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HENSTOCK'S VERSION OF ITÔ'S FORMULA

Abstract

Itô's Formula is the stochastic analogue of the change of variable formula for deterministic integrals. It is a useful tool in dealing with stochastic integration. In this paper, using Henstock's approach, we derive two versions of Itô's Formula. Henstock's or generalized Riemann approach has been successful in giving an alternative definition of stochastic integral, which is more explicit, intuitive and less measure theoretic. Henstock's approach provides a simpler and more direct proof of Itô's Formula, although we do not claim that it is a generalization of the classical results.

1 Introduction.

Stochastic calculus has been well developed in the study of stochastic integrals, see [3, 8, 14, 18, 19, 20, 21, 22, 31]. In the classical theory of stochastic integration, it has often been highlighted that it is impossible to use the Riemann approach to define the integral, since the Riemann approach uses uniform mesh. In the classical theory of non-stochastic integration, in the 1950s, J. Kurzweil and R. Henstock independently modified the Riemann integral by using non-uniform meshes; that is, mesh that varies from point to point. It turns out that this integral is more general than the classical Riemann integral and the Lebesgue integral, see [4, 5, 6, 9, 10, 11].

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Along this line of thought, the Henstock approach, also known as the generalized Riemann approach, has been used to study stochastic integrals, see [2, 7, 10, 12, 13, 15, 16, 17, 23, 24, 25, 26, 27, 30]. The advantage of the generalized Riemann approach is that it gives an explicit and intuitive definition of the stochastic integral using L^2 -convergence. Even for the stochastic integrals, it turns out that Henstock's definition encompasses the classical stochastic integrals (see [23, 24, 25, 26, 27]). The Henstock approach was also used to characterize stochastic integrable processes in [29] and an integration-by-part formula is also derived for stochastic integrals, see [28]. In this paper, we shall use the Henstock approach to derive the Itô Formula.

The Itô Formula is the stochastic analogue of the change of variable formula for deterministic integrals in the classical integration theory. It is a useful tool in the study of stochastic analysis. In the field of finance, it made its way into the financial models in 1973 when Black and Scholes discovered its use to find the price of an option. In this note we will use the Henstock approach to derive the Itô Formula. What is noteworthy is that we do not need to use stochastic calculus in obtaining the results. This approach is more direct, intuitive and less measure theoretic in its approach. Hopefully it will make the mathematics underlying the Itô Formula more comprehensible.

2 Setting.

Let Ω denote the set of all real-valued continuous functions on $[a, b]$ and \mathbb{R} the set of all real numbers.

The class of all Borel cylindrical sets B in Ω , denoted by \mathcal{C} , is a collection of all the sets B in Ω of the form

$$B = \{w : (w(t_1), w(t_2), \dots, w(t_n)) \in E\}$$

where $0 \leq t_1 < t_2 < \dots < t_n \leq 1$ and E is any Borel set in \mathbb{R}^n (n is not fixed). The Borel σ -field of \mathcal{C} is denoted by \mathcal{F} ; i.e. it is the smallest σ -field which contains \mathcal{C} . Let P be the Wiener measure defined on (Ω, \mathcal{F}) . Then (Ω, \mathcal{F}, P) is a probability space; that is, a measure space with $P(\Omega) = 1$.

Let $\{\mathcal{F}_t\}$ be an increasing family of σ -subfields of \mathcal{F} for $t \in [a, b]$, that is, $\mathcal{F}_r \subset \mathcal{F}_s$ for $a \leq r < s \leq b$ with $\mathcal{F}_b = \mathcal{F}$. The probability space together with its family of increasing σ -subfields is called a *standard filtering space* and denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

A process $\{\varphi(t, \omega) : t \in [a, b]\}$ on (Ω, \mathcal{F}, P) is a family of \mathcal{F} -measurable functions (which are called random variables) on (Ω, \mathcal{F}, P) . Very often, $\varphi(t, \omega)$ is denoted by $\varphi_t(\omega)$.

A process $\{\varphi_t(\omega) : t \in [a, b]\}$ is said to be adapted to the filtering $\{\mathcal{F}_t\}$ if for each $t \in [a, b]$, φ_t is \mathcal{F}_t -measurable.

Next, we shall consider a very special and important process, namely, the Brownian motion denoted by W .

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a standard filtering space as outlined above, and the stochastic process $W = \{W_t(\omega)\}_{a \leq t \leq b}$ be adapted to the filtering space $\{\mathcal{F}_t\}$. Then W is said to be a canonical Brownian motion if it satisfies the following properties:

1. $W_a(\omega) = 0$ for all $\omega \in \Omega$;
2. it has **Normal Increments**; that is, $W_t - W_s$ has a Normal distribution with mean 0 and variance $t - s$ for all $t > s$ (which naturally implies that W_t has a Normal distribution with mean 0 and variance t);
3. for all $0 \leq s < t < \infty$, the increment $W_t - W_s$ is independent of \mathcal{F}_s ; and
4. its sample paths are continuous; i.e., for each $\omega \in \Omega$, $W_t(\omega)$ as a function of t is continuous on $[a, b]$.

Definition 1. Let δ be a positive function on $[a, b]$. A finite collection D of interval-point pairs $\{((\xi_i, v_i), \xi_i) : i = 1, 2, 3, \dots, n\}$ is a δ -fine belated *partial* division of $[a, b]$ if

- (i) $(\xi_i, v_i), i = 1, 2, 3, \dots, n$, are disjoint left-open subintervals of $[a, b]$; and
- (ii) each (ξ_i, v_i) is δ -fine belated, that is, $(\xi_i, v_i) \subset [\xi_i, \xi_i + \delta(\xi_i)]$.

In Definition 1, the division is said to be *belated* as the tag ξ for each interval $(\xi, v]$ is always the left-hand point. It is *partial* in that the union of all the disjoint intervals $(\xi, v]$ of D may not cover the entire interval $(a, b]$.

Definition 2. Given $\eta > 0$, a given δ -fine belated partial division $D = \{((\xi, v), \xi)\}$ is to be a (δ, η) -fine belated partial division of $[a, b]$ if it fails to cover $[a, b]$ by at most Lebesgue measure η , that is,

$$\left| b - a - (D) \sum (v - \xi) \right| \leq \eta.$$

Remark 1. Given any positive function δ , we may not be able to find a δ -fine division that covers the entire interval $[a, b]$. For example, take $\delta(\xi) = (b - \xi)/2$. The point b is not covered by any *finite* collection of δ -fine belated intervals. However, from Vitali's covering theorem, we can always make the part of $[a, b]$ that is not covered arbitrarily small.

Definition 3. (See Definition 10 of [2]). Let $f = \{f_t : t \in [a, b]\}$ be a process adapted to the standard filtering space. Then f is said to be Itô-Henstock integrable on $[a, b]$ if there exists an $A \in L^2(\Omega)$ such that for any $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ and a positive number $\eta > 0$ such that for any (δ, η) -fine belated partial division $D = \{((\xi_i, v_i), \xi_i) : i = 1, 2, 3, \dots, n\}$ of $[a, b]$, we have

$$E \left(\sum_{i=1}^n f_{\xi_i} [W_{v_i} - W_{\xi_i}] - A \right)^2 \leq \varepsilon.$$

Thus, the definition of the Itô-Henstock integral makes use of Vitali covers as in [1]. For simplicity, we may let the Riemann sum $(D) \sum f_{\xi} [W_v - W_{\xi}]$ be denoted by the notation $S(f, D, \delta, \eta)$, where D is a (δ, η) -fine belated partial division of $[a, b]$. The standard properties of the integral, such as (i) the uniqueness of the integral, (ii) the additivity of the integral, (iii) integrability over subintervals and (iv) Cauchy criterion hold true for the Itô-Henstock integral. Let A in Definition 3 be denoted by $(IH) \int_a^b f_t dW_t$ throughout this paper. In this section, we shall only state and prove the following theorem, which perceives the Itô-Henstock integral can be seen as the limit of a sequence of Riemann sums.

If f is a given process on $[a, b]$, and if D is a (δ, η) -fine belated partial division of $[a, b]$, let $\tilde{S}(f, D, \delta, \eta)$ denote the Riemann sum $(D) \sum f(\xi)(v - \xi)$. We shall use this notation subsequently.

Theorem 4. *The stochastic process f is Itô-Henstock integrable on $[a, b]$ if and only if there exist $A \in L^2(\Omega)$, a decreasing sequence $\{\delta_n(\xi)\}$ of positive functions defined on $[a, b]$, and a decreasing sequence of positive numbers $\{\eta_n\}$, that is, $0 < \delta_{n+1}(\xi) < \delta_n(\xi)$ and $\eta_{n+1} < \eta_n$ for all n and all $\xi \in [a, b]$, such that we have*

$$\lim_{n \rightarrow \infty} E \left(|S(f, D_n, \delta_n, \eta_n) - A|^2 \right) = 0$$

for any (δ_n, η_n) -fine belated partial division of $[a, b]$ denoted by D_n . In this case,

$$A = (IH) \int_a^b f_t dW_t.$$

PROOF. Let f be Itô-Henstock integrable on $[a, b]$ with integral $A \in L^2(\Omega)$. For each $\varepsilon = 1/n, n = 1, 2, 3, \dots$, there exists a positive function δ_n on $[a, b]$

and a positive number η_n such that whenever D_n is any (δ_n, η_n) -fine belated partial division in $[a, b]$, we have

$$E \left(|S(f, D_n, \delta_n, \eta_n) - A|^2 \right) \leq \frac{1}{n},$$

$n = 1, 2, 3, \dots$

Hence we obtain

$$\lim_{n \rightarrow \infty} E \left(|S(f, D_n, \delta_n, \eta_n) - A|^2 \right) = 0.$$

Conversely, let us assume that there exists $A \in L^2(\Omega)$ and a decreasing sequence $\{\delta_n(\xi)\}$ of positive functions on $[a, b]$ and a decreasing sequence of positive numbers $\{\eta_n\}$ such that

$$\lim_{n \rightarrow \infty} E \left(|S(f, D_n, \delta_n, \eta_n) - A|^2 \right) = 0.$$

Suppose that f is not Itô-Henstock integrable to A on $[a, b]$. Then there exists $\varepsilon > 0$ such that for every positive function δ on $[a, b]$ and every positive number η there exists a (δ, η) -fine belated partial division D in $[a, b]$ with

$$E \left(|S(f, D, \delta, \eta) - A|^2 \right) \geq \varepsilon.$$

Hence for each δ_n and η_n , there exists a (δ_n, η_n) -fine belated partial division D_n in $[a, b]$ with

$$E \left(|S(f, D_n, \delta_n, \eta_n) - A|^2 \right) \geq \varepsilon,$$

leading to a contradiction. \square

Remark 2. 1. Note that Theorem 4 above gives an alternative definition of the Itô-Henstock integral in terms of the limits of sequence of Riemann sums.

2. In the classical non-stochastic integration theory, integrals can be defined by using the belated partial division of Definition 1 in this paper, which gives rise to McShane's belated non-stochastic integral. A real-valued function f defined on $[a, b]$ is McShane belated integrable to $A \in \mathbb{R}$ if given $\varepsilon > 0$, there exists a function $\delta(\xi) > 0$ on $[a, b]$ and a real constant $\eta > 0$ such that

$$|A - (D) \sum f(\xi)(v - \xi)| < \varepsilon$$

whenever D is any (δ, η) -fine belated partial division of $[a, b]$. In fact, McShane's belated non-stochastic integral is equivalent to the Lebesgue integral, (see [1] and [17]). We shall denote the Lebesgue integral A as $(L) \int_a^b f(t) dt$. To make our presentation clear, we shall use $(IH) \int_a^b$ to denote the Itô-Henstock integral on $[a, b]$ while $(L) \int_a^b$ to denote the Lebesgue or McShane belated non-stochastic integral on $[a, b]$ throughout our paper.

3. Analogous to the stochastic case of defining stochastic integral by means of limits of sequences of Riemann sums, as in Theorem 4, the Lebesgue or McShane's belated non-stochastic integral can also be defined in terms of limits of sequences of Riemann sums. If h is a real-valued function defined on $[a, b]$ and $D = \{(\xi, v), \xi\}$ a (δ, η) -fine belated partial division of $[a, b]$, let $\tilde{S}(h, D, \delta, \eta)$ denote the Riemann sum $(D) \sum h(\xi)(v - \xi)$. Consequently we have Theorem 5:

Theorem 5. *The function $h : [a, b] \rightarrow \mathbb{R}$ is Lebesgue integrable to $A \in \mathbb{R}$ if and only if there exists a decreasing sequence of positive functions $\{\delta_n\}$ and a sequence of decreasing positive constants $\{\eta_n\}$ such that*

$$\lim_{n \rightarrow \infty} \left| \tilde{S}(h, D_n, \delta_n, \eta_n) - A \right| = 0,$$

where D_n is any (δ_n, η_n) -fine belated partial division of $[a, b]$.

It is instructional to go through the proof of Theorem 4 again, with the obvious modifications, to complete the proof of Theorem 5.

3 Itô's Formula.

Itô's Formula is the stochastic analogue of the change of variable formula for deterministic integrals. It is a useful tool in the theory of stochastic integration. It made its way into the financial models in 1973 when Black and Scholes discovered that it was useful to find the price of an option. There are many versions of Itô's Formula. In this section, we shall only present two versions of Itô's Formula using Henstock's approach.

First we shall prove several lemmas which are related to Riemann sums of the stochastic Itô-Henstock integral $(IH) \int_a^b g_t dW_t$ and the pathwise Lebesgue integral $(L) \int_a^b g_t(\omega) dt$ for each $\omega \in \Omega$. In order to facilitate the following proofs, we shall use convergence in probability and pointwise convergence a.s.

In this paper, we shall establish our settings using convergence in probability. Note that pointwise convergence a.s. implies convergence in probability (note that probability is a finite measure). Thus, results like Lemma 6 can be established under norm-convergence, but we establish it under a.s. pointwise convergence (of a subsequence), which then implies convergence in probability.

Lemma 6. *Let $h = \{h_t : t \in [a, b]\}$ be an adapted stochastic process such that $E(h_t^2)$ is integrable over $[a, b]$. Then there exists a decreasing sequence of positive functions $\{\delta_n\}$ and a decreasing sequence of positive constants $\{\eta_n\}$ such that*

$$\lim_{n \rightarrow \infty} P(h, D_n, \delta_n, \eta_n) = 0 \text{ a.s.}, \quad (1)$$

where $P(h, D_n, \delta_n, \eta_n) = (D_n) \sum h_\xi [(W_v - W_\xi)^2 - (v - \xi)]$ and $\{D_n\}$ is any (δ_n, η_n) -fine belated partial division of $[a, b]$.

PROOF. Let $\{\delta_n\}$ be a decreasing sequence of functions on $[a, b]$ with $|\delta_n(\xi)| \leq 1/n$ for each $n = 1, 2, 3, \dots$. By Theorem 5, we can define the integral $\int_a^b E(h_t^2) dt$ in terms of a sequence of Riemann sums $\tilde{S}(E(h_t^2), D_n, \delta_n, \eta_n)$, which has to be bounded. Then

$$\begin{aligned} & E \left(\left| (D_n) \sum h_\xi [(W_v - W_\xi)^2 - (v - \xi)] \right|^2 \right) \\ &= E \left((D_n) \sum h_\xi^2 [(W_v - W_\xi)^2 - (v - \xi)]^2 \right) \\ &= E \left((D_n) \sum h_\xi^2 E \left([(W_v - W_\xi)^2 - (v - \xi)]^2 \mid \mathcal{F}_\xi \right) \right) \\ &= 2E \left((D_n) \sum h_\xi^2 (v - \xi)^2 \right) \\ &\leq 2 \frac{1}{n} E \left((D_n) \sum h_\xi^2 (v - \xi) \right) \\ &= 2 \frac{1}{n} \tilde{S}(E(h_t^2), D_n, \delta_n, \eta_n) \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence there exists a subsequence $\{P(h, D_{n_k}, \delta_{n_k}, \eta_{n_k})\}$ which converges to 0 a.s. By re-labeling, if necessary, we have

$$\lim_{n \rightarrow \infty} P(h, D_n, \delta_n, \eta_n) = 0,$$

thereby completing the proof. \square

Lemma 7. *Let $g = \{g_t : t \in [a, b]\}$ be an Itô-Henstock integrable process such that $E(g_t^2)$ is integrable over $[a, b]$. Given $\varepsilon > 0$, there exists a positive function δ and a positive constant η such that whenever $D = \{((\xi, v], \xi)\}$ is a (δ, η) -fine belated partial division of $[a, b]$, we have*

$$E \left(\left| (D \cup D^c) \sum g_\xi(W_v - W_\xi) - (IH) \int_a^b g_t dW_t \right|^2 \right) \leq \varepsilon \tag{2}$$

where $\{(\xi, v) : (\xi, v) \in D^c\}$ is the collection of all those subintervals of $[a, b]$ which are not included in the set $\{(\xi, v) : (\xi, v) \in D\}$.

PROOF. Given $\varepsilon > 0$, there exists a positive function δ and a positive number η such that whenever $D = \{((\xi, v], \xi)\}$ is a (δ, η) -fine belated partial division of $[a, b]$, we have

$$E \left(\left| (D) \sum g_\xi(W_v - W_\xi) - (IH) \int_a^b g_t dW_t \right|^2 \right) \leq \frac{\varepsilon}{4}. \tag{3}$$

Choose η small enough in the above such that whenever $D^c = \{((\xi, v], \xi)\}$ is a partial division of $[a, b]$ with $(D^c) \sum |v - \xi| \leq \eta$, we have

$$E \left(\left| (D^c) \sum g_\xi(W_v - W_\xi) \right|^2 \right) = E \left((D^c) \sum g_\xi^2(v - \xi) \right) \leq \frac{\varepsilon}{4}. \tag{4}$$

The result (4) is true and it is a direct consequence of Lemma 15 of [2]. Hence the proof is complete. □

Lemma 8. *Let $g = \{g_t : t \in [a, b]\}$ be an Itô-Henstock integrable process such that $E(g_t^2)$ is integrable over $[a, b]$. Then there exists a sequence $\{\delta'_n\}$ of positive functions and a sequence $\{\eta'_n\}$ of positive constants such that*

$$\lim_{n \rightarrow \infty} S(g, D_n \cup D_n^c, \delta'_n, \eta'_n) = (IH) \int_a^b g_t dW_t \text{ in probability} \tag{5}$$

where D_n is any (δ'_n, η'_n) -fine belated partial division of $[a, b]$, and

$$S(g, D_n \cup D_n^c, \delta'_n, \eta'_n) = (D_n \cup D_n^c) \sum g_\xi(W_v - W_\xi). \tag{6}$$

□

PROOF. The result follows from Lemma 7 since

$$\lim_{n \rightarrow \infty} S(g, D_n \cup D_n^c, \delta'_n, \eta'_n) = (IH) \int_a^b g_t dW_t$$

in L^2 -norm, the result holds true for convergence in probability for some subsequence. \square

Lemma 9. *Let $h = \{h_t : t \in [a, b]\}$ be a process such that for each $\omega \in \Omega$, the function $h(\cdot, \omega)$ is continuous on $[a, b]$ (hence integrable over $[a, b]$ for each ω). Then for each $\omega \in \Omega$, there exists a decreasing sequence of positive functions $\{\delta_n(\omega)\}$ and a decreasing sequence of positive constants $\{\eta_n(\omega)\}$ such that*

$$\lim_{n \rightarrow \infty} \tilde{S}(h, D_n(\omega), \delta_n(\omega), \eta_n(\omega)) = (L) \int_a^b h(t, \omega) dt \tag{7}$$

where

$$\tilde{S}(h, D_n(\omega), \delta_n(\omega), \eta_n(\omega)) = (D_n(\omega)) \sum h_\xi(\omega)(v - \xi)$$

and $D_n(\omega) = \{(\xi, v], \xi\}$ is any $(\delta(\omega), \eta(\omega))$ -fine belated partial division of $[a, b]$.

PROOF. Note that Lemma 9 is a consequence of Theorem 5 of defining the Lebesgue integral. We remark that in the statement of Lemma 9, the choices of $\delta_n(\omega)$, $\eta_n(\omega)$ and $D_n(\omega)$ are dependent on the choice of $\omega \in \Omega$. Hence, they can be seen as functions of Ω . Notice that the integral in Lemma 8 is the stochastic Itô-Henstock integral, while the integral in Lemma 9 is the pathwise Lebesgue integral. In the case of Lemma 9, the choice of δ and η are dependent on $\omega \in \Omega$; that is, the Lebesgue integral is defined pathwise. \square

Lemma 10. *Let $h = \{h_t : t \in [a, b]\}$ be an adapted process. Suppose that for each $\omega \in \Omega$, $h_t(\omega)$ is continuous on $[a, b]$. Then for each $\omega \in \Omega$, there exists a sequence $\{\delta_n(\xi, \omega)\}$ of positive functions and a sequence of positive constants $\{\eta_n(\omega)\}$, for which a sequence of (δ_n, η_n) -fine belated partial division $D_n(\omega)$ of $[a, b]$ is such that*

$$\lim_{n \rightarrow \infty} \hat{S}(h, D_n(\omega), \delta_n(\omega)) = (L) \int_a^b h_t(\omega) dt \tag{8}$$

where

$$\hat{S}(h, D_n(\omega), \delta_n(\omega)) = (D_n(\omega)) \sum h_\theta(\omega)(W_v - W_\xi)^2$$

and θ is any given point in the interval $[\xi, v]$.

PROOF. First, we comment that it is sufficient to prove (8) for $\theta = \xi$, since $h(\cdot, \omega)$ is continuous: for each positive integer n , let $\delta_n(\omega)$ be chosen such that

$$|h_\xi(\omega) - h_\theta(\omega)| \leq \frac{1}{n}$$

whenever $\theta \in [\xi, v]$ and $\{(\xi, v], \xi\}$ is $\delta_n(\xi)$ -fine. Now let $D_n(\omega) = \{((\xi, v], \xi)\}$ be a $\delta_n(\omega)$ -fine full division of $[a, b]$. Then

$$\begin{aligned} & \left| (D_n) \sum h_\theta(W_v - W_\xi)^2 - (D_n) \sum h_\xi(W_v - W_\xi)^2 \right| \\ & \leq \frac{1}{n} \left[(D_n) \sum (W_v - W_\xi)^2 \right]. \end{aligned}$$

Note that $(D_n) \sum (W_v - W_\xi)^2$ converges in probability. Hence we may assume that $(D_n) \sum (W_v - W_\xi)^2$ converges a.s. on Ω .

Notice that in this lemma, convergence in L^2 -norm is weakened to convergence in probability, as the choice of δ and η are each dependent on the choice of ω . □

Theorem 11 (Itô's Formula). *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable. Suppose that*

- (i) $F'(W_t)$ is Itô-Henstock integrable on $[a, b]$;
- (ii) $E(F'(W_t))^2$ is integrable over $[a, b]$;
- (iii) $E(F''(W_t))^2$ is integrable over $[a, b]$.

Then for almost all $\omega \in \Omega$, we have

$$F(W_b) - F(W_a) = (IH) \int_a^b F'(W_t) dW_t + \frac{1}{2}(L) \int_a^b F''(W_t) dt$$

where $\int_a^b F'(W_t) dW_t$ is the Itô integral of $F'(W_t)$.

PROOF. Applying Taylor's Theorem to F at $W_\xi(\omega)$ for $\xi \in [a, b]$ and $\omega \in \Omega$, we have for any $v > \xi$,

$$\begin{aligned} F(W_v(\omega)) - F(W_\xi(\omega)) &= F'(W_\xi(\omega))(W_v(\omega) - W_\xi(\omega)) \\ &\quad + \frac{1}{2}F''(\alpha(\omega))(W_v(\omega) - W_\xi(\omega))^2 \end{aligned} \tag{9}$$

where $\alpha(\omega)$ is between $W_v(\omega)$ and $W_\xi(\omega)$. Note that for each ω , $W_t(\omega)$ is continuous on $[\xi, v]$. Hence there exists $\theta(\omega)$ between ξ and v such that $\alpha(\omega) = W_{\theta(\omega)}(\omega)$. Therefore we have

$$F(W_v) - F(W_\xi) = F'(W_\xi)(W_v - W_\xi) + \frac{1}{2}F''(W_\theta)(W_v - W_\xi)^2. \tag{10}$$

Let δ_n and η_n be as defined in Lemma 7, and $D_n = \{(u, v], u\}$ be a (δ_n, η_n) -fine belated partial division of $[a, b]$, and D_n^c be as defined in Lemma 8.

Suppose the sequence $\{\delta_n(\omega)\}$ and $\{\eta_n(\omega)\}$ are chosen as in Lemma 10. We next sum up both sides of (10) over $D_n \cup D_n^c$, for which the left hand side of (10) always yields $F(W_b) - F(W_a)$. The sum of the first term on the right hand side of (10) given by $(D \cup D^c) \sum F'(W_\xi)(W_v - W_\xi)$ converges to $\int_a^b F'(W_t) dW_t$ in probability by Lemma 8. Let the sequence of positive functions $\{\delta'_n\}$ and the sequence of positive constants $\{\eta'_n\}$ be as defined in Lemma 8, with g_t replaced by $F'(W_t)$.

Choose $\delta_n(\omega)$ and $\eta_n(\omega)$ for Lemma 10 with h replaced by $F''(W_t)$ such that $\delta_n(\omega) \leq \delta'_n$ and $\eta_n(\omega) \leq \eta'_n$. By Lemmas 9 and 10, the sum of the second term on the right hand side of (10) converges to $\frac{1}{2} \int_a^b F''(W_t) dt$ in probability. Hence we have the result. \square

Example 12. Let $F(x) = x^m$, where $m \geq 2$. Then we have $F'(x) = mx^{m-1}$ and $F''(x) = m(m-1)x^{m-2}$. Hence

(i) $F'(W_t) = mW_t^{m-1}$ and

$$E(F'(W_t))^2 = m^2 E(W_t^{2(m-1)}) = m^2 \beta t^{m-1}$$

for some constant β . Thus $E(F'(W_t))^2$ is bounded over $[0, s]$ and hence it is integrable on $[0, s]$, showing that $F'(W_t)$ is Itô-Henstock integrable on $[0, s]$.

(ii) Similarly, $E(F''(W_t))^2$ is bounded over $[0, s]$. Hence we have

$$W_s^m = m \int_0^s W_t^{m-1} dW_t + \frac{m(m-1)}{2} \int_0^s W_t^{m-2} dt.$$

Theorem 13. Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function whose second order partial derivatives are continuous. Suppose that

(i) $F_2(t, W_t)$ is Itô-Henstock integrable;

- (ii) $E(F_2(t, W_t))^2$ is integrable over $[a, b]$;
- (iii) $E(F_{2,2}(t, W_t))^2$ is integrable over $[a, b]$.

Then for almost all $\omega \in \Omega$, we have

$$F(b, W_b) - F(a, W_a) = (R) \int_a^b \left[F_1(t, W_t) + \frac{1}{2} F_{2,2}(t, W_t) \right] dt + \int_a^b F_2(t, W_t) dW_t.$$

PROOF. We shall only outline the ideas of the proof. Applying Taylor's Theorem to F at $(\xi, W_\xi(\omega))$ for fixed $\omega \in \Omega$, we have

$$\begin{aligned} F(v, W_v) - F(\xi, W_\xi) &\approx F_1(\xi, W_\xi)(v - \xi) + F_2(\xi, W_\xi)(W_v - W_\xi) \\ &\quad + \frac{1}{2} [F_{1,1}(\xi, W_\xi)(v - \xi)^2 \\ &\quad + F_{1,2}(\xi, W_\xi)(v - \xi)(W_v - W_\xi) \\ &\quad + F_{2,2}(\xi, W_\xi)(W_v - W_\xi)^2]. \end{aligned} \tag{11}$$

In the above W_v and W_ξ represents $W_v(\omega)$ and $W_\xi(\omega)$ respectively. Let δ be a positive function and $D = \{((\xi, v], \xi)\}$ be a δ -fine partial division. Then

$$\left| (D) \sum F_{1,1}(\xi, W_\xi(\omega))(v - \xi)^2 \right| \leq \delta(D) \left| \sum F_{1,1}(\xi, W_\xi(\omega))(v - \xi) \right|. \tag{12}$$

Note that for fixed ω , $F(t, W_t(\omega))$ is continuous on $[a, b]$. More importantly, F admits all the continuous second order partial derivatives. Thus $F_{1,1}(t, W_t(\omega))$ is Riemann integrable on $[a, b]$. Therefore the sum on the left-hand side of (12) can be made arbitrarily small. Thus Itô's Formula does not involve $F_{1,1}$. Next

$$\begin{aligned} \left| (D) \sum F_{1,2}(\xi, W_\xi)(v - \xi)(W_v(\omega) - W_\xi(\omega)) \right| \\ \leq \alpha \left| (D) \sum F_{1,2}(\xi, W_\xi)(v - \xi) \right| \end{aligned} \tag{13}$$

where $|W_v(\omega) - W_\xi(\omega)| \leq \alpha$ whenever $((\xi, v], \xi)$ is δ -fine. Therefore, the sum on the left-hand side of (13) can also be made arbitrarily small. Thus Itô's Formula also does not involve $F_{1,2}$. Now using the same idea as in the proof of the first version of Itô Formula for $F_1, F_{2,2}$ and F_2 , we get the required result. □

4 Conclusions.

We have thus presented that the Henstock's approach, which has provided an easier approach to the study of the Itô integral, has also in this case given the Itô Formula a more direct approach by means of considering the Riemann sums.

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