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QUOTIENTS OF DARBOUX-LIKE FUNCTIONS

Abstract

We characterize the families of quotients of functionally connected, connected and almost continuous functions. We prove also theorems concerning common divisor for the families of the quotients of functionally connected (resp. connected, almost continuous) functions with respect to functional connectivity (resp. connectivity, almost continuity).

1 Introduction.

The letter $\mathbb{R}$ denotes the real line. The symbol $\mathrm{I}[a, b]$ denotes the closed interval with endpoints a and b. The family of all functions from a set X into Y is denoted by Y^X. For each set $A \subset \mathbb{R}$, the symbol χ_A denotes the characteristic function of A. We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals. The symbol $\operatorname{card} X$ stands for the cardinality of a set X; we write $\mathfrak{c} = \operatorname{card} \mathbb{R}$. For a cardinal number κ, we write $\operatorname{cf}(\kappa)$ for the *cofinality of* κ; i.e., for the smallest cardinality of a family of cardinals less than κ whose union equals κ. We say that κ is *regular* provided that $\kappa = \operatorname{cf}(\kappa)$. The projection of a set $U \subset \mathbb{R}^2$ onto the x-axis is denoted by $\operatorname{dom} U$. We say that a set $A \subset \mathbb{R}$ is *bilaterally* $\mathfrak{c}$*-dense in itself* if $\operatorname{card}(A \cap I) = \mathfrak{c}$ for every nondegenerate interval I with $A \cap I \neq \emptyset$.

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Let $f\colon \mathbb{R} \to \mathbb{R}$. For each $y \in \mathbb{R}$, let $[f < y] = \{x \in \mathbb{R}\colon f(x) < y\}$. Similarly we define the symbols $[f > y]$, $[f = y]$, etc.

A word *function* stands for a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. The following are classes of functions:

$\mathcal{D}$ consists of all *Darboux* functions; i.e., $f \in \mathcal{D}$ iff it has the intermediate value property,

$\mathcal{FC}$ consists of all *functionally connected* functions; i.e., $f \in \mathcal{FC}$ iff $g \cap f \neq \emptyset$ whenever $g\colon \mathrm{I}[a,b] \to \mathbb{R}$ is a continuous function with $f(a) < g(a)$ and $f(b) > g(b)$ (we make no distinction between a function and its graph),

$\mathcal{C}onn$ consists of all *connected* functions; i.e., $f \in \mathcal{C}onn$ iff f is a connected subset of $\mathbb{R}^2$,

$\mathcal{AC}$ consists of all *almost continuous* functions in the sense of Stallings [9]; i.e., $f \in \mathcal{AC}$ iff for every open set $V \subset \mathbb{R}^2$ containing f, there is a continuous function $h \subset V$.

Remark 1.1. One can easily see that $\mathcal{AC} \subset \mathcal{C}onn \subset \mathcal{FC} \subset \mathcal{D}$ (see also [9]).

There are several papers concerning theorems on a common summand (see, e.g., [2] or [1]) or factor [7]. In this paper we deal with theorems on a common divisor. Similar problems were studied by the first author in [3] and [4]. More precisely, we examine the cardinal

$$\mathrm{q}(\mathcal{A}) \overset{\mathrm{df}}{=} \min\bigl(\{\operatorname{card}\mathcal{F} \,:\, \mathcal{F} \subset {}^{\mathcal{A}}/_{\mathcal{A}} \;\&\; \neg\bigl(\exists_g \forall_{f\in\mathcal{F}}\; f/g \in \mathcal{A}\bigr)\} \cup \{(\operatorname{card}{}^{\mathcal{A}}/_{\mathcal{A}})^+\}\bigr)$$

for $\mathcal{A} \in \{\mathcal{FC}, \mathcal{C}onn, \mathcal{AC}\}$, where

$${}^{\mathcal{A}}/_{\mathcal{A}} \overset{\mathrm{df}}{=} \bigl\{f/g \,:\, f, g \in \mathcal{A} \;\&\; g(x) \neq 0 \text{ for each } x \in \mathbb{R}\bigr\}.$$

Remark 1.2. In the above definition, it is quite natural to restrict ourselves to subfamilies of ${}^{\mathcal{A}}/_{\mathcal{A}}$ only. Indeed, if there is a function g such that both f/g and $1/g$ are in $\mathcal{A}$, then $f \in {}^{\mathcal{A}}/_{\mathcal{A}}$.

So, before we can examine the value of $\mathrm{q}(\mathcal{A})$, we should know what the family ${}^{\mathcal{A}}/_{\mathcal{A}}$ is. In this paper, using the results of the second author [5], we characterize the families ${}^{\mathcal{FC}}/_{\mathcal{FC}}$, ${}^{\mathcal{C}onn}/_{\mathcal{C}onn}$, and ${}^{\mathcal{AC}}/_{\mathcal{AC}}$.

Recall the following characterization of ${}^{\mathcal{D}}/_{\mathcal{D}}$ given by Natkaniec and Orwat [8, Theorem 7].

Proposition 1.1. *A function f belongs to ${}^{\mathcal{D}}/_{\mathcal{D}}$ iff the following conditions hold*:

D1. *if* $f(a)f(b) < 0$, *then* $[f = 0] \cap \mathrm{I}[a, b] \neq \emptyset$,

D2. *both* $[f > 0]$ *and* $[f < 0]$ *are bilaterally* $\mathfrak{c}$*-dense in themselves.*

In [4], the cardinal $\mathrm{q}(\mathcal{D})$ was examined. We prove analogous results for $\mathrm{q}(\mathcal{A})$, where $\mathcal{A} \in \{\mathcal{AC}, \mathcal{C}onn, \mathcal{FC}\}$.

2 Main Results.

For brevity, we introduce two denotations more. The symbol $\mathcal{K}$ denotes the family of all closed subsets of $\mathbb{R}^2$. If f is a function, then let

$$A_f \stackrel{\mathrm{df}}{=} \bigl\{\langle x, y\rangle \in \mathbb{R}^2 \,:\, yf(x) > 0\bigr\};$$

i.e., $A_f = \bigl([f > 0] \times (0, \infty)\bigr) \cup \bigl([f < 0] \times (-\infty, 0)\bigr)$.

We start with a technical lemma.

Lemma 2.1. *Let* $\mathcal{F}$ *be any family of functions with* $\operatorname{card} \mathcal{F} \leq \mathfrak{c}$. *There exists a function* $g \colon \mathbb{R} \to (0, \infty)$ *such that for all* $K \in \mathcal{K}$ *and* $f \in \mathcal{F}$,

$$\text{if} \quad \operatorname{card} \operatorname{dom}(K \cap A_f) = \mathfrak{c}, \quad \text{then} \quad K \cap \frac{f}{g} \neq \emptyset. \tag{1}$$

Proof. If $\operatorname{card}[f \neq 0] < \mathfrak{c}$ for each function $f \in \mathcal{F}$, then we define $g \stackrel{\mathrm{df}}{=} \chi_{\mathbb{R}}$.

So, assume the opposite case. Put

$$\mathcal{B} \stackrel{\mathrm{df}}{=} \{\langle K, f\rangle \,:\, K \in \mathcal{K} \;\&\; f \in \mathcal{F} \;\&\; \operatorname{card} \operatorname{dom}(K \cap A_f) = \mathfrak{c}\}$$

and notice that $\operatorname{card} \mathcal{B} = \mathfrak{c}$.

Indeed, let $f \in \mathcal{F}$ be such that $\operatorname{card}[f \neq 0] = \mathfrak{c}$. Then

$$\operatorname{card}[f > 0] = \mathfrak{c} \qquad \text{or} \qquad \operatorname{card}[f < 0] = \mathfrak{c},$$

whence

$$\bigl\{\langle \mathbb{R} \times \{y\}, f\rangle \,:\, y > 0\bigr\} \subset \mathcal{B} \qquad \text{or} \qquad \bigl\{\langle \mathbb{R} \times \{y\}, f\rangle \,:\, y < 0\bigr\} \subset \mathcal{B},$$

and consequently $\operatorname{card} \mathcal{B} \geq \mathfrak{c}$. On the other hand, $\operatorname{card} \mathcal{B} \leq \operatorname{card} \mathcal{K} = \mathfrak{c}$.

Let $\bigl\{\langle K_\alpha, f_\alpha\rangle \,:\, \alpha < \mathfrak{c}\bigr\}$ be an enumeration of $\mathcal{B}$. For each $\alpha < \mathfrak{c}$, choose any

$$x_\alpha \in \operatorname{dom}(K_\alpha \cap A_{f_\alpha}) \setminus \{x_\beta \,:\, \beta < \alpha\},$$

and let y_α be such that $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$ and $y_\alpha f_\alpha(x_\alpha) > 0$. Define the function $g\colon \mathbb{R} \to (0,\infty)$ by

$$g(x) \overset{\mathrm{df}}{=} \begin{cases} f_\alpha(x)/y_\alpha & \text{if } x = x_\alpha,\ \alpha < \mathfrak{c}, \\ 1 & \text{otherwise.} \end{cases}$$

We shall prove that g possesses the required properties.

One can easily see that $g > 0$ on $\mathbb{R}$. Let $K \in \mathcal{K}$ and $f \in \mathcal{F}$ be such that $\operatorname{card}\operatorname{dom}(K \cap A_f) = \mathfrak{c}$. Then $\langle K, f\rangle = \langle K_\alpha, f_\alpha\rangle$ for some $\alpha < \mathfrak{c}$. Clearly $(f_\alpha/g)(x_\alpha) = y_\alpha$, whence

$$\langle x_\alpha, y_\alpha\rangle \in K_\alpha \cap \frac{f_\alpha}{g} = K \cap \frac{f}{g}. \qquad \square$$

The next proposition was proved by the second author in his paper devoted to separating of sets by Darboux-like functions [5, Proposition 4.1].

Proposition 2.2. *If a function $f \in {}^{\mathcal{D}}\!/_{\mathcal{D}}$ fulfills the condition*:

$$K \cap f \neq \emptyset \quad \textit{whenever} \quad K \in \mathcal{K} \textit{ and } \operatorname{card}\operatorname{dom}(K \cap A_f) = \mathfrak{c}, \tag{2}$$

then $f \in \mathcal{D}$. If, moreover,

for each closed set $F \subset [f \neq 0]$, if $\alpha \in [f<0] \cap F$ and $\beta \in [f>0] \cap F$, then there exist $a \in [f<0] \cap \mathrm{I}[\alpha,\beta]$ and $b \in [f>0] \cap \mathrm{I}[\alpha,\beta]$ such that $F \cap \mathrm{I}[a,b] = \emptyset$, (3)

then $f \in \mathcal{C}onn$. If, moreover,

for each closed set $F \subset [f \neq 0]$, if $\alpha \in [f<0] \cap F$ and $\beta \in [f>0] \cap F$, then there exist $a \in [f<0] \cap \mathrm{I}[\alpha,\beta]$ and $b \in [f>0] \cap \mathrm{I}[\alpha,\beta]$ such that $(b-a)(\beta-\alpha) > 0$ and $F \cap \mathrm{I}[a,b] = \emptyset$. (4)

then $f \in \mathcal{AC}$.

We will use the above proposition to prove another lemma. Its first part (concerning Darboux functions) is a repetition of [4, Theorem 2.2].

Lemma 2.3. *Let $\mathcal{F} \subset {}^{\mathcal{D}}\!/_{\mathcal{D}}$ be such that $\operatorname{card}\mathcal{F} \le \mathfrak{c}$. There exists a function $g\colon \mathbb{R} \to (0,\infty)$ such that $f/g \in \mathcal{D}$ for each function $f \in \mathcal{F}$.*

If, moreover, every function $f \in \mathcal{F}$ fulfills (3), *then we can conclude that $f/g \in \mathcal{C}onn$ for each function $f \in \mathcal{F}$.*

If, moreover, every function $f \in \mathcal{F}$ fulfills (4), *then we can conclude that $f/g \in \mathcal{AC}$ for each function $f \in \mathcal{F}$.*

PROOF. Construct g according to Lemma 2.1. Fix an $f \in \mathcal{F}$. Then

$$[f/g > 0] = [f > 0], \qquad [f/g < 0] = [f < 0], \qquad [f/g = 0] = [f = 0]. \tag{5}$$

Since $f \in {}^{\mathcal{D}}/_{\mathcal{D}}$, we conclude by Proposition 1.1 that $f/g \in {}^{\mathcal{D}}/_{\mathcal{D}}$. So by Proposition 2.2, $f/g \in \mathcal{D}$, and if, moreover, f fulfills (3) (resp. (4)), then f/g fulfills this condition, too, and $f/g \in \mathcal{C}onn$ (resp. $f/g \in \mathcal{AC}$). □

Corollary 2.4. *The following conditions are equivalent*:

a) $f \in {}^{\mathcal{C}onn}/_{\mathcal{C}onn}$;

b) $f \in {}^{\mathcal{FC}}/_{\mathcal{FC}}$;

c) $f \in {}^{\mathcal{D}}/_{\mathcal{D}}$ *and* f *fulfills condition* (3).

PROOF. a) ⇒ b). This implication follows from the inclusion $\mathcal{C}onn \subset \mathcal{FC}$.

b) ⇒ c). Let $f = g/h$, where $g, h \in \mathcal{FC}$. Since $h \in \mathcal{D}$, we may assume that $h > 0$ on $\mathbb{R}$. Then

$$[f > 0] = [g > 0], \qquad [f < 0] = [g < 0], \qquad [f = 0] = [g = 0]. \tag{6}$$

So by [5, Theorem 4.3], f fulfills (3). The relation '$f \in {}^{\mathcal{D}}/_{\mathcal{D}}$' follows from the inclusion $\mathcal{FC} \subset \mathcal{D}$.

c) ⇒ a). By Lemma 2.3, there exists a function $g\colon \mathbb{R} \to (0, \infty)$ such that $f/g \in \mathcal{C}onn$ and $1/g \in \mathcal{C}onn$. So, $f = (f/g)\big/(1/g) \in {}^{\mathcal{C}onn}/_{\mathcal{C}onn}$. □

Using Lemma 2.3, Corollary 2.4 and the inclusion $\mathcal{C}onn \subset \mathcal{FC}$, we obtain

Corollary 2.5. $\mathrm{q}(\mathcal{FC}) \geq \mathrm{q}(\mathcal{C}onn) > \mathfrak{c}$.

The proofs of the next two corollaries mimic the arguments used above.

Corollary 2.6. *The following conditions are equivalent*:

a) $f \in {}^{\mathcal{AC}}/_{\mathcal{AC}}$;

b) $f \in {}^{\mathcal{D}}/_{\mathcal{D}}$ *and* f *fulfills condition* (4).

Corollary 2.7. $\mathrm{q}(\mathcal{AC}) > \mathfrak{c}$.

Combining [4, Theorem 2.2] and Corollaries 2.5 and 2.7, we obtain the next corollary.

Corollary 2.8. *If* $\mathfrak{c}^+ = 2^{\mathfrak{c}}$ *and* $\mathcal{A} \in \{\mathcal{AC}, \mathcal{C}onn, \mathcal{FC}, \mathcal{D}\}$, *then* $\mathrm{q}(\mathcal{A}) = 2^{\mathfrak{c}}$.

In Theorem 2.9 we need another cardinal:

$$\mathrm{a}(\mathcal{D}) \stackrel{\mathrm{df}}{=} \min\bigl\{\operatorname{card}\mathcal{F} \,:\, \mathcal{F} \subset \mathbb{R}^{\mathbb{R}} \;\&\; \neg\bigl(\exists_g \forall_{f\in\mathcal{F}}\; f+g \in \mathcal{D}\bigr)\bigr\}.$$

This cardinal was defined by Natkaniec [6] and was thoroughly examined by Ciesielski and Miller [1].

Natkaniec observed that $\mathfrak{c} < \mathrm{a}(\mathcal{D}) \leq 2^{\mathfrak{c}}$. (See also [2].) Ciesielski and Miller [1] generalized this result by showing that $\mathrm{cf}(\mathrm{a}(\mathcal{D})) > \mathfrak{c}$. They also proved that it is pretty much all that can be said about $\mathrm{a}(D)$ in ZFC, by showing that $\mathrm{a}(\mathcal{D})$ can be equal to any regular cardinal between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$, and that it can be equal to $2^{\mathfrak{c}}$ independently of the cofinality of $2^{\mathfrak{c}}$.

We will compare the values of q for the families of Darboux-like functions with this cardinal. We start with the following

Theorem 2.9. *Let* $\mathcal{A} \in \{\mathcal{AC}, \mathcal{C}onn, \mathcal{FC}, \mathcal{D}\}$*. Then* $\mathrm{q}(\mathcal{A}) \leq \mathrm{a}(\mathcal{D})$*.*

PROOF. Pick a family $\mathcal{F} \subset (0,\infty)^{\mathbb{R}}$ of cardinality $\mathrm{a}(\mathcal{D})$ such that for each $g\colon \mathbb{R} \to \mathbb{R}\setminus\{0\}$, there exists an $f \in \mathcal{F}$ with $f/g \notin \mathcal{D}$ (see [4, Theorem 2.3]). Notice that $\mathcal{F} \subset (0,\infty)^{\mathbb{R}} \subset {}^{\mathcal{A}}\!/\!{}_{\mathcal{A}}$. (Cf. Proposition 1.1 and Corollaries 2.4 and 2.6.)

Let $g\colon \mathbb{R} \to \mathbb{R}\setminus\{0\}$. Since $\mathcal{A} \subset \mathcal{D}$, there is an $f \in \mathcal{F}$ such that $f/g \notin \mathcal{A}$. So, $\mathcal{F}$ witnesses that $\mathrm{q}(\mathcal{A}) \leq \operatorname{card}\mathcal{F} = \mathrm{a}(\mathcal{D})$. $\square$

To prove some results in the opposite direction, we need several new notions.

For a partially ordered set $(\mathbb{P}, \leq)$, we say that $G \subset \mathbb{P}$ is a $\mathbb{P}$-*filter*, if

- for all $p, q \in G$, there exists an $r \in G$ with $r \leq p$ and $r \leq q$,
- for all $p, q \in \mathbb{P}$, if $p \in G$ and $p \leq q$, then $q \in G$.

We define $D \subset \mathbb{P}$ to be *dense*, if for every $p \in \mathbb{P}$, there exists a $q \in D$ with $q \leq p$.

For a cardinal κ and a poset $\mathbb{P}$, we define the following statements.

$\mathrm{MA}_\kappa(\mathbb{P})$**:** for every family $\mathfrak{D}$ of dense subsets of $\mathbb{P}$ with $\operatorname{card}\mathfrak{D} < \kappa$, there exists a $\mathbb{P}$-filter G such that $D \cap G \neq \emptyset$ for every $D \in \mathfrak{D}$.

$\mathrm{Lus}_\kappa(\mathbb{P})$**:** there exists a sequence $\langle G_\alpha : \alpha < \kappa\rangle$ of $\mathbb{P}$-filters (called a κ-*Lusin sequence*), such that for every dense set $D \subset \mathbb{P}$,

$$\operatorname{card}\bigl\{\alpha < \kappa \,:\, G_\alpha \cap D = \emptyset\bigr\} < \kappa.$$

Recall that these statements are independent of ZFC. (See, e.g., [1].)

From now on, we let

$$\mathbb{P} \stackrel{\text{df}}{=} \left\{p \in (0,\infty)^X \,:\, X \subset \mathbb{R} \;\&\; \operatorname{card} X < \mathfrak{c}\right\},$$

and we define

$$p \le q \quad \text{iff} \quad q \subset p\,;$$

i.e., if p extends q as a partial function.

In the proofs of our theorems, we shall use methods similar to those due to Ciesielski and Miller [1].

Theorem 2.10. *Let $\kappa > \mathfrak{c}$. Assume $\mathrm{MA}_\kappa(\mathbb{P})$ and let $\mathcal{F}$ be a family of functions with $\operatorname{card}\mathcal{F} < \kappa$. There exists a function $g\colon \mathbb{R} \to (0,\infty)$ such that* (1) *holds for all $K \in \mathcal{K}$ and $f \in \mathcal{F}$.*

PROOF. First we will show that if $\operatorname{card}\operatorname{dom}(K \cap A_f) = \mathfrak{c}$ for some $K \in \mathcal{K}$ and $f \in \mathcal{F}$, then

$$D_{K,f} \stackrel{\text{df}}{=} \left\{q \in \mathbb{P} \,:\, \exists_{x \in \operatorname{dom} q}\, \langle x, (f/q)(x)\rangle \in K\right\}$$

is dense in $\mathbb{P}$.

Indeed, let $K \in \mathcal{K}$ and $f \in \mathcal{F}$ be such that $\operatorname{card}\operatorname{dom}(K \cap A_f) = \mathfrak{c}$. Fix a $p \in \mathbb{P}$. We can choose an $x \in \operatorname{dom}(K \cap A_f) \setminus \operatorname{dom} p$ and a $y \in \mathbb{R}$ such that $\langle x, y\rangle \in K \cap A_f$. Then $q \stackrel{\text{df}}{=} p \cup \{\langle x, f(x)/y\rangle\} \in D_{K,f}$ and $q \le p$.

Next observe that for each $x \in \mathbb{R}$, the set

$$D_x \stackrel{\text{df}}{=} \left\{q \in \mathbb{P} \,:\, x \in \operatorname{dom} q\right\}$$

is dense in $\mathbb{P}$.

Indeed, let $x \in \mathbb{R}$ and $p \in \mathbb{P}$. If $x \in \operatorname{dom} p$, then we put $q \stackrel{\text{df}}{=} p$, otherwise we let $q \stackrel{\text{df}}{=} p \cup \{\langle x, 1\rangle\}$. Then evidently $q \in D_x$ and $q \le p$.

Now define

$$\mathfrak{D} \stackrel{\text{df}}{=} \left\{D_x \,:\, x \in \mathbb{R}\right\} \cup \left\{D_{K,f} \,:\, K \in \mathcal{K} \;\&\; f \in \mathcal{F} \;\&\; \operatorname{card}\operatorname{dom}(K \cap A_f) = \mathfrak{c}\right\}.$$

Then $\mathfrak{D}$ is a family of dense subsets of $\mathbb{P}$ with $\operatorname{card}\mathfrak{D} < \kappa$. Applying $\mathrm{MA}_\kappa(\mathbb{P})$, we can find a $\mathbb{P}$-filter G such that $G \cap D \ne \emptyset$ for each $D \in \mathfrak{D}$.

Put $g \stackrel{\text{df}}{=} \bigcup G$. Evidently g is a function and g is positive. For every $x \in \mathbb{R}$, we have $G \cap D_x \ne \emptyset$, whence $\operatorname{dom} g = \mathbb{R}$.

Take $K \in \mathcal{K}$ and $f \in \mathcal{F}$ with $\operatorname{card}\operatorname{dom}(K \cap A_f) = \mathfrak{c}$. Since $G \cap D_{K,f} \ne \emptyset$, there is a function $q \subset g$ with $q \in D_{K,f}$. Hence

$$K \cap (f/g) \supset K \cap (f/q) \ne \emptyset. \qquad \square$$

Theorem 2.11. *Assume* $\mathrm{MA}_\kappa(\mathbb{P})$. *For each* $\mathcal{A} \in \{\mathcal{AC}, \mathcal{C}onn, \mathcal{FC}, \mathcal{D}\}$, *we have* $\mathrm{q}(\mathcal{A}) \geq \kappa$.

PROOF. Take any $\mathcal{F} \subset {}^{\mathcal{A}}\!/_{\mathcal{A}}$ with $\operatorname{card} \mathcal{F} < \kappa$. We will show that $\operatorname{card} \mathcal{F} < \mathrm{q}(\mathcal{A})$. For, we will prove that there exists a function g such that $f/g \in \mathcal{A}$ for each $f \in \mathcal{F}$.

By [4, Theorem 2.2] and Corollaries 2.5 and 2.7, we may assume that $\kappa > \mathfrak{c}$. Let g be a function constructed according to Theorem 2.10. Fix an $f \in \mathcal{F}$.

Notice that $f \in {}^{\mathcal{D}}\!/_{\mathcal{D}}$ and equalities (5) hold. Thus $f/g \in {}^{\mathcal{D}}\!/_{\mathcal{D}}$ (cf. Proposition 1.1). Using (1) and equalities (5), we conclude that f/g satisfies (2). Hence by Proposition 2.2, $f/g \in \mathcal{D}$.

If $f \in {}^{\mathcal{C}onn}\!/_{\mathcal{C}onn} = {}^{\mathcal{FC}}\!/_{\mathcal{FC}}$, then by Corollary 2.4, f satisfies (3). Thus f/g fulfills (3), too. Hence by Proposition 2.2, $f/g \in \mathcal{C}onn \subset \mathcal{FC}$.

Finally if $f \in {}^{\mathcal{AC}}\!/_{\mathcal{AC}}$, then by Corollary 2.6, f satisfies (4). Thus f/g fulfills (4), too. Hence by Proposition 2.2, $f/g \in \mathcal{AC}$. $\square$

In the proof of the next theorem we will use two other posets. Put

$$\mathbb{P}' \stackrel{\mathrm{df}}{=} \{p \in \mathbb{R}^X \,:\, X \subset \mathbb{R} \,\&\, \operatorname{card} X < \mathfrak{c}\},$$

and $p \leq q$ iff $q \subset p$. Moreover, let

$$\mathbb{P}^* \stackrel{\mathrm{df}}{=} \{\langle p, \mathcal{E}\rangle \,:\, p \in \mathbb{P}',\ \mathcal{E} \subset \mathbb{R}^{\mathbb{R}} \,\&\, \operatorname{card} \mathcal{E} < \mathfrak{c}\},$$

and $(p, \mathcal{E}) \leq (q, \mathcal{F})$ iff

$$q \subset p \quad \& \quad \mathcal{E} \supset \mathcal{F} \quad \& \quad p(x) \neq f(x) \quad \text{for all } x \in \operatorname{dom} p \setminus \operatorname{dom} q \text{ and } f \in \mathcal{F}.$$

Theorem 2.12. *Assume* $\mathrm{Lus}_\kappa(\mathbb{P}^*)$. *If* $\kappa > \mathfrak{c}$ *is regular, then* $\mathrm{q}(\mathcal{A}) = \mathrm{a}(\mathcal{D}) = \kappa$ *for each* $\mathcal{A} \in \{\mathcal{AC}, \mathcal{C}onn, \mathcal{FC}, \mathcal{D}\}$.

PROOF. The inequality $\kappa \geq \mathrm{a}(\mathcal{D})$ follows by [1, Lemma 3.2 and Theorem 2.1]. The inequality $\mathrm{a}(\mathcal{D}) \geq \mathrm{q}(\mathcal{A})$ follows by Theorem 2.9.

By [1, Lemma 3.3], $\mathrm{Lus}_\kappa(\mathbb{P}^*)$ implies $\mathrm{MA}_\kappa(\mathbb{P}')$. Since the posets $\mathbb{P}$ and $\mathbb{P}'$ are order isomorphic, $\mathrm{MA}_\kappa(\mathbb{P})$ holds. Thus by Theorem 2.11, $\mathrm{q}(\mathcal{A}) \geq \kappa$. $\square$

Ciesielski and Miller proved that the assumptions of Theorem 2.12 are independent of ZFC [1]. We want to present the following

Problem. Can any of the equalities

$$\mathrm{q}(\mathcal{AC}) = \mathrm{q}(\mathcal{C}onn) = \mathrm{q}(\mathcal{FC}) = \mathrm{q}(\mathcal{D}) = \mathrm{a}(\mathcal{D})$$

be proved in ZFC?

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