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ON APPROXIMATIONS OF SEMICONTINUOUS FUNCTIONS BY DARBOUX SEMICONTINUOUS FUNCTIONS

Abstract

In the whole article, the function is understood as the real valued function of real variables. It is well known that the construction of semicontinuous functions is possible either as a limit (in the sense of pointwise convergence) of the monotonous sequence of continuous functions or with assistance of the certain system of associated sets. In the first case a good understanding of the constructed function is obtained. However in the second case the Darboux property of semicontinuous function is assured when certain requirements on the system of associated sets are satisfied. Therefore the combination of both approaches seems to be the optimal method when a semicontinuous function with Darboux property is constructed. Such a combined method is used when the main theorem of this paper is proven. Later, the validity of the main theorem helps us to address the problem published by J. G. Ceder and T. L. Pearson in [2].

Let the family of Baire 1 functions be denoted by B_1, the family of functions having the Darboux property be denoted by D and the family of lower and upper semicontinuous functions be denoted by lsc and usc respectively. Notation DB_1 is used for $D \cap B_1$ and $Dlsc$ for $D \cap lsc$. The set A is bilaterally c-dense in the set B ($A \subset_c B$) iff for each $x \in A$ the sets $(x, x+\delta) \cap B$, $(x-\delta, x) \cap B$ are nondenumerable for every $\delta > 0$.

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Theorem 1. *Let E be a set of type F_σ such that the interval $I = [0,1]$ is bilaterally c-dense in E (for points 0 and 1 we require only unilateral c-dense). Then for every function $f \in lsc$ defined on $[0,1]$ there exists a function $g \in Dlsc$ such that*

$$g < f \text{ on } E$$
$$g = f \text{ on } [0,1] \setminus E.$$

In the proof of Theorem 1 the assertion of the following lemma is used.

Lemma 2. *Let a Borel set E be bilaterally c-dense-in-itself and let X be a closed subset of E. Then there exists a perfect set P such that $X \subset_c P \subset E$.*

Proof. The proof of the lemma follows from the fact that each nondenumerable Borel set contains a nonempty perfect set [3]. □

Now we can proceed with the proof of Theorem 1.

Proof. E is a set of type F_σ and thus $E = \bigcup_{i=1}^{\infty} F_i$, where each F_i is a closed sets. Without loss of generality it can be assumed that $F_1 \subset F_2 \subset \dots$ If the function $f \in lsc$ then a sequence of continuous functions $f_1 \leq f_2 \leq \dots$ exists, such that it pointwise converges to the function f. Let's construct an increasing sequence of functions g_n as

$$g_n = f_n - \frac{1}{n}, n = 1, 2, \dots$$

Evidently, the inequality $g_n < f$ holds and the sequence g_n again converges to the function f. Moreover, let ε_n, where $n = 1, 2, \dots$, be a sequence of positive real numbers such that $\varepsilon_n \to 0$. Functions g_n are uniformly continuous on $[0,1]$. Thus the sequence ε_n determines a sequence of positive numbers δ_n such that for every $x_1, x_2 \in [0,1]$

$$|x_1 - x_2| < \delta_n \Longrightarrow |g_n(x_1) - g_n(x_2)| < \varepsilon_n. \qquad (\star)$$

Following Lemma 1, let P_1 be a perfect set, $F_1 \subset_c P_1 \subset E$ and let P_1 be associated to the function g_1. Similarly, let P_2 be a perfect set, $(F_2 \cup P_1) \subset_c P_2 \subset E$ and let P_2 be associated to g_2 and continue inductively with P_i, $i = 3, 4, \dots$ Suppose P_i is already defined as a perfect set associated with the function g_i, then P_i satisfies the conditions

$$P_1 \subset_c P_2 \subset_c P_3 \subset_c \dots, \; E = \bigcup_{i=1}^{\infty} P_i$$

and a sequence of associated functions g_i satisfies conditions

$$g_1 < g_2 < g_3 < \ldots < f,\ g_i \to f.$$

The set E is dense in $[0,1]$, and therefore P_i can be required to satisfy the condition

$$dist(x, P_i) < \delta_i \text{ for every x } \in [0,1]\,. \qquad (\star\star)$$

If the perfect set P_{i+1} is in the form

$$P_{i+1} = [0,1] \setminus \bigcup_{k=1}^{\infty} (a_k^{i+1}, b_k^{i+1}),$$

then evidently it is true that

$$P_i \subset_c [0,1] \setminus \bigcup_{k=1}^{\infty} \left[a_k^{i+1}, b_k^{i+1}\right] \subset_c P_{i+1}.$$

According to Lemma 2, there exists a perfect set $P_{i+\frac{1}{2}}$ such that

$$P_i \subset_c P_{i+\frac{1}{2}} \subset_c P_{i+1}.$$

Let the function

$$g_{i+\frac{1}{2}} = g_i + \frac{1}{2}\left(g_{i+1} - g_i\right)$$

be associated to the set $P_{i+\frac{1}{2}}$. By continuing this way, perfect sets $P_{i+\frac{m}{2^n}}$ can be found for every i, n, m (where $i = 1, 2, \ldots; n = 1, 2, \ldots; 0 < m < 2^n$) such that

$$P_i \subset_c P_{i+\frac{1}{2^n}} \subset_c P_{i+\frac{2}{2^n}} \subset_c \ldots \subset_c P_{i+\frac{m}{2^n}} \subset_c P_{i+\frac{m+1}{2^n}} \subset_c \ldots \subset_c P_{i+1},$$

and associated functions

$$g_{i+\frac{m}{2^n}} = g_i + \frac{m}{2^n}\left(g_{i+1} - g_i\right),$$

for which following inequalities hold

$$g_i < g_{i+\frac{1}{2^n}} < g_{i+\frac{2}{2^n}} < \ldots < g_{i+\frac{m}{2^n}} < g_{i+\frac{m+1}{2^n}} < \ldots < g_{i+1}.$$

Finally, for each real $\alpha \geq 1, i \leq \alpha < i+1$, the closed set P_α can be defined as

$$P_\alpha = \bigcap_{\alpha < i+\frac{m}{2^n}} P_{i+\frac{m}{2^n}}$$

and associated function

$$g_\alpha = g_i + (\alpha - i)\,(g_{i+1} - g_i)\,.$$

Thus for every $\alpha \geq 1$ there is a closed set P_α and associated continuous function g_α satisfying conditions

$$P_{\alpha_1} \subset_c P_{\alpha_2}, \text{ and}$$
$$g_{\alpha_1} < g_{\alpha_2}, \text{ if } \alpha_1 < \alpha_2.$$

This follows from the definitions of sets $P_{\alpha_1}, P_{\alpha_2}$ and functions $g_{\alpha_1}, g_{\alpha_2}$. If $\alpha_1 < \alpha_2$, then let i, n, m be chosen such that

$$\alpha_1 < i + \frac{m}{2^n} < i + \frac{m+1}{2^n} < \alpha_2.$$

Then

$$P_{\alpha_1} \subset P_{i+\frac{m}{2^n}} \subset_c P_{i+\frac{m+1}{2^n}} \subset P_{\alpha_2}$$

and therefore $P_{\alpha_1} \subset_c P_{\alpha_2}$. A similar argument proves that $g_{\alpha_1} < g_{\alpha_2}$. Finally, let's define function g as

$$g(x) = g_{\alpha(x)}(x), \text{ if } x \in E,\ \alpha(x) = \min\{\alpha : x \in P_\alpha\}\,,$$
$$g(x) = f(x), \text{ if } \ x \notin E$$

and show that g has following properties:

(1) $g \in lsc$

(2) $g \in D$

(3) $g = f$ on $[0,1] \setminus E\, g < f$ on E.

and thus meets the assertion of Theorem 1.

(1) The function g is lsc iff for every $x_0 \in I$ and for arbitrary sequence $x_n \to x_0$, $\liminf_{x_n \to x_0} g(x_n) \geq g(x_0)$. Suppose that there exists a sequence x_n, where $n = 1, 2, \ldots$, that converges to a point x_0 in such a way that the sequence $g(x_n) \to \lambda < g(x_0)$. If $x_0 \in P_1$, then the definition of the function g together with continuity of function g_1 implies

$$g(x_n) \geq g_1(x_n) \to g_1(x_0) = g(x_0).$$

However, it contradicts the assumption that

$$g(x_n) \to \lambda < g(x_0). \tag{1}$$

If $x_0 \notin P_1$, then $g_1(x_0) < \lambda < g(x_0)$ and evidently $\lambda < g_\alpha(x_0) < g(x_0)$ for certain $\alpha > 1$. With respect to the continuity of the function g_α, for sufficiently large n, $n > n_0$, the inequality $g(x_n) < g_\alpha(x_n)$ holds for every $n > n_0$, which implies $x_n \in P_\alpha$. Then, $x_0 \in P_\alpha$ since the set P_α is closed, and by the definition of the function g we have $g_\alpha(x_0) > g(x_0)$. However, this contradicts $\lambda < g_\alpha(x_0) < g(x_0)$.

(2) Since the function $g \in lsc \subset B_1$, it is sufficient to show [1] , that for each x_0 there exist sequences $x_n \uparrow x_0$ and $y_n \downarrow x_0$ (for points 0 and 1 we require only one of these sequences) such that

$$g(x_0) = \lim_{x_n \to x_0} g(x_n) = \lim_{y_n \to x_0} g(y_n).$$

Two cases can be assumed. Either $x_0 \in E$ or $x_0 \notin E$. If $x_0 \in E$, then there exists an integer i_0 and a real number $\alpha_0 \in [0;1)$ such that $x_0 \in P_{i_0+\alpha_0}$ and $x_0 \notin P_\alpha$ for $\alpha < i_0 + \alpha_0$. Hence $g(x_0) = g_{_{i_0+\alpha_0}}(x_0)$. Assume a sequence α_n that satisfies

$$\alpha_n \downarrow \alpha_0,\ i_0 \leq i_0 + \alpha_0 < i_0 + \alpha_n < i_0 + 1,\ n = 1, 2, \ldots$$

Since $P_{i_0+\alpha_0} \subset_c P_{i_0+\alpha_n}$ we may find sequences $x_n \uparrow x_0$ and $y_n \downarrow x_0$ where x_n, $y_n \in P_{i_0+\alpha_n}$. The last implies $g(x_n) \leq g_{i_0+\alpha_n}(x_n)$ as well as $g(y_n) \leq g_{i_0+\alpha_n}(y_n)$. The sequence of continuous functions $g_{i_0+\alpha_n}, n = 1, 2, \ldots$ uniformly converges to the continuous function $g_{_{i_0+\alpha_0}}$. Therefore

$$\lim_{x_n \to x_0} g(x_n) = \lim_{n \to \infty} g(x_n) \leq \lim_{n \to \infty} g_{i_0+\alpha_n}(x_n) = g_{_{i_0+\alpha_0}}(x_0) = g(x_0).$$

Since the function $g \in lsc$, the following holds

$$\lim_{x_n \to x_0} g(x_n) \geq g(x_0)$$

which implies $\lim_{x_n \to x_0} g(x_n) = g(x_0)$. The same is true for the sequence y_n. If the second case $x_0 \notin E$ is assumed, then $x_0 \notin P_i$ for every $i = 1, 2, \ldots$ Let $P_i = [0;1] \setminus \bigcup_{k=1}^{\infty} \left(a_k^i; b_k^i\right)$ and let $x_0 \in \left(a_{k_i}^i; b_{k_i}^i\right)$. In this case x_i is chosen such that $x_i = a_{k_i}^i$. Evidently $x_i \in P_i$. However $x_i \notin P_\alpha$, $\alpha < i$; that is $g_i(x_i) = g(x_i)$. According to ($\star$) and ($\star\star$), for every $i = 1, 2, \ldots$

$$\begin{aligned} |x_i - x_0| &< \delta_i, \\ |g(x_i) - g_i(x_0)| = |g_i(x_i) - g_i(x_0)| &< \varepsilon_i. \end{aligned}$$

Thence $x_i \uparrow x_0$ and

$$\lim_{x_i \to x_0} g(x_i) = g(x_0).$$

because $g_i(x_0) \longrightarrow f(x_0) = g(x_0)$. If $y_i = b^i_{k_i}$ is chosen, then by the same arguments

$$y_i \downarrow x_0, \quad \lim_{y_i \to x_0} g(y_i) = g(x_0).$$

(3) If $x \in E$ then there exists i, such that $x \in F_i \subset_c P_i$. Consequently, by definition of the function g, the inequality

$$g(x) \leq g_i(x) < f(x)$$

holds. If $x \notin E$, then again by the definition of g, the equality $f(x) = g(x)$ holds.

□

In paper [4] Ibrahim Mustafa answers some of the questions posed by J.G. Ceder and T.L. Pearson in [2] by the theorems stated below.

Theorem 3. *If f is a bounded, lower semicontinuous function, then there exists a bounded, Darboux lower semicontinuous function g such that $\{x : f(x) \neq g(x)\}$ is a first category null subset of I and $f(x) \leq g(x)$ for all $x \in I$.*

Theorem 4. *Let g_1 and g_2 be two bounded, Darboux lower semicontinuous functions such that $g_1 < g_2$ on I. Then there exists a bounded, Darboux lower semicontinuous function g such that $g_1 < g < g_2$ on I.*

However, these theorems are valid even though the assumption on boundedness of function is relaxed.

Theorem 5. *If $f \in lsc$, then there exists a function $g \in Dlsc$ such that $\{x : f(x) \neq g(x)\}$ is a first category null subset of I and $f(x) \leq g(x)$ for all $x \in I$.*

Proof. The proof is an immediate consequence of Theorem 1. □

Theorem 6. *Let g_1 and g_2 be two Dlsc functions such that $g_1 < g_2$ on I. Then there exists a Dlsc function g such that $g_1 < g < g_2$ on I.*

Proof. Since the function $f = \frac{1}{2}(g_1 + g_2) \in lsc$, then there exists a sequence of continuous functions $f_1 \leq f_2 \leq f_3 \leq \dots,\ f_n \uparrow f$. A sequence of associated sets can be defined as

$$F_n = \left\{ x \in I; g_1(x) \leq f_n(x) - \frac{1}{n} \right\},\ n = 1, 2, \dots$$

It is true that $g_1 - f_n \in lsc$. Therefore the sets F_n are closed and moreover it is possible to prove that $F_n \subset_c F_{n+1}$ for every $n = 1, 2, \ldots$ The function $g_1 \in Dlsc \subset DB_1$, therefore for an arbitrary $x_0 \in F_n$ there exists a perfect road of the function g_1 at the point x_0 (see [1]); i.e., a perfect set P such that x_0 is a bilateral point of accumulation of P and $g_1/_P$ is continuous at x_0. Inequality $g_1(x_0) \leq f_n(x_0) - \frac{1}{n} < f_{n+1}(x_0) - \frac{1}{n+1}$ implies the existence of a neighborhood $U(x_0)$ such that $g_1(x) \leq f_{n+1}(x) - \frac{1}{n+1}$ for every $x \in U(x_0) \cap P$, hence $F_n \subset_c F_{n+1}$. It might be the case that the first elements of sequence F_n are empty sets. However, empty sets are not taken into account and only a subsequence of nonempty sets is assumed. Let sequences of positive real numbers $\varepsilon_n,\ \delta_n,\ n = 1, 2, \ldots$ be defined in the same way as in the proof of Theorem 1, which for every $x_1, x_2 \in [0, 1]$,

$$|x_1 - x_2| < \delta_n \Longrightarrow |f_n(x_1) - f_n(x_2)| < \varepsilon_n.$$

Now, a sequence of closed sets $P_n,\ n = 1, 2, \ldots$ is going to be constructed. Let P_1 be a closed null subset of F_1 such that

$$dist(x, P_1) < \delta_1 \text{ for every } x \in F_1.$$

In the next step let P_2 be a perfect null subset of F_2 such that

$$dist(x, P_2) < \delta_2 \text{ for every } x \in F_2,$$

and let $P_1 \subset_c P_2$. By iteration of this process a sequence of perfect sets $P_n \subset F_n, n = 2, 3, \ldots$

$$P_1 \subset_c P_2 \subset_c P_3 \subset_c P_4 \subset_c \ldots$$

$$dist(x, P_n) < \delta_n \text{ for every } x \in F_n,$$

and a sequence of associated continuous functions f_n,

$$g_1(x) < f_n(x) \text{ for every } x \in P_n,$$

is obtained. Similarly as in Theorem 1, we define for every real $\alpha \geq 1$ a closed set P_α and associated continuous function f_α and consequently a function g

$$g(x) = f_{\alpha(x)}(x), \text{ if } x \in \bigcup_{n=1}^{\infty} P_n,\ \alpha(x) = \min\{\alpha : x \in P_\alpha\},$$

$$g(x) = f(x), \text{ if } x \notin \bigcup_{n=1}^{\infty} P_n.$$

For the function g defined in such a way, it may be shown that $g \in Dlsc$, $g_1 < g \leq f < g_2$. Moreover, because we choose null sets P_n that are nowhere dense, then the functions g and f are identical except over the first category null subset of I. □

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