $\alpha^* < \Omega$ with $X \cap F_{\alpha^*} = \varnothing$. It follows that $F_\alpha = F_{\alpha^*}$ for $\alpha \geq \alpha^*$. □

4 Measures and the $\phi-$transform.

In this section we shall deal with *real* valued distributions and functions. We shall use the standard spaces of test functions $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ [10]. The $\phi-$transform [6, 21, 22] is defined as follows. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ be a fixed *normalized* test function; that is, one that satisfies

$$\int_{\mathbb{R}^n} \phi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1\,. \tag{4.1}$$

If $f \in \mathcal{D}'(\mathbb{R}^n)$ we introduce the function of $n+1$ variables $F = F_\phi\{f\}$ by the formula

$$F(\mathbf{x}, t) = \langle f(\mathbf{x} + t\mathbf{y}), \phi(\mathbf{y})\rangle\,, \tag{4.2}$$

where $(\mathbf{x},t)\in\mathbb{H}$, the half space $\mathbb{R}^n\times(0,\infty)$. Naturally the evaluation in (4.2) is with respect to the variable $\mathbf{y}$. We call F the $\phi-$transform of f. Whenever we consider $\phi-$transforms we assume that ϕ satisfies (4.1).

The definition of the $\phi-$transform tells us that if the distributional point value [14, 15] $f(\mathbf{x}_0)$ exists and equals γ then $F(\mathbf{x}_0,t)\to\gamma$ as $t\to0^+$, but actually $F(\mathbf{x},t)\to\gamma$ as $(\mathbf{x},t)\to(\mathbf{x}_0,0)$ in an *angular* or *non-tangential* fashion; that is if $|\mathbf{x}-\mathbf{x}_0|\le Mt$ for some $M>0$ (just replace $\phi(\mathbf{y})$ by $\phi(\mathbf{y}-M\omega)$ where $|\omega|=1$).

The $\phi-$transform converges to the distribution as $t\to0^+$: If $\phi\in\mathcal{D}(\mathbb{R}^n)$ and $f\in\mathcal{D}'(\mathbb{R}^n)$, then

$$\lim_{t\to0^+}F(\mathbf{x},t)=f(\mathbf{x}), \tag{4.3}$$

distributionally in the space $\mathcal{D}'(\mathbb{R}^n)$; that is, if $\rho\in\mathcal{D}(\mathbb{R}^n)$ then

$$\lim_{t\to0^+}\langle F(\mathbf{x},t),\rho(\mathbf{x})\rangle=\langle f(\mathbf{x}),\rho(\mathbf{x})\rangle. \tag{4.4}$$

We shall use the following nomenclature. A (Radon) measure would mean a *positive* functional in the space of continuous functions, which would be denoted by integral notation such as $\mathrm{d}\mu$, or by distributional notation, $f=f_\mu$, so that

$$\langle f,\phi\rangle=\int_{\mathbb{R}^n}\phi(\mathbf{x})\,\mathrm{d}\mu, \tag{4.5}$$

and $\langle f,\phi\rangle\ge0$ if $\phi\ge0$. A signed measure is a real bounded functional in the space of continuous functions, denoted as, say $\mathrm{d}\nu$, or as $g=g_\nu$. Observe that any signed measure can be written as $\nu=\nu_+-\nu_-$, where $\nu_\pm$ are measures concentrated on disjoint sets. We shall also use the Lebesgue decomposition, according to which any signed measure ν can be written as $\nu=\nu_{\mathrm{abs}}+\nu_{\mathrm{sig}}$, where ν_{abs} is absolutely continuous with respect to the Lebesgue measure, so that it corresponds to a regular distribution, while ν_{sig} is a signed measure concentrated on a set of Lebesgue measure zero. We shall also need to consider the measures $(\nu_{\mathrm{sig}})_\pm=(\nu_\pm)_{\mathrm{sig}}$, the positive and negative singular parts of ν.

If $\mathbf{x}_0\in\mathbb{R}^n$ we shall denote by $C_{\mathbf{x}_0,\theta}$ the cone in $\mathbb{H}$ starting at $\mathbf{x}_0$ of angle θ,

$$C_{\mathbf{x}_0,\theta}=\{(\mathbf{x},t)\in\mathbb{H}:|\mathbf{x}-\mathbf{x}_0|\le(\tan\theta)t\}. \tag{4.6}$$

If $f\in\mathcal{D}'(\mathbb{R}^n)$ and $\mathbf{x}_0\in\mathbb{R}^n$ then we consider the upper and lower angular values of its $\phi-$transform,

$$f^{+}_{\phi,\theta}(\mathbf{x}_0) = \limsup_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0)\\ (\mathbf{x},t)\in C_{\mathbf{x}_0,\theta}}} F(\mathbf{x},t)\,, \tag{4.7}$$

$$f^{-}_{\phi,\theta}(\mathbf{x}_0) = \liminf_{\substack{(\mathbf{x},t)\to(\mathbf{x}_0,0)\\ (\mathbf{x},t)\in C_{\mathbf{x}_0,\theta}}} F(\mathbf{x},t)\,. \tag{4.8}$$

The quantities $f^{\pm}_{\phi,\theta}(\mathbf{x}_0)$ are well defined at all points $\mathbf{x}_0$, but, of course, they could be infinite.

The following result was proved in [22].

Theorem 4.1. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set. Then f is a measure in U if and only if its $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ satisfies*

$$f^{-}_{\phi,\theta}(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in U\,, \tag{4.9}$$

for all angles θ. Moreover, if the support of ϕ is contained in a ball of radius R and center at the origin and if (4.9) holds for a single value of $\theta >$ arctan R, then f is a measure in U.

It is easy to see that the result is not true if we use *radial* limits instead of angular ones. An example is provided by taking $f(x) = -\delta'(x)$ and $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi'(0) > 0$. Actually this example shows that if (4.9) holds for a value of $\theta < \arctan R$, then f might not be a measure. Furthermore, one needs the inequality (4.9) to be true at *all* points of U, as the example $f(\mathbf{x}) = -\delta(\mathbf{x}-\mathbf{a})$, for any $\mathbf{a} \in U$, shows.

We should also point out that if there exists a constant $M > 0$ such that $f^{-}_{\phi,\theta}(\mathbf{x}) \geq -M$, $\forall \mathbf{x} \in U$, where $\theta > \arctan R$, then f is a signed measure in U, whose singular part is *positive* [22].

Using Romanovski's lemma we can prove the ensuing stronger result.

Theorem 4.2. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set. If the $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$ satisfies*

$$f^{+}_{\phi,\vartheta}(\mathbf{x}) \geq 0 \text{ almost everywhere in } U, \tag{4.10}$$

for some ϑ, while for each $\mathbf{x} \in U$ there is a constant $M_{\mathbf{x}} > 0$ such that

$$f^{-}_{\phi,\theta}(\mathbf{x}) \geq -M_{\mathbf{x}}\,, \tag{4.11}$$

where $\theta >$ arctan R, then f is a measure in U.

Proof. Let $\mathfrak{U}$ be the family of open subsets V of U such that the restriction $f|_V$ is a measure. Let us first show that $\mathfrak{U} \neq \emptyset$. Indeed, let $t_0 \geq 1$ be fixed and let

$$g_n(\mathbf{x}) = \min\left\{F(\mathbf{y}, t) : |\mathbf{y} - \mathbf{x}| \leq (\tan\theta)t,\ n^{-1} \leq t \leq t_0\right\}. \tag{4.12}$$

The functions g_n are continuous and because of (4.11), for each $\mathbf{x} \in U$ there exists a constant $M'_{\mathbf{x}} > 0$ such that $g_n(\mathbf{x}) \geq -M'_{\mathbf{x}}$. If we now employ the Baire theorem we obtain the existence of a non-empty open set $V \subset U$ and a constant $M > 0$ such that $F(\mathbf{x}, t) \geq -M$ for all $(\mathbf{y}, t) \in C_{\mathbf{x},\theta}$ with $\mathbf{x} \in V$ and $0 < t \leq t_0$, and hence $f^-_{\phi,\theta}(\mathbf{x}) \geq -M$ for $\mathbf{x} \in V$. It follows that $f|_V$ is a signed measure $\nu = \nu_{\text{abs}} + \nu_{\text{sig}}$, whose singular part ν_{sig} is a measure; it thus follows using (4.10) that the regular distribution f_{abs} corresponding to ν_{abs} satisfies $f_{\text{abs}}(\mathbf{x}) \geq 0$ a.e. in V, and consequently the signed measure ν_{abs} is actually a measure. Therefore $\nu \geq 0$.

It is clear that II_{v_1} and III_{v_1} are satisfied.

In order to prove IV_{v_1}, let $V \in \mathfrak{U}$, and suppose that $U \setminus V \neq \emptyset$. Then using the Baire theorem again, there exists a set Y open in $U \setminus V$ and a constant $M > 0$ such that $f^-_{\phi,\theta}(\mathbf{x}) \geq -M$ for all $\mathbf{x} \in Y$. But if Y is open in $U \setminus V$, then we can find W open in U such that $W \cap U \setminus V = Y$. Observe now that $f|_V$ is a measure, and thus $f^-_{\phi,\theta}(\mathbf{x}) \geq 0$ if $\mathbf{x} \in V$, in particular, if $\mathbf{x} \in W \cap V$. Therefore, $f^-_{\phi,\theta}(\mathbf{x}) \geq -M$ for $\mathbf{x} \in W = Y \cup (W \cap V)$. The argument used above shows that $f|_W$ is a measure; that is, $W \in \mathfrak{U}$ and this proves IV_{v_1}. □

The following useful result on the existence of almost everywhere limits follows from the Theorem 4.2.

Corollary 4.3. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Let U be an open set. If the $\phi-$transform $F = F_\phi\{f\}$ with respect to a given normalized, positive test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$ satisfies that for each $\mathbf{x} \in U$ there is a constant $M_{\mathbf{x}} > 0$ such that*

$$M_{\mathbf{x}} \geq f^+_{\phi,\theta}(\mathbf{x}) \geq f^-_{\phi,\theta}(\mathbf{x}) \geq -M_{\mathbf{x}}, \tag{4.13}$$

where $\theta > \arctan R$, while for some ϑ

$$f^+_{\phi,\vartheta}(\mathbf{x}) \geq 0 \geq f^-_{\phi,\vartheta}(\mathbf{x}), \tag{4.14}$$

almost everywhere in U, then $f = 0$ in U.

4.1 Analytic Functions and Existence of Derivatives A.E.

One can use these ideas, for example, to study the several equivalent definitions of holomorphy of a function f defined in a region $\Omega \subset \mathbb{C}$. If we say that f is holomorphic or analytic if it is equal to the sum of a convergent power series in the neighborhood of each point of Ω, then we easily obtain that f is analytic if and only if $\oint_{\partial R} f(z)\, \mathrm{d}z = 0$ for each rectangle R with $\overline{R} \subset \Omega$; this is Morera's theorem. On the other hand, if f is analytic then it is complex differentiable; that is,

$$f'(z) = \lim_{\xi \to 0} \frac{f(z+\xi) - f(z)}{\xi}\,, \tag{4.15}$$

exists at each $z \in \Omega$. Conversely, if f is complex differentiable then it is analytic, and this follows by proving that $\oint_{\partial R} f(z)\, \mathrm{d}z = 0$ for each rectangle R with $\overline{R} \subset \Omega$; earlier proofs of this fact used the continuity of f', but Goursat [12] gave a nice proof, now standard in the textbooks in complex variables [17, Sect. 1.2], that shows that the mere existence of $f'(z)$ implies that f is analytic. Actually Looman [16] proved that it is enough to assume that $f'(z)$ exists almost everywhere if we suppose that

$$f(z+\xi) = f(z) + O(\xi)\,, \text{ as } \xi \to 0\,, \tag{4.16}$$

everywhere in Ω. This follows from the Corollary 4.3. Indeed, if R is a fixed rectangle with $\overline{R} \subset \Omega$, we can find f_1 continuous in all $\mathbb{C}$ with $f_1 = f$ in R. Let $z_0 = x_0 + iy_0 \in R$ and define

$$g_{z_0}(z) = \oint_{\partial R_{z_0,z}} f_1(\zeta)\, \mathrm{d}\zeta\,, \tag{4.17}$$

where $R_{z_0,z}$ is the rectangle with vertices $z_0 = x_0 + iy_0$, $x + iy_0$, $z = x + iy$, and $x_0 + iy$.

The identity

$$g_{z_0}(x+iy) = g_{z_1}(x+iy) + g_{x_1+iy_0}(x+iy_1) + g_{x_0+iy_1}(x_1+iy)\,, \tag{4.18}$$

implies that $h = \partial^2 g_{z_0}/\partial x \partial y$ does not depend on z_0. The hypotheses of Corollary 4.3 are satisfied in R for h for *any* test function ϕ, since (4.16) yields

$$g_{z_0}(z_0+\xi) = O\left(\xi^2\right)\,, \tag{4.19}$$

for any $z_0 \in R$, while the existence of $f'(z)$ yields

$$g_{z_0}(z_0+\xi) = o\left(\xi^2\right)\,, \tag{4.20}$$

and this holds almost everywhere. It follows that $h = 0$ in R and, consequently, $g_{z_0} = 0$ in R for any z_0; this implies that f is analytic in R and thus, since R is arbitrary, in Ω.

It is also well known [5, Chapter 6] that if f is continuous in Ω, analytic in $\Omega \setminus Z$, where Z is a closed set with finite linear Haussdorf measure $H_1(Z) < \infty$, then f is actually analytic in Ω. We can use Romanovski's lemma, in the version given in the Theorem 2.1, to prove the following result of Besicovitch [2]: *If f is continuous in Ω, the derivative $f'(z)$ exists almost everywhere, and f satisfies (4.16) in $\Omega \setminus \bigcup_{n=1}^{\infty} Z_n$, where the Z_n are closed sets of finite linear Haussdorf measure, then f is analytic in Ω.* Indeed, one may take $\mathfrak{U}$ to be the class of open subsets of Ω where f is analytic. Conditions $\mathrm{II}_{\mathrm{v}_1}$ and $\mathrm{III}_{\mathrm{v}_1}$ are easy, while to prove $\mathrm{I}_{\mathrm{v}_1}$ and $\mathrm{IV}_{\mathrm{v}_1}$ one can use Baire's theorem in the decomposition of any K closed in Ω into a denumerable union of closed subsets as

$$K = \bigcup_{n=1}^{\infty} (X_n \cup Z_1 \cdots \cup Z_n) \cap K \,, \tag{4.21}$$

where

$$X_n = \left\{ z : \left| \frac{f(z+\xi) - f(z)}{\xi} \right| \leq n \text{ for } |\xi| \leq 1, \ z + \xi \in \Omega \right\}. \tag{4.22}$$

5 Continuous Extensions of Analytic Functions.

Let F be analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that F can be extended to an element $F_{\mathrm{ex}} \in C\left(\overline{\mathbb{D}}\right)$, the continuous functions in $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$. Then it is well known that the Cauchy integral formula

$$F(z) = \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - z} \,, \tag{5.1}$$

holds, where f is the restriction of F_{ex} to $\partial\mathbb{D}$. The set of all such restrictions forms a well known and much studied subalgebra A of $C(\partial\mathbb{D})$; one could say that in a sense A is of about "half the dimension" of $C(\partial\mathbb{D})$, since the elements of $C(\partial\mathbb{D})$ have distributionally convergent Fourier expansions of the type $g\left(e^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, for some constants a_n, $n \in \mathbb{Z}$, while the elements of A are those for which $a_n = 0$ for $n < 0$.

The Cauchy representation formula (5.1) holds not only for analytic functions that have continuous extensions, but also in many other spaces, such as the Hardy spaces H^p, $1 \leq p \leq \infty$, if one takes f as the *radial* limit

$f(\xi) = \lim_{r\to 1^-} F(r\xi)$, which in such a case exists almost everywhere in $\partial\mathbb{D}$ and defines a Lebesgue integrable function there. It holds, in particular, if $F \in H^\infty$; that is, if F is bounded in $\mathbb{D}$ [7, 19].

It is interesting that the mere existence of radial limits is usually not enough for the validity of (5.1). Indeed, the function

$$W(z) = (z-1)\, e^{i((z+1)/(z-1))^2}, \tag{5.2}$$

is analytic in $\mathbb{D}$, the radial limit $w(\xi) = \lim_{r\to 1^-} W(r\xi)$ exists for *all* elements $\xi \in \partial\mathbb{D}$, and actually the function w is continuous in $\partial\mathbb{D}$. However, $w \in C(\partial\mathbb{D}) \setminus \mathsf{A}$ and the Cauchy representation formula does not hold: this is clear because W is not bounded in $\mathbb{D}$, while all elements of $C\left(\overline{\mathbb{D}}\right)$ are.

If one considers functions with *radial limits almost everywhere* the situation is even more surprising [4, 19], since if $g \in C(\partial\mathbb{D})$ and X is given a subset of $\partial\mathbb{D}$ of first category (which one can take of full measure!) then there is a function G, analytic in $\mathbb{D}$, that satisfies $g(\xi) = \lim_{r\to 1^-} G(r\xi)$ for all $\xi \in X$.

We shall now use Romanovski's lemma to show that if some extra conditions are satisfied then if the almost everywhere radial limits of an analytic function in $\mathbb{D}$, F, are the restriction of a function $f \in C(\partial\mathbb{D})$ to a set of full measure, then actually F admits an extension in $C\left(\overline{\mathbb{D}}\right)$ and $f \in \mathsf{A}$. We shall assume that F is bounded on radial segments and we shall also suppose that F has *distributional* boundary values in the circle $\partial\mathbb{D}$ [1, 3, 9]. Notice that the existence of the distributional limit $f(\xi) = \lim_{r\to 1^-} F(r\xi)$, $f \in \mathcal{D}'(\partial\mathbb{D})$ is equivalent to the growth estimate

$$|F(z)| \le \frac{M}{(1-|z|)^\alpha}, \tag{5.3}$$

for some constants $M > 0$ and $\alpha \in \mathbb{R}$.

We need a couple of preliminary results. First, it is well known that if F is analytic in $\mathbb{D}$ and if the distributional boundary value $f(\xi) = \lim_{r\to 1^-} F(r\xi)$ exists, $f \in \mathcal{D}'(\partial\mathbb{D})$, then f cannot have jump discontinuities; i.e., if the lateral limits $f(\xi^+)$ and $f(\xi^-)$ exist at $\xi \in \partial\mathbb{D}$, then they have to coincide $f(\xi^+) = f(\xi^-)$ [8]. This yields the following result.

Lemma 5.1. *Let F be analytic in $\mathbb{D}$ with distributional boundary value $f(\xi) = \lim_{r\to 1^-} F(r\xi)$, $f \in \mathcal{D}'(\partial\mathbb{D})$. Suppose that there exists a bounded function f_0 in $\partial\mathbb{D}$ and a finite set $E \subset \partial\mathbb{D}$ such that $f = f_0$ in $\partial\mathbb{D} \setminus E$. Then $f = f_0$ in all of $\partial\mathbb{D}$.*

PROOF. Indeed, $f - f_0$ has support contained in E, and it is thus equal to a finite sum of Dirac delta functions and its derivatives at E, $f(\omega) - f_0(\omega) =$

$\sum_{\xi\in E}\sum_{j=0}^{n} a_{\xi,j}\delta^{(j)}(\omega-\xi)$. There is a constant a and a distribution $g \in \mathcal{D}'(\partial\mathbb{D})$ such that $f - f_0 = a + g^{(n+1)}$. Since g would have jumps of magnitude $a_{\xi,n}$ at each $\xi \in E$, it follows that $a_{\xi,n} = 0$ $\forall\xi \in E$, and, by induction, that $a_{\xi,j} = 0$ for $0 \le j \le n$ and $\xi \in E$. Hence $f = f_0$. □

We emphasize that the previous result does not hold if the analytic function does not have distributional boundary values: (5.2) is an example.

Using a conformal map we also obtain the following result.

Lemma 5.2. *Let F be analytic in $\mathbb{D}$ with distributional boundary value $f(\xi) = \lim_{r\to 1^-} F(r\xi)$, $f \in \mathcal{D}'(\partial\mathbb{D})$. Suppose there is an arc $I = \left\{e^{i\theta} : \theta_1 < \theta < \theta_2\right\}$ of $\partial\mathbb{D}$ such that f is a bounded function in I,*

$$\left|f\left(e^{i\theta}\right)\right| \le M,\ \theta_1 < \theta < \theta_2\,, \tag{5.4}$$

and suppose that

$$\left|F\left(re^{i\theta_j}\right)\right| \le M,\ 0 \le r < 1,\ j = 1, 2\,. \tag{5.5}$$

Then

$$\left|F\left(re^{i\theta}\right)\right| \le M,\ 0 \le r < 1,\ \theta_1 < \theta < \theta_2\,. \tag{5.6}$$

We can now give the main result of this section.

Theorem 5.3. *Let F be analytic in $\mathbb{D}$ and let f be continuous in $\partial\mathbb{D}$. Suppose the following three conditions are satisfied:*

1. $\lim_{r\to 1^-} F(r\xi) = f(\xi)$, *almost everywhere;*
2. *F has distributional boundary limits in $\partial\mathbb{D}$; and*
3. *there is a countable set E such that for all $\xi \in \partial\mathbb{D}\setminus E$ there is a constant $M_\xi < \infty$ with $|F(r\xi)| \le M_\xi$ for all $r \in [0, 1)$.*

Then $f \in \mathsf{A}$ and the function F_{ex} defined in $\overline{\mathbb{D}}$ as F in $\mathbb{D}$ and as f in $\partial\mathbb{D}$ belongs to $C\left(\overline{\mathbb{D}}\right)$.

Proof. Let $\mathfrak{B}$ be the set of open arcs I of $\partial\mathbb{D}$ such that the function given by F in $\mathbb{D}$ and by f in I is continuous in $\mathbb{D}\cup I$. We shall show that $\mathfrak{B}$ satisfies the hypotheses of the Theorem 2.2.

First we prove $\mathrm{I}_{\mathrm{v}_2}$; that is, that $\mathfrak{B} \ne \emptyset$. Let $r_n \in [0, 1)$ such that $r_n \nearrow 1$, and consider the functions $h_n(\xi) = \max\{|F(r\xi)| : 0 \le r \le r_n\}$; the h_n are continuous functions, and for each $\xi \in \partial\mathbb{D}\setminus E$ we have $h_n(\xi) \le M_\xi$. Therefore, using the Baire theorem, we can find a non-empty open arc I and a constant $M>0$ such that $h_n(\xi)\le M$ for all n and for $\xi\in I$, and thus $|F(r\xi)|\le M$ for all $r\in[0,1)$ and $\xi\in I$. Then $\lim_{r\to1^-}F(r\xi)=g(\xi)$ exists in a weak* sense in the space of bounded functions $L^\infty(J)$ for any open arc J with $\overline{J}\subset I$. But 1 implies that $f=g$ a.e. and therefore $\lim_{r\to1^-}F(r\xi)=f(\xi)$ uniformly in J. Hence $I\in\mathfrak{B}$.

Conditions $\mathrm{II}_{\mathrm{v}_2}$ and $\mathrm{III}_{\mathrm{v}_2}$ are clear. Finally we can establish $\mathrm{IV}_{\mathrm{v}_2}$ as follows. Let K be a non-empty closed proper subset of $\partial\mathbb{D}$, such that $\partial\mathbb{D}\setminus K=\bigcup_{n=1}^{\infty}I_n$, where $I_n\in\mathfrak{B}$ are disjoint. If K has an isolated point ξ_0, then the Lemma 5.1 immediately yields that if J is an open arc with $J\cap K=\{\xi_0\}$ then $J\in\mathfrak{B}$. When K is perfect, we can use Baire's theorem again to obtain an open arc J with $J\cap K\neq\emptyset$ such that for some constant M, we have $|F(r\xi)|\le M$ for $r\in[0,1)$ and for $\xi\in J\cap K$. We may suppose that $M\ge\max_{\omega\in\partial\mathbb{D}}|f(\omega)|$, and we may also suppose that the endpoints of J belong to K and that $|F(r\xi)|\le M$ for $r\in[0,1)$ when ξ is one of the endpoints. If I_n is one of the open arcs of $\partial\mathbb{D}\setminus K$ with $I_n\subset J$, then its endpoints belong to $\overline{J}\cap K$ and thus we can use the Lemma 5.2 to conclude that $|F(r\xi)|\le M$ for $r\in[0,1)$ and $\xi\in I_n$. Therefore $|F|$ is bounded by M in the sector $z=r\omega$, $r\in[0,1)$ and $\omega\in\overline{J}$. It follows that $J\in\mathfrak{B}$. □

References

[1] E. J. Beltrami and M. R. Wohlers, *Distributions and the Boundary Values of Analytic Functions,* Academic Press, New York-London, 1966.

[2] A. S. Besicovitch, *On sufficient conditions for a function to be analytic, and on the behaviour of analytic functions in the neighborhood of non-isolated singular points,* Proc. London Math. Soc., **32** (1931), 1–9.

[3] H. Bremermann, *Distributions, Complex Variables and Fourier Transforms,* Addison-Wesley, Reading, Massachusetts-London, 1965.

[4] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets,* Cambridge Tracts in Mathematics and Mathematical Physics, **56**, Cambridge University Press, Cambridge, 1966.

[5] W. F. Donoghue, *Distributions and Fourier Transforms,* Academic Press, New York, 1969.

[6] Y. N. Drozhzhinov and B. I. Zavyalov, *Tauberian theorems for generalized functions with values in Banach spaces*, Izv. Ross. Acad. Nauk Ser. Mat.,

66 (2002), 47–118 (Russian); translation in: Izv. Math., **66** (2002), 701–769.

[7] P. L. Duren, *The theory of H^p spaces,* Pure and Applied Mathematics, **38**, Academic Press, New York-London, 1970.

[8] R. Estrada, *One-sided cluster sets of distributional boundary values of analytic functions,* Complex Var. and Elliptic Eqns., **51** (2006), 661–673.

[9] R. Estrada and R. P. Kanwal, *Distributional boundary values of harmonic and analytic functions,* J. Math. Anal. Appls., **89** (1982), 262–289.

[10] R. Estrada and R. P. Kanwal, *A Distributional Approach to Asymptotics: Theory and Applications*, Second edition. Birkhäuser Advanced Texts: Basler Lehrbücher, [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Boston, Boston, MA, 2002.

[11] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock,* (English summary) Graduate Studies in Mathematics, **4** American Mathematical Society, Providence, RI, 1994.

[12] E. Goursat, *Sur la définition générale des functions analytiques d'après Cauchy,* Trans. Amer. Math. Soc., **1** (1900), 14–16.

[13] E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier series,* Vol. 1, Dover, New York, 1958.

[14] S. Łojasiewicz, *Sur la valuer et la limite d'une distribution en un point,* Studia Math., **16** (1957), 1–36.

[15] S. Łojasiewicz, *Sur la fixation des variables dans une distribution,* Studia Math., **17** (1958), 1–64.

[16] H. Looman, *Ueber eine Erweiterung des Cauchy-Goursatschen Integralsatzes,* Nieuw. Arch. Wiskde, **14** (1925), 234–239.

[17] R. Narasimhan and Y. Nievergelt, *Complex Analysis in One Variable,* (English summary) Second edition, Birkhäuser Boston, Boston, 2001.

[18] I. P. Natanson, *Theory of functions of a real variable,* Vol. II, Translated from the Russian by Leo F. Boron, Frederick Ungar Publishing, New York, 1961.

[19] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **299**, Springer-Verlag, Berlin, 1992.

[20] P. Romanovski, *Essai d'une exposition de l'integrale de Denjoy sans nombres transfini,* Fund. Math., **19** (1932), 38–44.

[21] J. Vindas and R. Estrada, *Distributionally regulated functions,* Studia Math., **181** (2007), 211–236.

[22] J. Vindas and R. Estrada, *Measures and the distributional* $\phi-$*transform,* Integral Transforms Spec. Funct., **20** (2009), 325–332.