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WEAK OPENNESS OF MULTIPLICATION IN THE SPACE $C(0,1)$

Abstract

Let $C(0,1)$ be the space of all continuous real-valued functions defined on an open interval $(0,1)$. We shall show that the multiplication is a weakly open operation in $C(0,1)$.

Let $C(0,1)$ be the space of all continuous real-valued functions in $(0,1)$ with the metric $\operatorname{dist}(f,g) = \min\{\sup\{|f(t) - g(t)|: t \in (0,1)\}, 1\}$. There are some natural operations on $C(0,1)$, for example, addition, multiplication, minimum and maximum. In [1, 4, 5] such operations were investigated in the space $C([0,1])$ of all continuous real-valued functions defined on $[0,1]$. All the operations are continuous but only addition, minimum and maximum are open as mappings from $C([0,1]) \times C([0,1])$ to $C([0,1])$. It is interested that multiplication is not continuous in $C(0,1)$. Namely, convergence in $C(0,1)$ is equivalent to the uniform convergence. Consider $f_n(x) = \frac{1}{n}$ and $g_n(x) = \frac{1}{x}$ for any $n \in \mathbb{N}$ and $x \in (0,1)$. Then the sequence $(f_n \cdot g_n)$ is not uniformly convergent to $\lim_{n\to\infty} f_n \cdot \lim_{n\to\infty} g_n = 0$.

Definition 1. *[1, 2] A map between topological spaces is* weakly open *if the image of every non-empty open set has a non-empty interior.*

In [1] it is shown that the multiplication in $C([0,1])$ is a weakly open operation. In [3] there are considered some properties of multiplication in the algebra $C(X)$ of real-valued continuous functions defined on a compact topological space X.

During 22^{th} *SUMMER CONFERENCE ON REAL FUNCTION THEORY*, Stará Lesná, Slovakia 31.08-05.09 2009 Artur Wachowicz showed that multiplication in $C(0,1)$ is not an open operation and asked the question:

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Does multiplication is a weakly open mapping in the space $C(0,1)$**?**

In the present paper we give a positive answer to this question.

Let $B(f,r)$ ($\overline{B}(f,r)$) denote an open (respectively, closed) ball centered at f and of radius $r>0$ in $C(0,1)$ and let $||\cdot||$ stand for the standard euclidean norm in $\mathbb{R}^2$.

Theorem 1. *Let* $f,g\in C(0,1)$ *and* $\varepsilon>0$ *be such that* $||(f(t),g(t))||\geq\varepsilon$ *for every* $t\in(0,1)$*. Then for every* $h\in C(0,1)$ *satisfying condition* $\operatorname{dist}(h,fg)\leq\frac{\varepsilon^2}{2}$*, there exist* $f_1,g_1\in C(0,1)$ *such that* $\operatorname{dist}(f,f_1)\leq\varepsilon$*,* $\operatorname{dist}(g,g_1)\leq\varepsilon$ *and* $f_1g_1=h$*.*

PROOF. Let $D=\{(x,y)\in\mathbb{R}^2\colon ||(x,y)||\geq\varepsilon\}$. We define a function $\alpha\colon D\to\mathbb{R}^2$

$$\alpha(x,y)=\left(x+\varepsilon\frac{y}{\sqrt{x^2+y^2}},y+\varepsilon\frac{x}{\sqrt{x^2+y^2}}\right).$$

Next, for every $(x,y)\in D$ we define a function $\varphi_{(x,y)}\colon[0,1]\to\mathbb{R}$

$$\varphi_{(x,y)}(t)=\left(x+t\varepsilon\frac{y}{\sqrt{x^2+y^2}}\right)\left(y+t\varepsilon\frac{x}{\sqrt{x^2+y^2}}\right).$$

The function $\varphi_{(x,y)}$ is just the restriction of the multiplication to the line segment starting at (x,y) and ending at $\alpha(x,y)$. We will show that the following properties are fulfilled:

a1) α is continuous,

a2) $||\alpha(x,y)-(x,y)||=\varepsilon$ for every $(x,y)\in D$,

a3) for every $(x,y)\in D$ the function $\varphi_{(x,y)}$ is strictly increasing,

a4) $\left(x+\varepsilon\frac{y}{\sqrt{x^2+y^2}}\right)\left(y+\varepsilon\frac{x}{\sqrt{x^2+y^2}}\right)-xy\geq\frac{\varepsilon^2}{2}$ (equivalently, $\varphi_{(x,y)}(1)-\varphi_{(x,y)}(0)\geq\frac{\varepsilon^2}{2}$) for every $(x,y)\in D$.

Property $a1$) follows directly from the definition of α.

a2) We have

$$||\alpha(x,y)-(x,y)||=\left|\left|\left(\varepsilon\frac{y}{\sqrt{x^2+y^2}},\varepsilon\frac{x}{\sqrt{x^2+y^2}}\right)\right|\right|=\varepsilon\sqrt{\frac{y^2}{x^2+y^2}+\frac{x^2}{x^2+y^2}}=\varepsilon$$

for every $(x,y)\in D$.

a3) We easily compute $\varphi_{(x,y)}(t) = xy + t\varepsilon\frac{x^2+y^2}{\sqrt{x^2+y^2}} + t^2\varepsilon^2\frac{xy}{x^2+y^2}$. Hence

$$\varphi'_{(x,y)}(t) = \varepsilon\frac{x^2+y^2}{\sqrt{x^2+y^2}} + 2t\varepsilon^2\frac{xy}{x^2+y^2} = \varepsilon\left(\sqrt{x^2+y^2} + t\varepsilon\frac{2xy}{x^2+y^2}\right).$$

And since $\frac{2|xy|}{x^2+y^2} \leq 1$ and $\varepsilon \leq \sqrt{x^2+y^2}$ for every $(x,y) \in D$ we get

$$\varphi'_{(x,y)}(t) \geq \varepsilon\left(\sqrt{x^2+y^2} - t\varepsilon\frac{2|xy|}{x^2+y^2}\right) \geq$$
$$\geq \varepsilon\left(\sqrt{x^2+y^2} - t\sqrt{x^2+y^2}\right) \geq \varepsilon\sqrt{x^2+y^2}(1-t).$$

Hence $\varphi'_{(x,y)}(t) \geq 0$ for $t \in [0,1]$ and $\varphi'_{(x,y)}(t) > 0$ for $t \in [0,1)$. Therefore $\varphi_{(x,y)}$ is a strictly increasing function for every $(x,y) \in D$.

a4) For every $(x,y) \in D$, we have

$$\left(x + \varepsilon\frac{y}{\sqrt{x^2+y^2}}\right)\left(y + \varepsilon\frac{x}{\sqrt{x^2+y^2}}\right) - xy = xy + \varepsilon\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \varepsilon^2\frac{xy}{x^2+y^2} - xy =$$
$$= \varepsilon\left(\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \varepsilon\frac{xy}{x^2+y^2}\right) \geq \varepsilon\left(\sqrt{x^2+y^2} - \varepsilon\frac{|xy|}{x^2+y^2}\right) \geq$$
$$\geq \varepsilon\left(\sqrt{x^2+y^2} - \frac{\varepsilon}{2}\right) \geq \varepsilon\left(\sqrt{x^2+y^2} - \frac{\sqrt{x^2+y^2}}{2}\right) = \frac{\varepsilon}{2}\sqrt{x^2+y^2} \geq \frac{\varepsilon^2}{2}.$$

Thus properties $a1) - a4)$ are proven.

Similarly, define a function $\beta\colon D \to \mathbb{R}^2$:

$$\beta(x,y) = \left(x - \varepsilon\frac{y}{\sqrt{x^2+y^2}}, y - \varepsilon\frac{x}{\sqrt{x^2+y^2}}\right),$$

and for every $(x,y) \in D$, define a function $\psi_{(x,y)}\colon [0,1] \to \mathbb{R}$,

$$\psi_{(x,y)}(t) = \left(x - t\varepsilon\frac{y}{\sqrt{x^2+y^2}}\right)\left(y - t\varepsilon\frac{x}{\sqrt{x^2+y^2}}\right).$$

The function $\psi_{(x,y)}$ is just the restriction of the multiplication to the line segment starting at (x,y) and ending at $\beta(x,y)$. We shall show that the following properties are fulfilled:

b1) β is continuous,

b2) $||\beta(x,y) - (x,y)|| = \varepsilon$ for every $(x,y) \in D$, b3) for every $(x,y) \in D$ the function $\psi_{(x,y)}$ is strictly decreasing,

b4) $xy - \left(x - \varepsilon\frac{y}{\sqrt{x^2+y^2}}\right)\left(y - \varepsilon\frac{x}{\sqrt{x^2+y^2}}\right) \geq \frac{\varepsilon^2}{2}$ (equivalently, $\psi_{(x,y)}(0) - \psi_{(x,y)}(1) \geq \frac{\varepsilon^2}{2}$) for every $(x,y) \in D$.

The proofs of $b1) - b4)$ are analogous to those for $a1) - a4)$ and we omit them.

Now, let us take any $h \in C(0,1)$ such that $\operatorname{dist}(h, fg) \leq \frac{\varepsilon^2}{2}$. For every $t \in (0,1)$ we have

$$f(t)g(t) - \tfrac{\varepsilon^2}{2} \leq h(t) \leq f(t)g(t) + \tfrac{\varepsilon^2}{2},$$

hence by $a4)$ and $b4)$

$$\left(f(t) - \varepsilon\frac{g(t)}{\sqrt{(f(t))^2+(g(t))^2}}\right)\left(g(t) - \varepsilon\frac{f(t)}{\sqrt{(f(t))^2+(g(t))^2}}\right) \leq h(t) \leq \left(f(t) + \varepsilon\frac{g(t)}{\sqrt{(f(t))^2+(g(t))^2}}\right)\left(g(t) + \varepsilon\frac{f(t)}{\sqrt{(f(t))^2+(g(t))^2}}\right)$$

(or equivalently $\psi_{(f(t),g(t))}(1) \leq h(t) \leq \varphi_{(f(t),g(t))}(1)$). Therefore (by $a1)$, $a3)$, $b1)$, $b3)$ and the Darboux property) for every $t \in (0,1)$ there exists exactly one point $v^t = (v^t_x, v^t_y)$ lying on the broken line with vertices $\beta\big(f(t), g(t)\big)$, $\big(f(t), g(t)\big)$ and $\alpha\big(f(t), g(t)\big)$ such that $v^t_x v^t_y = h(t)$. Now, we may define functions $f_1, g_1 \colon (0,1) \to \mathbb{R}$ as $f_1(t) = v^t_x$ and $g_1(t) = v^t_y$ for every $t \in (0,1)$. It follows directly from the definitions of f_1 and g_1 that $f_1 g_1 = h$ and by $a2)$ and $b2)$

$$\begin{aligned}&\left|\left|\big(f_1(t), g_1(t)\big) - \big(f(t), g(t)\big)\right|\right| \leq \\ &\leq \max\left\{\left|\left|\alpha\big(f(t), g(t)\big) - \big(f(t), g(t)\big)\right|\right|, \left|\left|\beta\big(f(t), g(t)\big) - \big(f(t), g(t)\big)\right|\right|\right\} = \varepsilon\end{aligned}$$

for every $t \in (0,1)$. Hence $\operatorname{dist}(f, f_1) \leq \varepsilon$ and $\operatorname{dist}(g, g_1) \leq \varepsilon$. It remains to show that f_1 and g_1 are continuous.

Let $t_0 \in (0,1)$ and let $(t_n)_{n\in\mathbb{N}}$ be any sequence convergent to t_0. By the continuity of f, g, α and β, we have $\lim_{n\to\infty}\big(f(t_n), g(t_n)\big) = \big(f(t_0), g(t_0)\big)$, $\lim_{n\to\infty}\alpha\big(f(t_n), g(t_n)\big) = \alpha\big(f(t_0), g(t_0)\big)$ and $\lim_{n\to\infty}\beta\big(f(t_n), g(t_n)\big) = \beta\big(f(t_0), g(t_0)\big)$. Every point v^{t_n} lies on the broken line with vertices $\beta\big(f(t_n), g(t_n)\big)$, $\big(f(t_n), g(t_n)\big)$ and $\alpha\big(f(t_n), g(t_n)\big)$. Hence $\left(v^{t_n}\right)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^2$. Thus it has a convergent subsequence $\left(v^{t_{n_k}}\right)_{k\in\mathbb{N}}$. Let $v_0 = (v^0_x, v^0_y) = \lim_{k\to\infty} v^{t_{n_k}}$. Again, using the facts that every point $v^{t_{n_k}}$ lies on the broken line with vertices $\beta\big(f(t_{n_k}), g(t_{n_k})\big)$, $\big(f(t_{n_k}), g(t_{n_k})\big)$ and $\alpha\big(f(t_{n_k}), g(t_{n_k})\big)$

and that vertices of those broken lines converge to $\beta\big(f(t_0),g(t_0)\big)$, $\big(f(t_0),g(t_0)\big)$ and $\alpha\big(f(t_0),g(t_0)\big)$ respectively, we get that v^0 lies on the broken line with vertices $\beta\big(f(t_0),g(t_0)\big)$, $\big(f(t_0),g(t_0)\big)$ and $\alpha\big(f(t_0),g(t_0)\big)$. Next, by the continuity of h and by the continuity of multiplication we get

$$v_x^0 v_y^0 = \lim_{k\to\infty} v_x^{t_{n_k}} v^{t_{n_k}} = \lim_{k\to\infty} h(t_{n_k}) = h(t_0).$$

But on the broken line with vertices $\beta\big(f(t_0),g(t_0)\big)$, $\big(f(t_0),g(t_0)\big)$ and $\alpha\big(f(t_0),g(t_0)\big)$ there is only one point v^{t_0} such that $v_x^{t_0}v_y^{t_0} = h(t_0)$. Thus $v_0 = v^{t_0}$. It follows that $\lim_{k\to\infty} f_1(t_{n_k}) = f_1(t_0)$ and $\lim_{k\to\infty} g_1(t_{n_k}) = g_1(t_0)$. It proves that f_1 and g_1 are continuous functions. □

Corollary 1. *Let $f,g \in C(0,1)$ and $\varepsilon > 0$ be such that $||(f(t),g(t))|| \geq \varepsilon$ for every $t \in (0,1)$. Then $\overline{B}(fg,\frac{\varepsilon^2}{2}) \subset \overline{B}(f,\varepsilon)\overline{B}(g,\varepsilon)$, where*

$$\overline{B}(f,\varepsilon)\overline{B}(g,\varepsilon) = \{\tilde f\tilde g \colon \tilde f \in \overline{B}(f,\varepsilon), \tilde g \in \overline{B}(g,\varepsilon)\}.$$

Lemma 1. *For any continuous functions $f,g\colon (0,1)\to\mathbb{R}$ and for every $\varepsilon>0$ there exist continuous functions $\widetilde f,\widetilde g\colon (0,1)\to\mathbb{R}$ such that $\operatorname{dist}(f,\widetilde f)\leq 2\varepsilon$, $\operatorname{dist}(g,\widetilde g)\leq 2\varepsilon$ and $\big|\big|\big(\widetilde f(t),\widetilde g(t)\big)\big|\big| \geq \varepsilon$ for every $t\in(0,1)$.*

Proof. Let $A = \{t\in(0,1)\colon |f(t)|<\varepsilon\}$ and $B = \{t\in(0,1)\colon |g(t)|<\varepsilon\}$. Since f and g are continuous functions, the sets A and B are open. Hence $A = \bigcup_{k\in K}(a_k,b_k)$, where intervals (a_k,b_k) are pairwise disjoint and K is a countable set. Moreover $|f(a_k)| = |f(b_k)| = \varepsilon$ for every $k\in K$. Next, let

$$K_1 = \{k\in K\colon (a_k,b_k)\cap B = \emptyset\},$$
$$K_2 = \{k\in K\colon (a_k,b_k)\cap B \neq \emptyset \wedge f(a_k) = f(b_k)\},$$

and

$$K_3 = \{k\in K\colon (a_k,b_k)\cap B \neq \emptyset \wedge f(a_k) \neq f(b_k)\}.$$

Obviously, $K = K_1\cup K_2\cup K_3$ and K_1, K_2, K_3 are pairwise disjoint. Since f is continuous, $f\big((a_k,b_k)\big)\supset(-\varepsilon,\varepsilon)$ for every $k\in K_3$. Therefore, again by the continuity of f, the family $\{(a_k,b_k)\}_{k\in K_3}$ is locally finite in $(0,1)$. For every $k\in K_3$ we may choose open intervals (α_k,β_k) and (γ_k,δ_k) such that

$$[\gamma_k,\delta_k]\subset(\alpha_k,\beta_k)\subset[\alpha_k,\beta_k]\subset(a_k,b_k)$$

and $(\alpha_k,\beta_k)\subset B$. Now, we may define functions $\widetilde f,\widetilde g\colon(0,1)\to\mathbb{R}$:

$$\widetilde f(t) = \begin{cases} f(x) & for \quad t\in\big((0,1)\setminus A\big)\cup\bigcup_{k\in K_1}(a_k,b_k),\\ f(a_k) & for \quad t\in(a_k,b_k) \text{ and } k\in K_2,\\ f(a_k) & for \quad t\in(a_k,\gamma_k] \text{ and } k\in K_3,\\ f(b_k) & for \quad t\in[\delta_k,b_k) \text{ and } k\in K_3,\\ \text{linear on intervals } [\gamma_k,\delta_k] \text{ for every } k\in K_3, \end{cases}$$

and

$$\widetilde{g}(t) = \begin{cases} g(t) & for \quad t \in (0,1) \setminus \bigcup_{k\in K_3}(\alpha_k, \beta_k), \\ \varepsilon & for \quad t \in \bigcup_{k\in K_3}[\gamma_k, \delta_k], \\ \text{linear on intervals } [\alpha_k, \gamma_k] \text{ and } [\delta_k, \beta_k] \text{ for every } k \in K_3. \end{cases}$$

(It may happen that $a_k = 0$ or $b_k = 1$ for some $k \in K$ and then $f(a_k)$ or $f(b_k)$ do not exist. In this case, we take simply $\lim_{t\to a_k} f(t)$ and $\lim_{t\to b_k} f(t)$ instead of $f(a_k)$ and $f(b_k)$ in the definition of $\widetilde{f}$.)

Since $\widetilde{f}_{|(0,1)\setminus A} = f_{|(0,1)\setminus A}$ and $|\widetilde{f}(t)| \leq \varepsilon \geq |f(t)|$ for $t \in A$, we immediately get $\text{dist}(f, \widetilde{f}) \leq 2\varepsilon$. Similarly, since $\widetilde{g}_{|(0,1)\setminus B} = g_{|(0,1)\setminus B}$ and $|\widetilde{g}(t)| \leq \varepsilon \geq |g(t)|$ for $x \in B$, we have $\text{dist}(g, \widetilde{g}) \leq 2\varepsilon$. Obviously, $||(\widetilde{f}(t), \widetilde{g}(t))|| \geq \varepsilon$ for every $t \in (0,1)$, because if $|\widetilde{f}(t)| < \varepsilon$ then $|\widetilde{g}(t)| \geq \varepsilon$, and if $|\widetilde{g}(t)| < \varepsilon$ then $|\widetilde{f}(t)| \geq \varepsilon$. It remains to prove the continuity of $\widetilde{f}$ and $\widetilde{g}$. By definition, the restrictions of $\widetilde{g}$ to $\bigcup_{k\in K_3}[\alpha_k, \beta_k]$ and to $(0,1) \setminus \bigcup_{k\in K_3}(\alpha_k, \beta_k)$ are continuous, and by local finiteness of $\{[\alpha_k, \beta_k]\}_{k\in K_3}$, $\widetilde{g}$ is continuous on the whole interval $(0,1)$. Similarly, the function $\widetilde{f}$ is continuous on $\big((0,1) \setminus A\big) \cup \bigcup_{k\in K_1}(a_k, b_k)$ and on $\bigcup_{k\in K_2\cup K_3}[a_k, b_k]$. Since $\widetilde{f}\big((a_k, b_k)\big) = \{\lim_{t\to a_k} f(t)\}$ for $k \in K_2$ and since $\{[a_k, b_k]\colon k \in K_3\}$ is locally finite, we get that the function $\widetilde{f}$ is continuous on $(0,1)$. □

Theorem 2. *Multiplication in the space $C(0,1)$ is a weakly open mapping.*

PROOF. Let U be any nonempty open subset of $C(0,1) \times C(0,1)$. There exist $f, g \in C(0,1)$ and $\varepsilon > 0$ such that $B(f, 4\varepsilon) \times B(g, 4\varepsilon) \subset U$. By Lemma 1, we can find $\widetilde{f}, \widetilde{g} \in C(0,1)$ for which $\text{dist}(\widetilde{f}, f) \leq 2\varepsilon$, $\text{dist}(\widetilde{g}, g) \leq 2\varepsilon$ and $||(\widetilde{f}(t), \widetilde{g}(t))|| \geq \varepsilon$ for $t \in (0,1)$. Then by Theorem 1,

$$B(\widetilde{f}\widetilde{g}, \tfrac{\varepsilon^2}{2}) \subset \overline{B}(\widetilde{f}, \varepsilon)\overline{B}(\widetilde{g}, \varepsilon) \subset \overline{B}(f, 3\varepsilon)\overline{B}(g, 3\varepsilon) \subset B(f, 4\varepsilon)B(g, 4\varepsilon).$$

Hence

$$B(\widetilde{f}\widetilde{g}, \tfrac{\varepsilon^2}{2}) \subset Int\{f_1 f_2 \colon (f_1, f_2) \in U\} \neq \emptyset.$$

It follows that the multiplication in the space $C(0,1)$ is a weakly open mapping. □

Corollary 2. *From Theorem 2 it easily follows that multiplication is weakly open in $C([0,1])$ (this yields a new proof of the known result). Namely, consider an open set $U = B(f,r) \times B(g,r)$ where $B(f,r)$ and $B(g,r)$ are balls in $C([0,1])$. By the Tietze Extension Theorem we extend f, g to $f^*, g^* \in C(-1,2)$. Then applying Theorem 2 to the set $U^* = B(f^*,r) \times B(g^*,r)$ open in $C(-1,2) \times C(-1,2)$ we find a ball $B(h,\varepsilon)$ in $C(-1,2)$ witnessing that the respective interior is not nonempty. Finally, we "restrict" this ball to $C([0,1])$.*

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