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ON THE SUMS OF TWO QUASI-CONTINUOUS FUNCTIONS WITH CLOSED GRAPHS

Abstract

In 2000 J. Borsík, J. Doboš, and M. Repický characterized sums of quasi-continuous functions with closed graphs. More precisely, they showed that such a sum must be Baire one star, and proved that each Baire one star function defined on a separable metric space which is Baire in the strong sense is the sum of three quasi-continuous functions with closed graphs. They showed also that not every Baire one star function defined on $\mathbb{R}$ is the sum of two quasi-continuous functions with closed graphs, and asked for characterization of such sums. The goal of this article is to present the required characterization.

1 Preliminaries.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of all positive integers, respectively. The symbol ω_1 denotes the first uncountable ordinal.

Throughout the paper we consider a fixed separable metric space (X, d) which is Baire in the strong sense; i.e., each closed subset of X is a Baire space. Let $B(x, r)$ stand for the open ball with radius r centered at x. If $x \in X$ and A is a nonvoid subset of X, then we define

$$\operatorname{dist}(x, A) \overset{\text{df}}{=} \inf\{d(x, t) \colon t \in A\}.$$

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If $A \subset X$, then the symbols $\operatorname{int} A$ and $\operatorname{cl} A$ denote the interior and the closure of A, respectively. The word *function* denotes a mapping from a subset of X into $\mathbb{R}$. The symbol $\mathcal{C}_f$ stands for the set of points of continuity of a function f, and $\mathcal{D}_f \stackrel{\mathrm{df}}{=} X \setminus \mathcal{C}_f$.

Let $f\colon X \to \mathbb{R}$. We say that f is a *Baire one star function* [7] (see also [6]), if for each nonempty closed set $P \subset X$, there exists an open set $U \subset X$ such that $P \cap U \neq \emptyset$ and $f\restriction_{P\cap U}$ is continuous. We say that f is *quasi-continuous* in the sense of Kempisty (cf. [4]), if for each $x \in X$, each neighborhood U of x and each $\varepsilon > 0$, there is a nonvoid open set $V \subset U$ such that $\operatorname{diam} f[V \cup \{x\}] < \varepsilon$.

J. Borsík proved in 2002 that each function with closed graph defined on a complete metric space is Baire one star [2]. Clearly the sum of Baire one star functions is Baire one star as well. Moreover each Baire one star function is the sum of two functions with closed graphs [2], and it is the sum of two quasi-continuous functions [1]. However, there are Baire one star functions defined on $\mathbb{R}$ which cannot be written as the sum of two quasi-continuous functions with closed graphs [3]. We shall prove the following theorem.

Theorem. *Let $f\colon X \to \mathbb{R}$. The following conditions are equivalent:*

1. *there are quasi-continuous functions with closed graphs $f_1, f_2\colon X \to \mathbb{R}$ such that $f = f_1 + f_2$ on X,*
2. *f is Baire one star and for each $x \in X$,*

$$\limsup_{t\to x,\, t\in\mathcal{C}_f} |f(t)| = \infty \qquad \text{or} \qquad \liminf_{t\to x,\, t\in\mathcal{C}_f} |f(t) - f(x)| = 0. \tag{1}$$

We shall divide the proof into two parts. The proof of necessity is quite simple, see below. The proof of sufficiency is postponed to the next section.

PROOF OF NECESSITY. Let $f_1, f_2\colon X \to \mathbb{R}$ be quasi-continuous functions with closed graphs such that $f = f_1 + f_2$ on X. By [3], f is Baire one star.

Fix an $x \in X$ such that

$$\limsup_{t\to x,\, t\in\mathcal{C}_f} |f(t)| < \infty. \tag{2}$$

Since f_1 is a quasi-continuous function defined on a Baire space X, there is a sequence $(x_n) \subset \mathcal{C}_f$ such that $x_n \to x$ and $f_1(x_n) \to f_1(x)$. (Cf., e.g., [5].) By (2), the sequence $(f_2(x_n))$ is bounded. So, it has a subsequence, say $(f_2(x_{n_k}))$, convergent to some $y \in \mathbb{R}$. Since function f_2 has closed graph, then $y = f_2(x)$. Consequently,

$$\begin{aligned}\liminf_{t\to x, t\in \mathcal{C}_f} |f(t) - f(x)| &\le \lim_{k\to\infty} |f(x_{n_k}) - f(x)| \\ &\le \lim_{k\to\infty} |f_1(x_{n_k}) - f_1(x)| + \lim_{k\to\infty} |f_2(x_{n_k}) - f_2(x)| = 0.\end{aligned}$$

□

2 Proof of Sufficiency.

First we define some transfinite sequence of closed subsets of X. Put

$$F_0 \stackrel{\text{df}}{=} \operatorname{cl} \mathcal{D}_f.$$

Assume that we have already defined the closed sets F_β for each $\beta < \alpha$, where α is some ordinal. If $\alpha = \gamma + 1$ for some ordinal γ, then we put

$$F_\alpha \stackrel{\text{df}}{=} \operatorname{cl} \mathcal{D}_{f\restriction_{F_\gamma}},$$

and otherwise we let

$$F_\alpha \stackrel{\text{df}}{=} \bigcap_{\beta<\alpha} F_\beta.$$

It can be readily verified that $F_{\alpha_1} \supset F_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$. So, $F_\xi = F_{\xi+1}$ for some ordinal ξ.

Since f is Baire one star, $F_{\alpha+1}$ is nowhere dense in F_α for each ordinal α. Using the fact that X is second countable, we conclude that there is an ordinal ξ for which $F_\xi = \emptyset$, and the least such ordinal is countable.

Let $(\lambda, \varrho)\colon \mathbb{N} \to \xi \times \mathbb{N}$ be any bijection. For brevity, for each closed set $A \subset X$, we define the function $h_A\colon X \setminus A \to [0, \infty)$ as follows:

$$h_A(x) \stackrel{\text{df}}{=} \begin{cases} \dfrac{1}{\operatorname{dist}(x, A)} & \text{if } A \text{ is nonvoid,} \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{N}$, we define the function $M_n\colon X \setminus F_{\lambda(n)+1} \to [0, \infty)$ by

$$M_n(t) \stackrel{\text{df}}{=} |f(t)| + n + h_{F_{\lambda(n)+1}}(t).$$

Moreover for each $n \in \mathbb{N}$ and each $t \in X \setminus F_{\lambda(n)+1}$, we define

$$U_n(t) \stackrel{\text{df}}{=} (f(t) - (2n)^{-1}, f(t) + (2n)^{-1}) \cup (-\infty, -2M_n(t)) \cup (2M_n(t), \infty).$$

2.1 Families $\mathcal{L}_{\varrho(n)}^{\lambda(n)}$.

For each $n \in \mathbb{N}$, we shall construct the family $\mathcal{L}_{\varrho(n)}^{\lambda(n)}$, consisting of pairwise disjoint open sets, so that the following conditions are satisfied:

(E1) $\operatorname{diam} K < (2n)^{-1}$ for each $K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$,

(E2) the set $E_n \stackrel{\text{df}}{=} \bigcup_{K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}} \operatorname{cl} K$ is a subset of $X \setminus F_0$ which is closed in X,

(E3) $E_n \subset \bigcup_{x \in F_{\lambda(n)}} B(x, 2n^{-1}) \setminus \big(F_0 \cup \bigcup_{i<n} E_i\big)$,

(E4) $B(x, 2n^{-1}) \cap E_n \neq \emptyset$ for each $x \in F_{\lambda(n)} \setminus F_{\lambda(n)+1}$,

(E5) $\operatorname{cl} K \subset B(x, 2\operatorname{dist}(x, K))$ for each $x \in F_0$ and each $K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$,

(E6) for each $K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$, there exists an $a_K \in F_{\lambda(n)} \setminus F_{\lambda(n)+1}$ such that $\operatorname{dist}(a_K, K) < n^{-1}$ and

$$(\forall z \in K)\, |f(a_K) - f(z)| < n^{-1} \quad \text{or} \quad (\forall z \in K)\, |f(z)| > M_n(a_K),$$

(E7) for each $\alpha < \xi$ and each $x \in F_0 \setminus F_\alpha$, there exists an $r > 0$ such that

$$\lambda(n) \geq \alpha \Rightarrow B(x, r) \cap E_n = \emptyset.$$

Fix an $n \in \mathbb{N}$ and assume that we have already defined the families $\mathcal{L}_{\varrho(i)}^{\lambda(i)}$ for all $i < n$. Put

$$T_n \stackrel{\text{df}}{=} \bigcup_{x \in F_{\lambda(n)} \setminus F_{\lambda(n)+1}} f^{-1}(U_n(x)) \cap B(x, n^{-1}) \setminus \Big(F_0 \cup \bigcup_{i<n} E_i\Big) \subset \mathcal{C}_f.$$

For each $t \in T_n$, choose $\varphi_n(t) > 0$ so that

$$\varphi_n(t) < \operatorname{dist}\Big(t, F_0 \cup \bigcup_{i<n} E_i\Big)/4 \text{ and } \operatorname{diam} f[B(t, \varphi_n(t))] < (2n)^{-1}. \tag{3}$$

Observe that for each $t \in T_n$,

$$\varphi_n(t) < \operatorname{dist}\Big(t, F_0 \cup \bigcup_{i<n} E_i\Big)/4 \leq \operatorname{dist}(t, F_{\lambda(n)})/4 < (4n)^{-1}. \tag{4}$$

Let S_n be a maximal (with respect to inclusion) subset of T_n with the property that

$$d(x,y) > n^{-1} \text{ whenever } x,y \in S_n \text{ and } x \neq y. \tag{5}$$

Define

$$\mathcal{L}_{\varrho(n)}^{\lambda(n)} \stackrel{\text{df}}{=} \{B(t,\varphi_n(t)) \colon t \in S_n\}.$$

We shall verify that the family $\mathcal{L}_{\varrho(n)}^{\lambda(n)}$ fulfills (E1)–(E7).

Let $t \in S_n$. Then by (4), $\operatorname{diam} B(t,\varphi_n(t)) \le 2\varphi_n(t) < (2n)^{-1}$.

To prove (E2) fix an $x \in \operatorname{cl} E_n$. Let $(x_k) \subset E_n$ be convergent to x. For each k, choose a $t_k \in S_n$ such that $x_k \in \operatorname{cl} B(t_k,\varphi_n(t_k))$. Observe that there is a $k_0 \in \mathbb{N}$ such that for each $k > k_0$, by (4),

$$d(t_k,t_{k+1}) \le d(t_k,x_k) + d(x_k,x) + d(x,x_{k+1}) + d(x_{k+1},t_{k+1}) < n^{-1}.$$

So by (5), $t_k = t_{k_0}$ for $k > k_0$, and consequently, $x \in \operatorname{cl} B(t_{k_0},\varphi_n(t_{k_0})) \subset E_n$.

Let $t \in S_n$ and $z \in \operatorname{cl} B(t,\varphi_n(t))$. Then $t \in T_n$, so by (4),

$$\operatorname{dist}(z,F_{\lambda(n)}) \le d(z,t) + \operatorname{dist}(t,F_{\lambda(n)}) < \varphi_n(t) + n^{-1} < 2n^{-1}.$$

On the other hand,

$$\operatorname{dist}\Big(z, F_0 \cup \bigcup_{i<n} E_i\Big) \ge \operatorname{dist}\Big(t, F_0 \cup \bigcup_{i<n} E_i\Big) - d(z,t) > 3\varphi_n(t) > 0.$$

It follows that $z \in \bigcup_{x \in F_{\lambda(n)}} B(x,2n^{-1}) \setminus \big(F_0 \cup \bigcup_{i<n} E_i\big)$.

Now let $x \in F_{\lambda(n)} \setminus F_{\lambda(n)+1}$. Take any $m > n$ with

$$m^{-1} \le \operatorname{dist}\Big(x, \bigcup_{i<n} E_i\Big).$$

By (1), there is a $z \in f^{-1}(U_n(x)) \cap B(x,m^{-1}) \setminus F_0$. Clearly $z \in T_n$. If $z \in S_n$, then we are done. So, assume that $z \notin S_n$. Then by the maximality of S_n, there exists a $t \in S_n$ with $d(z,t) \le n^{-1}$, whence $t \in B(x,2n^{-1}) \cap E_n$.

Fix $x \in F_0$ and $K = B(t,\varphi_n(t)) \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$. Then

$$\operatorname{dist}(x,K) \ge d(x,t) - \varphi_n(t) \ge \operatorname{dist}(t,F_0) - \varphi_n(t) > 3\varphi_n(t) > \operatorname{diam} K$$

and

$$\operatorname{cl} K \subset \operatorname{cl} B(x, \operatorname{dist}(x,K) + \operatorname{diam} K) \subset B(x, 2\operatorname{dist}(x,K)).$$

Take any $t \in S_n$ and $z \in K \overset{\text{df}}{=} B(t, \varphi_n(t))$. By definition, there is an $a_K \in F_{\lambda(n)} \setminus F_{\lambda(n)+1}$ such that $t \in f^{-1}(U_n(a_K)) \cap B(a_K, n^{-1})$. Then clearly $\operatorname{dist}(a_K, K) < n^{-1}$. If $|f(t) - f(a_K)| < (2n)^{-1}$, then by (3),

$$|f(a_K) - f(z)| \le |f(a_K) - f(t)| + \operatorname{diam} f[K] < n^{-1}.$$

Otherwise $|f(t)| > 2M_n(a_K) > M_n(a_K) + n$. Using again (3), we obtain

$$|f(z)| \ge |f(t)| - \operatorname{diam} f[K] > M_n(a_K).$$

Now fix $\alpha < \xi$ and $x \in F_0 \setminus F_\alpha$. Then $\operatorname{dist}(x, F_\alpha) > 4m^{-1}$ for some $m \in \mathbb{N}$. By (E2) and (E3),

$$V \overset{\text{df}}{=} B(x, 2m^{-1}) \setminus \bigcup_{n<m} E_n \tag{6}$$

is an open neighborhood of x. Choose an $r \in (0, m^{-1})$ so that $B(x, r) \subset V$.

Finally fix an $n \in \mathbb{N}$ with $\lambda(n) \ge \alpha$ and take any $K \in \mathcal{L}^{\lambda(n)}_{\varrho(n)}$. If $n < m$, then by (6),

$$B(x, r) \cap \operatorname{cl} K \subset V \cap E_n = \emptyset.$$

In the opposite case by (E3),

$$B(x, r) \cap \operatorname{cl} K \subset B(x, r) \cap \bigcup_{x \in F_\alpha} B(x, 2m^{-1}) = \emptyset.$$

2.2 The Main Part of the Proof of the Theorem.

First observe that

$$\text{for each } t \in X \setminus F_0, \text{ there exists an } r > 0 \text{ such that } \operatorname{cl} K \cap B(t, r) \neq \emptyset \text{ for at most one } K \in \textstyle\bigcup_{n\in\mathbb{N}} \mathcal{L}^{\lambda(n)}_{\varrho(n)}. \tag{7}$$

For this, fix a $t \in X \setminus F_0$. Then $\operatorname{dist}(t, F_0) > 4m^{-1}$ for some $m \in \mathbb{N}$. Observe that by (E3),

$$\bigcup_{n \ge m} E_n \subset \bigcup_{n \ge m} \bigcup_{x \in F_{\lambda(n)}} B(x, 2n^{-1}) \subset \bigcup_{n \ge m} \bigcup_{x \in F_{\lambda(n)}} B(x, 2m^{-1}),$$

whence $B(t, 2m^{-1}) \cap \bigcup_{n\ge m} E_n = \emptyset$. By (E2), the set $\bigcup_{n<m} E_n$ is closed. So, if $t \notin \bigcup_{n\in\mathbb{N}} E_n$, then $B(t, r) \cap \bigcup_{n\in\mathbb{N}} E_n = \emptyset$ for some $r > 0$. Otherwise since by (E3), the sets $E_1, E_2, \dots$ are pairwise disjoint, there is a unique $k < m$

such that $t \in E_k$. By (5), there is a unique $K \in \mathcal{L}^{\lambda(k)}_{\varrho(k)}$ with $t \in \operatorname{cl} K$. Using again (5) we can find an $r > 0$ such that

$$B(t,r) \cap \Big((E_k \setminus K) \cup \bigcup_{n \neq k} E_n \Big) = \emptyset.$$

Observe also that by (E5), for each $x \in F_0$ and each $K \in \bigcup_{n\in\mathbb{N}} \mathcal{L}^{\lambda(n)}_{\varrho(n)}$,

$$\operatorname{diam}(\{x\} \cup K) \leq \operatorname{diam} B(x, 2\operatorname{dist}(x,K)) \leq 4\operatorname{dist}(x,K). \tag{8}$$

The following notation is standard. For $x \in X$, we define

$$f^+(x) \overset{\text{df}}{=} \max\{f(x),0\},\ \ f^-(x) \overset{\text{df}}{=} \max\{-f(x),0\}.$$

Define the functions $f_1, f_2 \colon X \to \mathbb{R}$ as follows:

$$f_1(x) \overset{\text{df}}{=} \begin{cases} f^+(x) + h_D(x) & \text{if } x \in X \setminus D, \\ f^+(x) + h_{F_{\alpha+1}}(x) & \text{if } x \in F_\alpha \setminus F_{\alpha+1},\ \alpha < \xi, \\ f^+(a_K) + h_{F_{\lambda(n)+1}}(a_K) & \text{if } x \in \operatorname{cl} K,\ K \in \mathcal{L}^{\lambda(n)}_{\varrho(n)},\ \varrho(n) \text{ even}, \\ f(x) + f^-(a_K) + h_{F_{\lambda(n)+1}}(a_K) & \text{if } x \in \operatorname{cl} K,\ K \in \mathcal{L}^{\lambda(n)}_{\varrho(n)},\ \varrho(n) \text{ odd}, \end{cases}$$

$$f_2(x) \overset{\text{df}}{=} \begin{cases} -f^-(x) - h_D(x) & \text{if } x \in X \setminus D, \\ -f^-(x) - h_{F_{\alpha+1}}(x) & \text{if } x \in F_\alpha \setminus F_{\alpha+1},\ \alpha < \xi, \\ f(x) - f^+(a_K) - h_{F_{\lambda(n)+1}}(a_K) & \text{if } x \in \operatorname{cl} K,\ K \in \mathcal{L}^{\lambda(n)}_{\varrho(n)},\ \varrho(n) \text{ even}, \\ -f^-(a_K) - h_{F_{\lambda(n)+1}}(a_K) & \text{if } x \in \operatorname{cl} K,\ K \in \mathcal{L}^{\lambda(n)}_{\varrho(n)},\ \varrho(n) \text{ odd}, \end{cases}$$

where

$$D \overset{\text{df}}{=} F_0 \cup \bigcup_{n\in\mathbb{N}} \bigcup_{K \in \mathcal{L}^{\lambda(n)}_{\varrho(n)}} \operatorname{cl} K = F_0 \cup \bigcup_{n\in\mathbb{N}} E_n.$$

Observe that D is closed.

Indeed, let $(t_k) \subset D$ be convergent to some $t_0 \in X$. If there is a subsequence $(t_{k_m}) \subset F_0$, then $t_0 \in \operatorname{cl} F_0 \subset D$.

So, assume that for each k, there is an $n_k \in \mathbb{N}$ such that $t_k \in E_{n_k}$. If $n_k = n_0$ for infinitely many k, then by (E2), $t_0 \in E_{n_0} \subset D$.

Finally if $n_k \to \infty$, then by (E3), for each k, there is an $x_k \in F_{\lambda(n_k)} \subset F_0$ such that $t_k \in B(x_k, 2n_k^{-1})$. Consequently,

$$d(x_k, t_0) \leq d(x_k, t_k) + d(t_k, t_0) \leq 2n_k^{-1} + d(t_k, t_0) \to 0$$

and $t_0 \in \operatorname{cl} F_0 \subset D$.

Clearly $f = f_1 + f_2$ on X. We shall verify that f_1 is quasi-continuous and its graph is closed. The proofs of the analogous statements for f_2 are similar, we shall omit them.

2.2.1 f_1 is Quasi-continuous.

Fix an $x \in X$, a neighborhood U of x, and an $\varepsilon \in (0,1)$. We consider several cases.

If $x \notin D$, then f_1 is continuous at x. So, the set

$$V \stackrel{\text{df}}{=} U \cap f_1^{-1}((f_1(x) - \varepsilon/3, f_1(x) + \varepsilon/3))$$

is an open neighborhood of x contained in U such that $\operatorname{diam} f_1[V] < \varepsilon$.

If $x \in D \setminus F_0$, then there is an $n \in \mathbb{N}$ and a $K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$ with $x \in \operatorname{cl} K$. Since $f_1\restriction_{\operatorname{cl} K}$ is continuous, the set

$$V \stackrel{\text{df}}{=} U \cap K \cap f_1^{-1}((f_1(x) - \varepsilon/3, f_1(x) + \varepsilon/3))$$

is a nonvoid and open subset of U such that $\operatorname{diam} f_1[V \cup \{x\}] < \varepsilon$.

Finally assume that $x \in F_\alpha \setminus F_{\alpha+1}$ for some $\alpha < \xi$. Since $f\restriction_{F_\alpha \setminus F_{\alpha+1}}$ is continuous, we can choose an $r \in (0,1)$ such that $B(x, 3r) \subset U \setminus F_{\alpha+1}$ and $\operatorname{diam} f[B(x,2r) \cap F_\alpha] < \varepsilon/2$. Let $n > 6r^{-2}\varepsilon^{-1}$ be such that $\varrho(n)$ is even and $\lambda(n) = \alpha$. By (E4), there is a $K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$ such that $V \stackrel{\text{df}}{=} B(x, 2n^{-1}) \cap K \neq \emptyset$. Notice that V is a nonvoid open subset of U. Choose $a_K \in F_\alpha \setminus F_{\alpha+1}$ according to (E6). Then by (8),

$$d(x, a_K) \le \operatorname{diam}(\{x\} \cup K) + \operatorname{dist}(a_K, K) \le 4 \operatorname{dist}(x, K) + n^{-1} < 9n^{-1} < 2r.$$

Hence $\operatorname{dist}(a_K, F_{\alpha+1}) \ge \operatorname{dist}(x, F_{\alpha+1}) - d(x, a_K) > r$ and

$$\begin{aligned}
\operatorname{diam} f_1[V \cup \{x\}] &= |f^+(a_K) + h_{F_{\lambda(n)+1}}(a_K) - f^+(x) - h_{F_{\alpha+1}}(x)| \\
&\le |f^+(a_K) - f^+(x)| + |h_{F_{\alpha+1}}(a_K) - h_{F_{\alpha+1}}(x)| \\
&< \varepsilon/2 + \frac{|\operatorname{dist}(x, F_{\alpha+1}) - \operatorname{dist}(a_K, F_{\alpha+1})|}{\operatorname{dist}(a_K, F_{\alpha+1}) \operatorname{dist}(x, F_{\alpha+1})} \\
&< \varepsilon/2 + \frac{d(x, a_K)}{3r^2} < \varepsilon/2 + \frac{3}{nr^2} < \varepsilon.
\end{aligned}$$

2.2.2 The Graph of f_1 is Closed.

Now take any point $\langle x, y \rangle$ from the closure of the graph of f_1. Let $(x_k) \subset X$ be such that $x_k \to x$ and $f_1(x_k) \to y$. We shall prove that $y = f_1(x)$. We consider several cases.

• If there is a subsequence $(x_{k_m}) \subset F_0$, then $x \in F_0$, whence $x \in F_\alpha \setminus F_{\alpha+1}$ for some $\alpha < \xi$. For each $m \in \mathbb{N}$, there is a unique $\alpha_m < \xi$ such that $x_{k_m} \in F_{\alpha_m} \setminus F_{\alpha_m+1}$. Since $F_{\alpha+1}$ is closed, $\alpha_m \le \alpha$ for sufficiently big m.

Notice that $f_1 \geq 0$ on F_0, whence $y \geq 0$. For sufficiently big m, we have $d(x_{k_m}, x) < (y+1)^{-1}$ and $f_1(x_{k_m}) < y+1$. Then $\operatorname{dist}(x_{k_m}, F_{\alpha_m+1}) = \dfrac{1}{h_{F_{\alpha_m+1}}(x_{k_m})} \geq \dfrac{1}{f_1(x_{k_m})} > \dfrac{1}{y+1} > d(x_{k_m}, x)$, whence $x \notin F_{\alpha_m+1}$ and $\alpha_m = \alpha$. Since $f\restriction_{F_\alpha \setminus F_{\alpha+1}}$ is continuous,

$$y = \lim_{k\to\infty} f_1(x_k) = \lim_{m\to\infty} \big(f^+(x_{k_m}) + h_{F_{\alpha+1}}(x_{k_m})\big) = f^+(x) + h_{F_{\alpha+1}}(x) = f_1(x).$$

• If there is a subsequence $(x_{k_m}) \subset X \setminus D$, then $\limsup_{m\to\infty} h_D(x_{k_m}) \leq \lim_{m\to\infty} f_1(x_{k_m}) = y < \infty$. Hence $x = \lim_{m\to\infty} x_{k_m} \in X \setminus D \subset \mathcal{C}_{f_1}$ and $y = f_1(x)$.

• Finally assume that $(x_k) \subset D \setminus F_0$. For each k, there are unique $n_k \in \mathbb{N}$ and $K_k \in \mathcal{L}_{\varrho(n_k)}^{\lambda(n_k)}$ such that $x_k \in K_k$. If $x \notin F_0$, then by (7), there is an $n \in \mathbb{N}$ and a $K \in \mathcal{L}_{\varrho(n)}^{\lambda(n)}$ such that $K_k = K$ for sufficiently big k. By (E2), $f_1\restriction_{\operatorname{cl} K}$ is continuous, whence $y = f_1(x)$. Now assume that $x \in F_\alpha \setminus F_{\alpha+1}$ for some $\alpha < \xi$. Use (E7) to find an $r \in (0, (2|y|+1)^{-1})$ such that

$$(\forall n \in \mathbb{N})\ \big(\lambda(n) > \alpha \Rightarrow B(x,r) \cap E_n = \emptyset\big). \tag{9}$$

For each k, choose a_{K_k} according to (E6). Notice that by (E2), there is no constant subsequence of (n_k). Consequently, $n_k \to \infty$ and by (8),

$$d(x, a_{K_k}) \leq \operatorname{diam}(\{x\} \cup K_k) + \operatorname{dist}(a_{K_k}, K_k) \leq 4d(x, x_k) + n_k^{-1} \to 0. \tag{10}$$

So, there is a $k_0 \in \mathbb{N}$ such that

$$d(x_k, x) < r \quad \text{and} \quad |f_1(x_k)| < |y| + 2^{-1} < n_k \qquad \text{for } k > k_0. \tag{11}$$

By (9), $\lambda(n_k) \leq \alpha$ for each $k > k_0$. We consider two subcases.

▲ Assume that there is a subsequence (n_{k_m}) such that $\varrho(n_{k_m})$ is even for each m. For sufficiently big m, we have $k_m > k_0$ and $n_{k_m} > 2|y|+1$. Then

$$\begin{aligned} d(x_{k_m}, F_{\lambda(n_{k_m})+1}) &\geq d(a_{K_{k_m}}, F_{\lambda(n_{k_m})+1}) - d(x_{k_m}, a_{k_m}) \\ &> \frac{1}{f_1(x_{k_m})} - n_{k_m}^{-1} > \frac{1}{2|y|+1} > d(x_{k_m}, x), \end{aligned}$$

whence $x \notin F_{\lambda(n_{k_m})+1}$ and by (9), $\lambda(n_{k_m}) = \alpha$. Since $f\restriction_{F_\alpha \setminus F_{\alpha+1}}$ is continuous, $y = \lim_{m\to\infty}\big(f^+(x_{k_m}) + h_{F_{\lambda(n_{k_m})+1}}(x_{k_m})\big) = f^+(x) + h_{F_{\alpha+1}}(x) = f_1(x)$.

▲ So, assume that $\varrho(n_k)$ is odd for each k. Then for each $k > k_0$, by (11),

$$\begin{aligned} |f(x_k)| &\leq |f(x_k) + f^-(a_{K_k}) + h_{F_{\lambda(n_k)+1}}(a_{K_k})| + |f^-(a_{K_k})| + h_{F_{\lambda(n_k)+1}}(a_{K_k}) \\ &= |f_1(x_k)| + |f^-(a_{K_k})| + h_{F_{\lambda(n_k)+1}}(a_{K_k}) < M_{n_k}(a_{K_k}), \end{aligned}$$

whence by (E6),

$$|f(a_{K_k}) - f(x_k)| < n_k^{-1}. \tag{12}$$

It follows that

$$\begin{aligned}|y| + 2^{-1} &> f_1(x_k) = f(x_k) + f^-(a_{K_k}) + h_{F_{\lambda(n_k)+1}}(a_{K_k}) \\ &\geq h_{F_{\lambda(n_k)+1}}(a_{K_k}) - |f(x_k) - f(a_{K_k})| + (f(a_{K_k}) + f^-(a_{K_k})) \\ &> h_{F_{\lambda(n_k)+1}}(a_{K_k}) - n_k^{-1}.\end{aligned}$$

By (10), we conclude that $x \notin F_{\lambda(n_k)+1}$, and by (9), that $\lambda(n_k) = \alpha$. Recall that $f^+\restriction_{F_\alpha \setminus F_{\alpha+1}}$ is continuous. So by (12) and (10),

$$\begin{aligned}|y - f_1(x)| &= \lim_{k\to\infty} |f_1(x_k) - f_1(x)| \\ &= \lim_{k\to\infty} |f(x_k) + f^-(a_{K_k}) + h_{F_{\lambda(n_k)+1}}(a_{K_k}) - f^+(x) - h_{F_{\alpha+1}}(x)| \\ &\leq \lim_{k\to\infty} |f(x_k) - f(a_{K_k})| + \lim_{k\to\infty} |f^+(a_{K_k}) - f^+(x)| \\ &\qquad + \lim_{k\to\infty} |h_{F_{\alpha+1}}(a_{K_k}) - h_{F_{\alpha+1}}(x)| = 0.\end{aligned}$$

This completes the proof.

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