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ALMOST ISOMETRY-INVARIANT SETS AND SHADINGS

Abstract

An *almost isometry-invariant* set $A \subset \mathbb{R}$ satisfies $|gA \triangle A| < \mathbf{c}$ for any isometry g acting on $\mathbb{R}$, where $\mathbf{c}$ is the cardinality of the continuum. A *shading* is any set $S \subseteq \mathbb{R}$ in which $\frac{\mu(S \cap I)}{\mu(I)}$ has the same constant value for every finite interval I, for any Banach measure μ. (A Banach measure is a finitely additive, isometry-invariant extension of the Lebesgue measure to $2^{\mathbb{R}}$.) In this paper we prove several theorems that show how these two types of sets are related. We also prove several sum and difference set results for almost isometry-invariant sets. Finally, we completely solve a problem involving subsets of Archimedean sets first posed by R. Mabry and partially solved by K. Neu.

1 Introduction.

In this paper, we use the standard notations $S + t = \{s + t | s \in S\}$, $S + S = \{s_1 + s_2 | s_1, s_2 \in S\}$, and $S - S = \{s_1 - s_2 | s_1, s_2 \in S\}$. We also write $A \doteqdot B$ if $|A \triangle B| < \mathbf{c}$.

In [4], R. Mabry first demonstrated the existence of *shadings*, or sets $S \subseteq \mathbb{R}$ that give the expression $\frac{\mu(S \cap I)}{\mu(I)}$ the same constant value for every finite interval I and every Banach measure μ. This constant value can be made to be any number in $[0, 1]$. Recall that a Banach measure is a finitely additive,

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isometry-invariant extension to $2^{\mathbb{R}}$ of the Lebesgue measure. All of the shadings constructed in Mabry's paper are built using *Archimedean sets*. These are sets that satisfy $A + t = A$ for densely many $t \in \mathbb{R}$. One of the important results proven in the paper states that for an Archimedean set A, the ratio $\frac{\mu(A\cap I)}{\mu(I)}$ has the same constant value for every interval I, for a given Banach measure μ. Note that unlike shadings, in the case of an Archimedean set A, the value of $\frac{\mu(A\cap I)}{\mu(I)}$ might depend on the Banach measure chosen. Any set in which $\frac{\mu(A\cap I)}{\mu(I)}$ has the same constant value for every interval I, for a given Banach measure μ, is called a μ-*shading*. Thus, Archimedean sets are examples of μ-shadings. For a μ-shading A, the ratio $\frac{\mu(A\cap I)}{\mu(I)}$ is called the μ-*shade* of A and is denoted $\mathrm{sh}_\mu A$. For a shading S, the ratio $\frac{\mu(S\cap I)}{\mu(I)}$ is called the *shade* of S and is denoted $\mathrm{sh}S$.

An *almost isometry-invariant set* $A \subset \mathbb{R}$ is any set satisfying $|gA \triangle A| < \mathbf{c}$ for any isometry g of $\mathbb{R}$, where $\mathbf{c}$ is the cardinality of the continuum. An *almost translation-invariant set* is similar to an almost isometry-invariant set, except that it is almost invariant under any translation, instead of any isometry. That is, an almost translation-invariant set A satisfies $|A \triangle (A + r)| < \mathbf{c}$ for any $r \in \mathbb{R}$. It is important to mention that almost isometry-invariant sets and almost translation-invariant sets, like Archimedean sets, are μ-shadings for any Banach measure μ. This result is proven in [5], Theorem 5.3.

2 Almost Isometry-Invariant and Almost Translation-Invariant Sets.

The next few results involve both almost isometry-invariant and almost translation-invariant sets and require the following fact, demonstrated in the proof of Lemma 6.1 of [5].

Lemma 2.1. *(Mabry) Let μ be any Banach measure and let A be an almost isometry-invariant set satisfying* $\mathrm{sh}_\mu A \neq 0$. *Then there exists a Banach measure ν satisfying $\nu(E \cap I) = \frac{\mu(E\cap I\cap A)}{\mathrm{sh}_\mu A}$ for any set $E \subseteq \mathbb{R}$ and any bounded interval I.*

PROOF. For any bounded set S, define $\phi(S) = \frac{\mu(S\cap A)}{\mathrm{sh}_\mu A}$. For arbitrary sets $X \subseteq \mathbb{R}$ define

$$\nu(X) = \sum_{i\in\mathbb{Z}} \phi(X \cap [i, i+1)).$$

As shown in the proof of Lemma 6.1 in [5], ν is well-defined, an extension of the Lebesgue measure, and isometry-invariant. This implies that ν is a

Banach measure. Since $E \cap I$ is bounded for any set $E \subset \mathbb{R}$ and any bounded interval I, $\sum_{i\in\mathbb{Z}} \phi(E \cap I \cap [i, i+1))$ is a finite sum, and so the finite additivity of ϕ implies the result. $\square$

In [5], Mabry introduces the concept of a *shade-almost invariant* set, or a set S in which $\mathrm{sh}(S \triangle g(S)) = 0$ for every isometry g of $\mathbb{R}$. Since almost isometry-invariant sets are also shade-almost invariant sets (Lemma 4.5 of [4] states that $|Y| < \mathbf{c}$ implies $\mathrm{sh}Y = 0$), we can use Lemma 6.2 of [5] to conclude that almost isometry-invariant sets can have *any* μ-shade desired, for the right μ, assuming those sets are not shadings of shade 0 or 1. (A special case of Lemma 6.2 of [5] states that for any $t > 0$ and any shade-almost invariant set B that is not a shading of shade 0 or 1, there exists a Banach measure μ for which $\mathrm{sh}B = t$. The measure given in Lemma 2.1 is an integral part of the proof.) Because almost isometry-invariant sets that are not shadings of shade 0 or 1 can have any μ-shade desired for the right μ, almost isometry-invariant sets are only shadings when they are of shade 0 or shade 1. This fact is proven in a slightly different way in Theorem 2.2.

Theorem 2.2. *Let S be an almost isometry-invariant shading. Then* $\mathrm{sh}S = 0$ *or* $\mathrm{sh}S = 1$.

Proof. Assume $\mathrm{sh}S \neq 0$ and let μ be any Banach measure. By Lemma 2.1, there exists a Banach measure ν satisfying $\nu(E \cap [0,1]) = \frac{\mu(E\cap[0,1]\cap S)}{\mathrm{sh}S}$ for any $E \subset \mathbb{R}$. Setting $E = S \cap [0,1]$ gives us $\nu(S \cap [0,1]) = \frac{\mu((S\cap[0,1])\cap S)}{\mathrm{sh}S}$. Since S is a shading, $\nu(S \cap [0,1]) = \mu(S \cap [0,1]) = \mathrm{sh}S$, hence $\mathrm{sh}S = \frac{\mathrm{sh}S}{\mathrm{sh}S} = 1$. $\square$

Using essentially the same method, we can also show that almost translation-invariant sets S are only shadings when $\mathrm{sh}S$ takes on one of three values.

Theorem 2.3. *Let S be an almost translation-invariant shading. Then* $\mathrm{sh}S = 0, 1,$ *or* $\frac{1}{2}$.

Proof. Assume $\mathrm{sh}S \neq 0$ or 1 and let μ be any Banach measure. Using a similar method to the one used in the proof of Lemma 2.1, it can be shown that there exists a *t-Banach* measure ν satisfying $\nu(E\cap[0,1]) = \frac{\mu(E\cap[0,1]\cap S)}{\mathrm{sh}S}$ for any $E \subseteq \mathbb{R}$. (A t-Banach measure is a Banach measure that is only translation-invariant, as opposed to isometry-invariant.) Define $w(E) = \frac{1}{2}(\nu(E)+\nu(-E))$. Clearly w is a Banach measure. Setting $E = S \cap [0,1]$ gives us $w(S \cap [0,1]) = \frac{1}{2}(\nu(S\cap[0,1])+\nu(-(S\cap[0,1])))$. Since $w(S\cap[0,1]) = \nu(S\cap[0,1]) = \mathrm{sh}S$, we have $\mathrm{sh}S = \frac{1}{2}\left(\frac{\mathrm{sh}S+\mu((-(S\cap[0,1]))\cap S)}{\mathrm{sh}S}\right)$. This implies $2\mathrm{sh}S = 1 + \frac{\mu((-(S\cap[0,1]))\cap S)}{\mathrm{sh}S}$, so that sh$S \geq \frac{1}{2}$. Since S is a shading, so is S^c. Since S is almost translation-invariant, so is S^c. Finally, sh$S \neq 1$ means sh$S^c \neq 0$, so we can reuse the above work to conclude sh$S^c \geq \frac{1}{2}$. But sh$S \geq \frac{1}{2}$ and sh$S^c \geq \frac{1}{2}$ together imply that sh$S = \frac{1}{2}$. □

Note: The existence of an almost translation-invariant shading S satisfying sh$S = \frac{1}{2}$ is demonstrated in Example 3.4. That set S is almost translation-invariant and therefore a μ-shading for any Banach μ. In the example, $-S = S^c$, so the reflection-invariance of Banach measures gives us $\text{sh}_\mu S = \mu(S \cap [-\frac{1}{2}, \frac{1}{2}]) = \mu(-S \cap [-\frac{1}{2}, \frac{1}{2}]) = \mu(S^c \cap [-\frac{1}{2}, \frac{1}{2}]) = \text{sh}_\mu S^c$ for any Banach μ. This forces $\text{sh}_\mu S = \text{sh}_\mu S^c = \frac{1}{2}$ and so sh$S = \frac{1}{2}$.

Corollary 2.4. *If S is an almost translation-invariant shading satisfying* sh$S = \frac{1}{2}$, *then* $\text{sh}(S \cap (-S)) = 0$.

PROOF. Clearly $S \cap (-S)$ is a μ-shading for any Banach μ, because it is almost translation-invariant. From the above proof we know that $2\text{sh}S = 1 + \frac{\mu((-(S \cap [0,1])) \cap S)}{\text{sh}S}$ and also that sh$S = \frac{1}{2}$. This implies $\mu((-(S \cap [0,1])) \cap S) = \mu(S \cap (-S) \cap [-1,0]) = 0$, which means $\text{sh}(S \cap (-S)) = 0$, since μ was arbitrary. □

The following result, also an easy consequence of Lemma 2.1, will be mentioned again after Theorem 4.5 when we discuss the intersection of an arbitrary μ-shading and a shading.

Theorem 2.5. *Let S be a shading, let A be an almost isometry-invariant set, and let μ be a Banach measure. Then* $\text{sh}_\mu(S \cap A) = (\text{sh}S)(\text{sh}_\mu A)$.

PROOF. If $\text{sh}_\mu A = 0$ the conclusion is obvious since $S \cap A \subseteq A$. Assume then that $\text{sh}_\mu A \neq 0$. Using $\nu(E \cap I) = \frac{\mu(E \cap I \cap A)}{\text{sh}_\mu A}$ from Lemma 2.1 and setting $E = S \cap I$ gives us $(\text{sh}S)(\mu(I)) = \frac{\mu(A \cap S \cap I)}{\text{sh}_\mu A}$, which means $\frac{\mu(A \cap S \cap I)}{\mu(I)} = (\text{sh}_\mu A)(\text{sh}S)$, so $A \cap S$ is a μ-shading and $\text{sh}_\mu(S \cap A) = (\text{sh}S)(\text{sh}_\mu A)$. □

3 Sum and Difference Sets.

In [6] it is shown that if μ is a Banach measure and A is an Archimedean set satisfying $\text{sh}_\mu A > \frac{1}{k+1}$ for some integer $k \geq 1$, then $\text{sh}_\mu(A - A) \geq \frac{1}{k}$. This result applies to almost isometry-invariant and almost translation-invariant sets, because in terms of Banach measure, these sets behave like Archimedean sets. Their additional invariant properties, however, allow us to more definitively characterize their sum and difference sets in certain cases.

Theorem 3.1. *Let S be an almost translation-invariant set. Then* $\mathrm{sh}(S+S^c)=1$.

PROOF. The main construction used in this proof is similar to that in the proof of Lemma 6.3, p. 186, in [8]. Assume S and S^c are both nonempty; otherwise the conclusion is obvious. Then there exists a real number r satisfying $S\cap(S^c-r)\neq\emptyset$. Define $C=[S^c\cup(S+r)]+[S\cap(S^c-r)]$ and let $a\in S^c\cup(S+r), b\in S\cap(S^c-r)$. If $a\in S^c$, then clearly $a+b\in S^c+S$. If $a\in S+r$, then $a=s+r$ for some $s\in S$, and since $b\in S^c-r$, $b=s_c-r$ for some $s_c\in S^c$. This implies $a+b=s+r+s_c-r=s_c+s\in S^c+S$. So in both cases, $a+b\in S^c+S$, whence $C\subseteq S+S^c$. It is therefore enough to show $\mathrm{sh}C=1$, which will be true if $\mathrm{sh}(S^c\cup(S+r))=1$. Observe that $|S\bigtriangleup(S+r)|<\mathbf{c}$ and $S\cap S^c=\emptyset$ together imply $|S^c\cap(S+r)|<\mathbf{c}$. This implies $\mathrm{sh}(S^c\cap(S+r))=0$ by Lemma 4.5 of [4]. By additivity, we then have $\mathrm{sh}_\mu(S^c\cup(S+r))=\mathrm{sh}_\mu S^c+\mathrm{sh}_\mu S=1$ for any Banach measure μ, which implies $\mathrm{sh}(S^c\cup(S+r))=1$. $\square$

We note that the above result applies to almost isometry-invariant sets too, because every almost isometry-invariant set is also almost translation-invariant. (An almost translation-invariant set is not necessarily fully isometry-invariant however.) Having proven that $\mathrm{sh}(S+S^c)=1$ for almost translation-invariant S, what can we say about $S+S$?

Theorem 3.2. *Let S be an almost translation-invariant set with $|S\cap -S|=\mathbf{c}$. Then $S+S=\mathbb{R}$.*

PROOF. Since S is almost translation-invariant, so is $-S$, whence $|-S\cap(-S+x)|=\mathbf{c}$ for every $x\in\mathbb{R}$. Since $|S\cap -S|=\mathbf{c}$, $|S\cap(-S+x)|=\mathbf{c}$ for every $x\in\mathbb{R}$; in particular, so has $S\cap(-S+x)$ for every $x\in\mathbb{R}$; in particular $S\cap(-S+x)$ is nonempty, which immediately gives $x\in S+S$. $\square$

Note: The proof above is due to one of the anonymous referees. It generalizes an earlier version of the theorem.

Corollary 3.3. *Let S be an almost isometry-invariant set with $|S|=\mathbf{c}$. Then $S\pm S=\mathbb{R}$.*

PROOF. Since $S \doteqdot -S$ and $|S|=\mathbf{c}$, Theorem 3.2 gives us $S+S=\mathbb{R}$. $S-S=\mathbb{R}$ follows from the fact that $S\cap(S+x)$ is nonempty for every $x\in\mathbb{R}$. $\square$

Note: A similar result is given in Lemma 2.3, p. 1834 in [1].

One might wonder if every almost translation-invariant set S with cardinality the continuum satisfies $S + S = \mathbb{R}$, even without the assumption $|S \cap -S| = \mathbf{c}$. The answer is 'no', as demonstrated in the following example due to Kharazishvili, mentioned after Corollary 7.7 of [4].

Example 3.4. Let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, let $\{h_\alpha\}_{\alpha<\mathbf{c}}$ be an injective well-ordering of H, and let $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$. Define $S = \{\sum_{i=1}^n q_i h_{\alpha_i} + q^+ h_\beta : h_{\alpha_i}, h_\beta \in H, q_i \in \mathbb{Q}, q^+ \in \mathbb{Q}^+, \alpha_i < \beta\}$. Clearly $|S \cap -S| \neq \mathbf{c}$ and $S + S = \mathbb{R}$ is false.

The following is an example of an almost translation-invariant set satisfying the hypotheses of Theorem 3.2.

Example 3.5. Again let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, and let $\{h_\alpha\}_{\alpha<\mathbf{c}}$ be an injective well-ordering of H. Define $S = \{\sum_{i=1}^n q_i h_{\alpha_i} + q h_\beta + (-q) h_\gamma : h_{\alpha_i}, h_\beta, h_\gamma \in H, \alpha_i < \beta < \gamma, q \in \mathbb{Q}, q \neq 0\}$. Clearly S is almost isometry-invariant and $|S| = \mathbf{c}$. Since $S = -S$, $S + S = \mathbb{R}$.

The next result is a generalization of Proposition 1, p. 126, by Kharazishvili [2]. Recall from the proof of Theorem 2.3 that a t-Banach measure is a finitely additive extension of the Lebesgue measure to $2^{\mathbb{R}}$ that is translation-invariant (but might not be fully isometry-invariant).

Theorem 3.6. *Let I be an interval of finite positive length, and let S and T be two sets satisfying $\mu(S \cap I) + \mu(T \cap I) > \lambda(I)$, where μ is a t-Banach measure. Then $(S \cap I) - (T \cap I)$ contains an interval about zero.*

PROOF. By contradiction. Assume there is no interval about zero in the difference set $(S \cap I) - (T \cap I)$. Let $\varepsilon > 0$ satisfy $\mu(S \cap I) + \mu(T \cap I) = \lambda(I) + \varepsilon$, and let $r = \dfrac{\varepsilon}{2}$. Since $(S \cap I) - (T \cap I)$ contains no interval about zero, there exists an r' satisfying $0 < r' < r, r' \notin (S \cap I) - (T \cap I)$. This implies that $(S \cap I)$ and $(T \cap I) + r'$ are disjoint. But $\mu(((T \cap I) + r') \cap I) \geq \mu(T \cap I) - r'$, hence $\lambda(I) \geq \mu(S \cap I) + \mu(((T \cap I) + r') \cap I) \geq \mu(S \cap I) + \mu(T \cap I) - r' > \lambda(I) + \varepsilon - r > \lambda(I)$. This contradiction proves the result. $\square$

The following related result uses the construction given in the proof of Theorem 6.1, p. 186, of [8].

Theorem 3.7. *Let S and T be two μ-shadings satisfying* $\mathrm{sh}_\mu S + \mathrm{sh}_\mu T > 1$, *where μ is a t-Banach measure. Then $S - T = \mathbb{R}$.*

PROOF. Assume $S - T \neq \mathbb{R}$ and choose $r \in (S-T)^c$. Then S and $T + r$ are disjoint μ-shadings, with μ-shades $\mathrm{sh}_\mu S$ and $\mathrm{sh}_\mu T$, respectively. Disjointness then implies $1 \geq \mathrm{sh}_\mu S + \mathrm{sh}_\mu T$, a contradiction. $\square$

Corollary 3.8. *Let S and T be two μ-shadings satisfying $\mathrm{sh}_\mu S + \mathrm{sh}_\mu T > 1$, where μ is a Banach measure. Then $S \pm T = \mathbb{R}$.*

PROOF. Since every Banach measure is a t-Banach measure, $S - T = \mathbb{R}$ follows from Theorem 3.7. Since μ is a Banach measure, $-T$ is a μ-shading with $\mathrm{sh}_\mu(-T) = \mathrm{sh}_\mu T$. Again using Theorem 3.7, we conclude that $S - (-T) = S + T = \mathbb{R}$. $\square$

4 A Few More Results on Archimedean Sets and Shadings.

In [4], a problem was posed: If A is an Archimedean set of μ-shade a, and b satisfies $0 < b < a$, does there exist an Archimedean subset $B \subseteq A$ such that $\mathrm{sh}_\mu B = b$? In [6], it was shown that the answer is 'yes,' provided the Archimedean set has two rationally independent translators. We now finish solving the problem by proving this subset result holds for *any* Archimedean set.

Theorem 4.1. *Let A be an Archimedean set and let μ be a Banach measure. For every b satisfying $0 < b < \mathrm{sh}_\mu A$, there exists an Archimedean set $B \subseteq A$ satisfying $\mathrm{sh}_\mu B = b$.*

PROOF. By Theorem 2.1 of [6], the theorem is true whenever $\tau(A)$, the group of translators of A, contains two rationally independent numbers. (A real number t is a *translator* of A provided $A + t = A$.) So assume $\tau(A)$ contains exactly one rationally independent translator. Then $\tau(A)$ is contained in $t\mathbb{Q}$ for some real number t. Choose r so that t and r are rationally independent and define the following equivalence relation: $x \sim y$ if and only if $x - y \in \tau(A) + r\mathbb{Q}$. Note that every real number has a unique representation in $\Gamma + \tau(A) + r\mathbb{Q}$, where Γ is a selector set of the equivalence relation. For $x \in [0,1]$ define $A_x = \Gamma + \tau(A) + r(([0,x] + \mathbb{Z}) \cap \mathbb{Q})$ and $f(x) = \mathrm{sh}_\mu(A \cap A_x)$. (The function $f(x)$ is well-defined because $A \cap A_x$ is an Archimedean set with translators $\tau(A)$.) Since $A \cap A_x$ is an Archimedean subset of A, it is enough to show that $f(x)$ takes on every value between 0 and $\mathrm{sh}_\mu A$. To do this, we use the Intermediate Value Theorem for continuous functions. Using the same proof given in Example 4.7 of [4], it can be shown that $\mathrm{sh}A_x = x$ for every $x \in [0,1]$. In particular, $\mathrm{sh}A_0 = 0$ and $A_1 = \mathbb{R}$, which implies that $f(0) = 0$ and $f(1) = \mathrm{sh}_\mu A$. To complete the proof then, we need only show that f is continuous. This will be true if $\lim_{x\to x_0}(f(x) - f(x_0)) = 0$ for every $x_0 \in [0,1]$. We will prove $\lim_{x\to x_0^+}(f(x) - f(x_0)) = 0$ for $x_0 \in (0,1)$. The other one-sided limit, as well as the special cases $x_0 = 0$ and $x_0 = 1$, can be proven similarly. Clearly for $x > x_0$, $A_x \setminus A_{x_0} = \Gamma + \tau(A) + r(([x_0,x]+\mathbb{Z})\cap\mathbb{Q})$ is a set of shade $x - x_0$. This implies $\mathrm{sh}(A_x \setminus A_{x_0}) \to 0$ as $x \to x_0^+$, which implies $\mathrm{sh}_\mu(A\cap(A_x\setminus A_{x_0})) \to 0$ as $x \to x_0^+$. But $\mathrm{sh}_\mu(A\cap(A_x\setminus A_{x_0})) = \mathrm{sh}_\mu(A\cap A_x) - \mathrm{sh}_\mu(A\cap A_{x_0}) = f(x) - f(x_0)$. This completes the proof. □

Note: The case where $\tau(A)$ contains two rationally independent translators can also be proven using the above method.

A *G-measure* μ is a countably-additive measure on a sigma-algebra of sets (the domain of the measure) that is invariant under members of a group G. That is, $\mu(g(X)) = \mu(X)$ for any $g \in G$ and any set X in the domain. The sigma-algebra on which μ is defined is invariant, meaning if X is in the sigma-algebra, so is $G(X)$. A set X is said to have the *uniqueness property* (see p. 114, [3]) if it is in the domain of some G-measure and if it has the same measure for every G-measure containing it in its domain. (That is, $\mu(X) = \nu(X)$ for any two G-measures μ, ν with X in their domains.) Kharazishvili proved (p. 110, [3]) that if a set X has the uniqueness property, then there exist countably many $\{g_i\} \subseteq G$ satisfying $\nu(\mathbb{R} \setminus \cup g_i(X)) = 0$ whenever ν is a G-measure containing only sets with the uniqueness property in its domain. Shadings are analogous to sets with the uniqueness property in the sense that they yield the same measure for all different Banach measures. In the theorem below, we prove that the union of a finite number of isometries of an Archimedean shading can have as large a μ-shading as we want, for a given Banach μ.

First we need the following lemma. The main idea used in the construction of the measure was conveyed to the author by R. Mabry and J. Roberts.

Lemma 4.2. *Let G be the group of isometries of $\mathbb{R}$ and let μ_G be a measure on G. (This means μ_G is finitely additive, G-invariant, $\mu_G(G) = 1$, and μ_G measures every subset of G. Such a μ_G exists since G is amenable.) For any Banach measure μ and any μ-shading S satisfying $\mathrm{sh}_\mu S \neq 0$, there exists a Banach measure ν such that*

$$\nu(E) = \frac{1}{\mathrm{sh}_\mu S}\int_G \mu\left(T^{-1}(E)\cap S\right) d\mu_G(T) \tag{1}$$

for any bounded set E.

PROOF. Many of the ideas used in the construction of the measure are similar to those given in the proof of Lemma 2.1. For any bounded set E, define $\phi(E) = \frac{1}{\mathrm{sh}_\mu S} \int_G \mu\left(T^{-1}(E) \cap S\right) d\mu_G(T)$. For arbitrary sets $X \subseteq \mathbb{R}$ define

$$\nu(X) = \sum_{i \in \mathbb{Z}} \phi(X \cap [i, i+1)). \tag{2}$$

Since E is bounded, $\mu\left(T^{-1}(E) \cap S\right) \leq \mu\left(T^{-1}(E)\right) = \mu(E) < \infty$. This implies that for fixed and bounded E, $\mu\left(T^{-1}(E) \cap S\right)$ is a bounded function of T. This boundedness ensures that our integral defining ϕ is well defined, and as such is additive and G-invariant (see [7, pp. 146–147]). Toward showing ν is an extension of the Lebesgue measure, first assume that E is a bounded Lebesgue measurable set. Then

$$\begin{aligned}\phi(E) =& \frac{1}{\mathrm{sh}_\mu S} \int_G \mu\left(T^{-1}(E) \cap S\right) d\mu_G(T) \\ =& \frac{1}{\mathrm{sh}_\mu S} \int_G (\mathrm{sh}_\mu S)(\lambda(E)) d\mu_G(T) \\ =& \lambda(E).\end{aligned}$$

The fact that $\mu\left(T^{-1}(E) \cap S\right) = (\mathrm{sh}_\mu S)(\lambda(E))$ follows from Theorem 3.11 in [4]. Using (2), we can show that ν is isometry-invariant, additive, and extends the Lebesgue measure for arbitrary sets. Equation (1) follows from the finite additivity of μ. $\square$

Theorem 4.3. *Let A be an Archimedean shading, let $0 < \alpha < \mathrm{sh}A$, and let μ be a Banach measure. For any β satisfying $\mathrm{sh}A < \beta < 1$, there exist 2^n isometries $\{S_i\}$ satisfying*

$$\mathrm{sh}_\mu \left(\bigcup_{i=1}^{2^n} S_i(A) \right) > \beta,$$

where $n = \left\lceil \dfrac{\beta - \alpha}{\alpha - \alpha\beta} \right\rceil$.

PROOF. Let μ_G be a measure on the group $G = T$ of isometries (which is amenable). By Lemma 4.2, there exists a Banach measure ν satisfying

$$\nu(E) = \frac{1}{\mathrm{sh}(A^c)} \int_G \mu(T^{-1}(E) \cap A^c) d\mu_G(T)$$

for every bounded set $E \subset \mathbb{R}$. We have

$$\nu(A \cap [0,1]) = \frac{1}{\mathrm{sh}(A^c)} \int_G \mu(T^{-1}(A \cap [0,1]) \cap A^c) d\mu_G(T),$$

and since ν is a Banach measure, $(\mathrm{sh}A)(\mathrm{sh}A^c) = \int_G \mu(T^{-1}(A \cap [0,1]) \cap A^c) d\mu_G(T) \leq \sup_{\{T\}} \mu(T^{-1}(A \cap [0,1]) \cap A^c)$, hence $\alpha\ \mathrm{sh}(A^c) < \mu(T_1(A \cap [0,1]) \cap A^c)$ for some isometry T_1. Since $\mu(T_1(A \cap [0,1]) \cap A^c) = \mu(T_1(A) \cap A^c \cap T_1([0,1])) = \mathrm{sh}_\mu(T_1(A) \cap A^c)$, we have $\alpha\ \mathrm{sh}(A^c) < \mathrm{sh}_\mu(T_1(A) \cap A^c)$. (We note that $T_1(A) \cap A^c$ is a μ-shading because $T_1(A)$ is an Archimedean set with the same set of translators as A, and so is A^c. The intersection of these two sets also has that same set of dense translators and so is a μ-shading.) Now let $A_1 = A \cup (T_1(A) \cap A^c)$. Since A_1 is a μ-shading, there exists a Banach measure ν_1 satisfying $\nu_1(E) = \frac{1}{\mathrm{sh}_\mu(A_1^c)} \int_G \mu(T^{-1}(E) \cap A_1^c) d\mu_G(T)$ for every bounded $E \subset \mathbb{R}$. (A_1 is a μ-shading because the union of two Archimedean sets with the same set of translators is also Archimedean with that same set of translators.) Setting $E = A_1 \cap [0,1]$ and repeating the steps in the first paragraph gives us $\alpha\ \mathrm{sh}_\mu(A_1^c) < \mathrm{sh}_\mu(T_2(A_1) \cap A_1^c)$ for some isometry T_2. Now let $A_2 = A_1 \cup (T_2(A_1) \cap A_1^c)$ and continue the process. We can thus create a sequence of isometries T_i and sets $A_{i+1} = A_i \cup (T_{i+1}(A_i) \cap A_i^c)$ satisfying $\alpha\ \mathrm{sh}_\mu(A_i^c) < \mathrm{sh}_\mu(T_{i+1}(A_i) \cap A_i^c)$ for $i = 0, 1, 2, \cdots$. (Define $A_0 = A$.) Since $T_{i+1}(A_i) \cap A_i^c$ is added at every step in the iteration, we can say that after $n = \left\lceil \frac{\beta - \alpha}{\alpha - \alpha\beta} \right\rceil$ iterations, we have $\mathrm{sh}_\mu A_n > \alpha + \alpha\ \mathrm{sh}_\mu A^c + \alpha\ \mathrm{sh}_\mu A_1^c + \alpha\ \mathrm{sh}_\mu A_2^c + \cdots + \alpha\ \mathrm{sh}_\mu A_{n-1}^c$. If $\mathrm{sh}_\mu A_i^c = 0$ for any $i \leq n-1$, then we are done because then $\mathrm{sh}_\mu A_i = 1$. If $\mathrm{sh}_\mu A_{n-1}^c < 1 - \beta$, then we are done because then $\mathrm{sh}_\mu A_{n-1} > \beta$. So assume $\mathrm{sh}_\mu A_{n-1}^c \geq 1 - \beta$. Since $\mathrm{sh}_\mu A_i^c$ is a non-increasing sequence, $\mathrm{sh}_\mu A_i^c \geq 1 - \beta$ for $i = 0, 1, 2, \cdots, n-1$. This implies $\mathrm{sh}_\mu A_n > \alpha + \alpha n(1-\beta) = \alpha + \alpha \left\lceil \frac{\beta - \alpha}{\alpha - \alpha\beta} \right\rceil (1-\beta) \geq \alpha + \alpha \frac{\beta - \alpha}{\alpha - \alpha\beta}(1-\beta) = \alpha + \beta - \alpha = \beta$. Each of the n iterations doubles the number of isometries on A (which may not be distinct), so A_n is formed by 2^n isometries S_i of A, including the identity. □

Corollary 4.4. *If A is an Archimedean shading of positive shade, μ is a Banach measure, and β satisfies $\mathrm{sh}A < \beta < 1$, then there exist a finite number of isometries T_i of $\mathbb{R}$ satisfying $\mathrm{sh}_\mu(\cup T_i(A)) > \beta$.*

The question of what happens when two arbitrary shadings are intersected is discussed in Example 5.3 and Example 5.6 of [4]. The answer is anything but straightforward. In those examples Mabry demonstrates that the intersection of two shadings need not even be a μ-shading, let alone have any nice formula or pattern. However, in Corollary 4.6 below, we show that if we are allowed to translate and/or reflect one of the shadings, then the μ-measure of their intersection is very nearly the product of the two shades. Theorem 4.5 is a more general result.

Theorem 4.5. *Let S_1 be a shading, let S_2 be a μ-shading for some Banach μ, and let I be a bounded interval. Then if T is an isometry of $\mathbb{R}$,*

$$\inf_{\{T\}} \frac{\mu\left(S_1 \cap T(S_2) \cap I\right)}{\mu\left(I\right)} \leq \left(\mathrm{sh} S_1\right)\left(\mathrm{sh}_\mu S_2\right) \leq \sup_{\{T\}} \frac{\mu\left(S_1 \cap T(S_2) \cap I\right)}{\mu\left(I\right)}.$$

Proof. If $\mathrm{sh}_\mu S_2 = 0$, then the conclusion is obvious, since $\mathrm{sh}_\mu T(S_2) = 0$ for any isometry T of $\mathbb{R}$. So assume $\mathrm{sh}_\mu S_2 \neq 0$. From Lemma 4.2, there exists a Banach measure ν satisfying $\nu\left(E\right) = \frac{1}{\mathrm{sh}_\mu S_2} \int_G \mu\left(T^{-1}\left(E\right) \cap S_2\right) d\mu_G\left(T\right)$ for any bounded set E. Letting $E = S_1 \cap I$ gives us

$$\nu\left(S_1 \cap I\right) = \frac{1}{\mathrm{sh}_\mu S_2} \int_G \mu\left(T^{-1}\left(S_1 \cap I\right) \cap S_2\right) d\mu_G\left(T\right),$$

which implies

$$\begin{aligned} \inf_{\{T\}} \mu\left(T^{-1}\left(S_1 \cap I\right) \cap S_2\right) \leq\ & \left(\mathrm{sh} S_1\right)\left(\mathrm{sh}_\mu S_2\right) \mu\left(I\right) \\ & \leq \sup_{\{T\}} \mu\left(T^{-1}\left(S_1 \cap I\right) \cap S_2\right), \end{aligned}$$

hence

$$\begin{aligned} \inf_{\{T\}} \mu\left(S_1 \cap I \cap T(S_2)\right) \leq\ & \left(\mathrm{sh} S_1\right)\left(\mathrm{sh}_\mu S_2\right) \mu\left(I\right) \\ & \leq \sup_{\{T\}} \mu\left(S_1 \cap I \cap T(S_2)\right). \end{aligned}$$

□

The inequality is tight, because if S_1 is a shading and S_2 is an almost isometry-invariant set, then $\frac{\mu(S_1 \cap T(S_2) \cap I)}{\mu(I)} = \frac{\mu(S_1 \cap S_2 \cap I)}{\mu(I)}$ for any isometry T of $\mathbb{R}$. This forces the upper and lower bounds to be the same. It also implies $\frac{\mu(S_1 \cap S_2 \cap I)}{\mu(I)} = (\mathrm{sh} S_1)(\mathrm{sh}_\mu S_2)$, the conclusion of Theorem 2.5. Also, the inequality will not always reduce to an equality because setting $S_1 = S_2$, where $-S_1 = S_1^c$ and $\mathrm{sh} S_1 = \dfrac{1}{2}$, gives us a strict inequality on both the infimum and supremum. (Such a set S_1 is given in Example 3.4.)

Because every shading is also a μ-shading for any fixed Banach μ, we have the following corollary.

Corollary 4.6. *If S_1 and S_2 are shadings, μ is a Banach measure, I is a bounded interval, and T is an isometry of $\mathbb{R}$, then*

$$\inf_{\{T\}} \frac{\mu(S_1 \cap T(S_2) \cap I)}{\mu(I)} \leq (\mathrm{sh} S_1)(\mathrm{sh} S_2) \leq \sup_{\{T\}} \frac{\mu(S_1 \cap T(S_2) \cap I)}{\mu(I)}.$$

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