# INROADS

Surinder Pal Singh Kainth,* Department of Mathematics, IIT Bombay,
Mumbai, Maharastra 400076, India.   email: `singh84@math.iitb.ac.in`

Inder K. Rana, Department of Mathematics, IIT Bombay, Mumbai,
Maharastra 400076, India.   email: `ikr@math.iitb.ac.in`

# SOME ALTERNATIVE APPROACHES TO THE MCSHANE INTEGRAL

**Abstract**

The aim of this paper is to give some equivalent ways of defining
McShane integral for vector valued functions.

## 1 Introduction.

Let $I \subset \mathbb{R}^k$ be a compact interval, $\mu$ be the Lebesgue measure on $I$ and $X$
be a Banach space. A collection $P := \{(t_i, J_i), i = 1, p\}$ is said to be an $\mathcal{M}$-
*partition* of $I$, if $J_i$'s are non-overlapping compact subintervals in $I$ such that
$\cup_{i=1}^p J_i = I$ and each $t_i \in I$. Given a gauge $\delta : I \to (0, \infty)$, we say that $P$ is
*$\delta$-fine* if $J_i \subset B(t_i, \delta(t_i))$ for every $1 \le i \le p$.

A function $f : I \to X$ is said to be *McShane-integrable* with $A \in X$ as its
*McShane-integral* over $I$ if for every $\varepsilon > 0$ there is a gauge $\delta : I \to (0, \infty)$ such
that the inequality

$$\|\sum_{i=1}^p f(t_i)\mu(J_i) - A\|_X < \varepsilon$$

holds, for all $\delta$-fine $\mathcal{M}$-partitions $\{(t_i, J_i), i = 1, p\}$ of $I$.

It is shown in section 4.2 of [1] that in the above definition of McShane
integrability, if $J_i$'s are replaced by disjoint Lebesgue measurable sets, then

---

the class of integrable functions does not change. This provides an alternative approach to McShane integral. Also, in the above result, as only the outer regularity of the Lebesgue measure is used, the arguments remain valid for any other outer regular measure $\mu$ defined on a $\sigma$-algebra containing the *Borel* sets.

   The aim of this note is to show that the class of McShane integrable functions does not change when one considers some other classes of partitions.

## 2   Definitions and Notation.

Let $I \subset \mathbb{R}^k$ be a compact interval with a $\sigma$- algebra $\mathcal{B}$ containing Borel sets, $\mu$ be a non-negative outer regular measure defined on $\mathcal{B}$ with $\mu(\{t\}) = 0$ for all $t \in I$, and $X$ be a Banach space. We denote by $\mathcal{M}_\mu^*$ the collection of all tagged-partitions $\{(t_i, E_i), i = 1, \ldots, p\}$, of $I$ such that $\mu(E_i \cap E_j) = 0 \ \forall \ i \neq j$, and $\mathcal{HK}_\mu^*$ denote the collection of partitions $\{(t_i, E_i), i = 1, \ldots, p\}$ from $\mathcal{M}_\mu^*$ such that $t_i \in E_i$ for every $i$. Let $\mathcal{M}^*$ $(\mathcal{HK}^*)$ denote the collection of all $\mathcal{M}_\mu^*(\mathcal{HK}_\mu^*)$-partitions $\{(t_i, E_i), i = 1, .., p\}$, of $I$ such that the sets $E_1, E_2, \ldots, E_p$ are pair-wise disjoint. We shall denote by $\mathcal{M}$ ( $\mathcal{HK}$) denote the collection of all partitions $\{(t_i, E_i), i = 1, \ldots, p\}$, in $\mathcal{M}_\mu^*$ $(\mathcal{HK}_\mu^*)$ such that $E_1, E_2, \ldots, E_p$ is a collection of non-overlapping subintervals of $I$. Finally, Let $\overline{\mathcal{M}}_\mu^*$ $(\overline{\mathcal{HK}}_\mu^*)$ denote the collection of all partition $\{(t_i, E_i) : i = 1, .., p\}$ in $\mathcal{M}_\mu^*$ $(\mathcal{HK}_\mu^*)$ such that each $E_i$ is a closed subset of $I$.

**Definition 1.** Let

(i)  Given a gauge $\delta : I \to (0, \infty)$, on $I$, we say that a tagged partition $P := \{(t_i, E_i), i = 1, \ldots, p\}$, is $\delta$-*fine* if $E_i \subset B(t_i, \delta(t_i))$ for every $1 \leq i \leq p$.

(ii) Let $f : I \to X$ and $\mathcal{A}$ be a collection of partitions of $I$. We say $f$ is $\mathcal{A}$-*integrable* over $I$ with respect to $\mu$ if there exists $x \in X$ such that for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon : I \to (0, \infty)$ with the property that

$$\| \sum_{i=1}^p f(t_i) \mu(E_i) - x \|_X < \varepsilon \tag{1}$$

for every $\delta_\varepsilon$-fine partition $\{(t_i, E_i), i = 1, \ldots, p\}$, in $\mathcal{A}$.

   In section 4.2 [1], it has been shown that McShane-integral is equivalent to $\mathcal{M}^*$-integral. The aim of this note is to show that McShane integral is also equivalent to $\mathcal{A}$-integral, where $\mathcal{A}$ is any one of the: $\mathcal{M}_\mu^*$, $\overline{\mathcal{M}}_\mu^*$, $\mathcal{M}^*$, $\mathcal{HK}_\mu^*$, $\overline{\mathcal{HK}}_\mu^*$, $\mathcal{HK}^*$.

## 3    The Main Result.

**Theorem 2.** *Let $f : I \to X$ be any given function. Then the following are equivalent:*

(a) *$f$ is McShane-integrable.*

(b) *$f$ is $\overline{\mathcal{HK}}^{*}_{\mu}$-integrable.*

(c) *$f$ is $\mathcal{HK}^{*}_{\mu}$-integrable.*

(d) *$f$ is $\mathcal{M}^{*}$-integrable.*

(e) *$f$ is $\overline{\mathcal{M}}^{*}_{\mu}$-integrable.*

(f) *$f$ is $\mathcal{M}^{*}_{\mu}$-integrable.*

(g) *$f$ is $\mathcal{HK}^{*}$-integrable.*

*Further, all the integrals coincide. In other words, the classes of integrable functions coincide and the values of the respective integrals are the same.*

PROOF. For a tagged partition $P := \{(t_i, E_i), i = 1, \ldots, p\}$, of $I$, let $S(P; f) := \sum_{i=1}^{p} f(t_i)\mu(E_i)$.

To prove: (d) $\iff$ (g), assume that $f$ is $\mathcal{HK}^{*}$-integrable. Let $\varepsilon > 0$ be given. Choose a gauge $\delta$ on $I$, corresponding to this $\varepsilon$, as per the definition of $\mathcal{HK}^{*}$-integrability. Pick any $\delta$-fine $P \in \mathcal{M}^{*}$. We take the union of measurable sets having same tag. Note that this will not effect the $\delta$-fineness of the partition. Thus we can assume, without loss of generality, that all $t_i$'s are distinct. Now consider the set of all tags

$$T = \{t_i : 1 \leq i \leq p\} \quad \text{and} \quad E'_i = (E_i \setminus T) \cup \{t_i\} \quad \text{for all} \;\; 1 \leq i \leq p.$$

Define $P' \in \mathcal{HK}^{*}$, $P' := \{(t_i, E'_i), i = 1, ..., p\}$. Since each $\mu(\{t_i\}) = 0$, we have $\mu(E_i) = \mu(E'_i)$ for each $1 \leq i \leq p$, implying that $S(P; f) = S(P'; f)$. Also, since $P$ is $\delta$-fine, $E_i \subset B(t_i; \delta(t_i))$ for each $i$ implies that $E'_i \subset B(t_i; \delta(t_i))$ for each $1 \leq i \leq p$. Thus $P' \in \mathcal{HK}^{*}$ is $\delta$-fine, and

$$\|S(P; f) - (\mathcal{HK}^{*}) \int_I f\| = \|S(P'; f) - (\mathcal{HK}^{*}) \int_I f\| < \varepsilon.$$

Hence $f$ is $\mathcal{M}^{*}$-integrable with $(\mathcal{M}^{*}) \int_I f = (\mathcal{HK}^{*}) \int_I f$. Converse is trivial since $\mathcal{HK}^{*} \subset \mathcal{M}^{*}$. This proves that (d) $\Leftrightarrow$ (g)

To prove:(d) $\iff$ (f), note that since $\mathcal{M}^* \subset \mathcal{M}_\mu^*$, (f)$\Rightarrow$(d). For converse, let $P \in \mathcal{M}_\mu^*$. Define $P'' \in \mathcal{M}^*$ as follows:

$P'' := \{(t_i, E_i''), i = 1, ..., p\}$, where $E_i'' = E_i \setminus \cup_{j<i} E_j$ for all $1 \le i \le p$.

Then, for $1 \le i \le p$,

$$\mu(E_i'') = \mu(E_i) - \mu(E_i \cap_{j<i} E_j) = \mu(E_i).$$

This proves that Riemann sums corresponding to $P$ and $P''$ are equal. Since each $E_i' \subset E_i$, for any gauge $\delta$, if $P$ is $\delta-$fine then $P''$ is $\delta$-fine. Now using the same type of arguments, as above, we can prove that every $\mathcal{M}^*$-integrable function is $\mathcal{M}_\mu^*$-integrable and the integrals coincide. This proves (d)$\Leftrightarrow$(f)
Similarly, we can prove that (g)$\Leftrightarrow$(c).
To Prove:(b) $\iff$ (e), since $\overline{\mathcal{HK}}_\mu^* \subset \overline{\mathcal{M}}_\mu^*$, so clearly (e)$\Rightarrow$(b). For converse, let $P \in \overline{\mathcal{M}}_\mu^*$. Define

$P''' := \{(t_i, E_i'''), i = 1, ..., p\}$ where $E_i''' = E_i \cup \{t_i\}$ for all $1 \le i \le p$.

Since each $\mu(\{t_i\}) = 0$, each $E_i'''$ is a closed set with $\mu(E_i) = \mu(E_i''')$. Hence $P''' \in \overline{\mathcal{HK}}_\mu^*$ and has the same Riemann-sum as that of $P$. It is also clear that for any gauge, the fine-ness of $P$ remains same for $P'''$. Thus, as above, we have (b)$\Leftrightarrow$(e)

Section 4.2 [1] proves that (d) $\iff$ (a).
Now, since by definition we have $\mathcal{M} \subset \overline{\mathcal{M}}_\mu^* \subset \mathcal{M}_\mu^*$. This proves (f)$\Rightarrow$ (e) $\Rightarrow$(a).

Hence all of the statements (a)-(g) are equivalent and the corresponding integrals coincide.

$\square$

**Note 3.** For a finite dimensional space $X$, it is well known that a function $f : I \to X$ is McShane integrable if and only if both $f$ and $||f||$ are Henstock-Kurzweil integrable. Using theorem 2, the same holds for all of the above defined integrals. Whether it holds when $dim(X) = \infty$ is not known.

**Remark 4.** In [1])(page 96) it is shown that if a function $f : I \to X$ is McShane integrable then $f \cdot \chi_E : I \to X$ is McShane integrable for all Lebesgue measurable sets $E \subset I$. Since, one can extend the notions of $\mathcal{M}_\mu^*$-partitions etc., to bounded measurable sets $E \subset \mathbb{R}^k$, it is natural to ask the following:

*Can one define McShane integral of a function $f : E \to X$ directly by using some type of Riemann sums?*

When $E$ is a bounded closed set, this is possible since the famous Cousin's lemma (ch 2, [2]) can be generalized as follows:

Let $E$ be a bounded subset of $\mathbb{R}^k$. Then the following are equivalent:

(a) For every gauge $\delta : E \to (0, \infty)$, there exists a $\delta$-fine $\mathcal{M}_\mu^*$-partition of $E$.

(b) $E$ is closed.

For a general bounded measurable $E$, the above question remains open.

## References

[1] Stefan Schwabik and Guoju Ye, *Topics in Banach Space Integration*, World Scientific (2005).

[2] Robert G. Bartle, *A Modern Theory of Integration*, Amer. Math. Soc., Providence, (2001).