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HOW TO CONCENTRATE IDEMPOTENTS

Abstract

Call a sum of exponentials of the form $f(x) = \exp(2\pi i N_1 x) + \exp(2\pi i N_2 x) + \cdots + \exp(2\pi i N_m x)$, where the N_k are distinct integers, an *idempotent*. We have L^p *interval concentration* if there is a positive constant a, depending only on p, such that for each interval $I \subset [0,1]$ there is an idempotent f so that $\int_I |f(x)|^p \, dx \big/ \int_0^1 |f(x)|^p \, dx > a$. We will explain how to produce such concentration for each $p > 0$. The origin of this question and the history of the development of its solution will be surveyed.

1 Idempotents.

An idempotent is a function

$$\iota(x) = \sum_{n \in S} e(nx),$$

where S is a finite set of integers and $e(x) = e^{2\pi i x}$. It takes its name from the identity

$$(\iota * \iota)(x) = \int_0^1 \iota(y)\, \iota(x-y)\, dy = \iota(x).$$

Another reason for the term "idempotent" is that the operator A_ι defined on $L^1(\mathbb{T})$ by $A_\iota : f \to \iota * f$ maps the function with Fourier series $\sum_{n=-\infty}^{\infty} a_n e(nx)$ to $\sum_{n \in S} a_n e(nx)$. So A_ι is a projection operator onto the finite dimensional subspace spanned by $\{e(nx) : n \in S\}$ and $A_\iota^2 = A_\iota$.

The absolute value of an idempotent is an even function since

$$\left| \sum_{n \in S} e\left(n\left(-x\right)\right) \right| = \left| \overline{\sum_{n \in S} e\left(nx\right)} \right| = \left| \sum_{n \in S} e\left(nx\right) \right|.$$

Write

$$\sum_{n \in S} e\left(nx\right) = z^m \left(\sum_{n \in S} z^{n-m} \right)$$

where $m = \min\{n : n \in S\}$ and $z = e^{2\pi i x}$ and apply the fundamental theorem of algebra to see that the set where an idempotent is zero must be finite.

2   Concentration.

Fix a real number $p > 0$ and consider only functions in $L^p\left(\mathbb{T}\right)$. These are all measurable complex valued functions f on the torus $\mathbb{T}$ satisfying $\int_{\mathbb{T}} |f\left(x\right)|^p \, dx < \infty$, where we take $\mathbb{T}$ to be the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with the endpoints identified. We consider a series of four questions.

Question 1. Given an open interval $I \subset T$, can you fully concentrate some function on I? In other words, can you find $f = f_I$ so that

$$\frac{\int_I |f\left(x\right)|^p \, dx}{\int_{\mathbb{T}} |f\left(x\right)|^p \, dx} = 1?$$

The characteristic function of I, $\chi_I\left(x\right) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \in \mathbb{T} \setminus I \end{cases}$ instantly shows the answer to be yes.

Question 2. Given an open interval $I \subset T$, can you find an *even* function of concentration .5 on I? In other words, can you find an even $f = f_I$ so that

$$\frac{\int_I |f\left(x\right)|^p \, dx}{\int_{\mathbb{T}} |f\left(x\right)|^p \, dx} = .5?$$

If $I = [.1, .2]$, it is clear that the best possible even function is $\chi_{[-.2,-.1] \cup [.1,.2]}$. Looking at this example also makes it clear why we lowered the bar to only demand a concentration of .5.

Question 3. Given an open interval $I \subset T$, can you find an *even almost everywhere non zero* function of concentration .5 on I?

The answer is just barely no; for example given $I = [.1, .2]$ again and defining $f\left(x\right) = \begin{cases} 1 & \text{if } x \in [-.2, .1] \cup [.1, .2] \\ \epsilon & \text{if } x \in T \setminus ([-.2, .1] \cup [.1, .2]) \end{cases}$ produces a function of

concentration less than .5. However f can be made to have concentration as close to .5 as desired by choosing ϵ small enough.

This example motivates us to slightly change the definition of concentration.

Definition 1. *A set of L^p functions $\mathcal{I}$ has concentration c on I if given any $\epsilon > 0$, there is an $f \in \mathcal{I}$ so that*

$$\frac{\int_I |f(x)|^p \, dx}{\int_{\mathbb{T}} |f(x)|^p \, dx} > c - \epsilon.$$

Now the answer to Question 3 is yes.

Our last question is the real issue of this paper.

Question 4. Does there exist a positive number c_p (depending only on p) such that for *any* open interval $I \subset \mathbb{T}$, the set of idempotents has concentration c_p on I?

Another phrasing of Question 4: Does there exist a positive number c_p such that for each interval I and each $\epsilon > 0$, there is an idempotent $\iota(x) = \iota_{I,p,\epsilon}(x)$ such that

$$\frac{\int_I |\iota(x)|^p \, dx}{\int_{\mathbb{T}} |\iota(x)|^p \, dx} > c_p - \epsilon? \tag{2.1}$$

If the answer to Question 4 is yes, we say that L^p concentration holds. From the above discussion we already know that the largest conceivable value for c_p is .5. If c_p can be .5, then we say that full L^p concentration holds. The answer to Question 4 is this.

Theorem 1. *L^p concentration holds for all $p > 0$. Full concentration fails only when p is an even integer.*

Actually, even for p an even integer, the concentration is substantial. The maximum possible value for c_{2n} is between .25 and a lower bound quite close to .25 for each even integer $2n$; the maximal value for c_2 is exactly $\max_{x>0} \frac{\sin^2 x}{\pi x} = .2306....$

3 The History of the Concentration Question.

The question of concentration originated with a problem in functional analysis.

Because of Plancherel's Theorem, we may define $L^2(\mathbb{Z})$ by

$$L^2(\mathbb{Z}) = \left\{ f : \mathbb{T} \to \mathbb{C} : \|f\| = \sqrt{\int_{\mathbb{T}} |f(x)|^2 \, dx} < \infty \right\}.$$

Let T be an operator defined from a subset $\mathcal{S}$ of $L^2(\mathbb{Z})$ into $L^2(\mathbb{Z})$ in such a way that $Tf(x) = \int_{\mathbb{T}} K(y) f(x-y) \, dy$ for some measurable function $K : \mathbb{T} \to \mathbb{C}$ and

$$\|Tf\| \leq C \|f\| \tag{3.1}$$

for every $f \in \mathcal{S}$ for some constant C. Then T is called a convolution operator and the function K is called the kernel of T. When $\mathcal{S}$ is equal to all of $L^2(\mathbb{Z})$, say that T is strong type $(2,2)$, or, equivalently, that T is bounded on all of $L^2(\mathbb{Z})$. If, however, inequality (3.1) is only known to be true for functions $f \in \mathcal{S}$, where $\mathcal{S}$ is the set of all idempotents, say that T is restricted type $(2,2)$. Obviously, if T is strong type $(2,2)$, then T is restricted type $(2,2)$. The primal functional analysis question referred to at the beginning of this section is this possible converse: If T is restricted type $(2,2)$, is T necessarily also strong type $(2,2)$?

When I looked at this question in the middle 1970s, I was unable to solve it, but I was able to formulate an almost equivalent question. Here is that equivalence.

Theorem 2. *If restricted type $(2,2)$ implies strong type $(2,2)$, then L^2 concentration for intervals holds. If L^2 concentration for sets holds (i.e., if inequality (2.1) continues to hold when $p = 2$, even if I is allowed to vary over all sets of positive Lebesgue measure), then restricted type $(2,2)$ implies strong type $(2,2)$.*

At about the same time, Michael Cowling, working in a much more general context, proved that an even weaker assumption than T being restricted type $(2,2)$ was sufficient to force T to be strong type $(2,2)$ [11]. This provided a non-constructive proof that L^2 concentration for intervals was true. So I published a paper pointing out that L^2 concentration for intervals was true, but the amount of concentration was unknown [5]. The connections with functional analysis are surveyed in some detail in [7].

There followed a series of direct proofs producing quantitative estimates for c_2.

(1) The referee of paper [5] pointed out that $\sqrt{c_2}$ must be at least $1/8 = .125$, so $c_2 \geq 1/64 = .0016$.

This is a good place to bring up a notational point. In determining concentrations it is equally natural (and obviously equivalent) to study the ratios

$$\frac{\int_I |\iota(x)|^p \, dx}{\int_{\mathbb{T}} |\iota(x)|^p \, dx} \quad \text{and} \quad \left(\frac{\int_I |\iota(x)|^p \, dx}{\int_{\mathbb{T}} |\iota(x)|^p \, dx} \right)^{1/p}.$$

In comparing concentrations announced in the references to this article, the reader must note which of these two definitions is used in each paper.

(2) S. Pichorides [18] obtained $\sqrt{c_2} \geq .14$, or $c_2 \geq .0196$.

(3) H. L. Montgomery [17], and

(4) J.-P. Kahane [16] obtained several better lower bounds. (The ideas of H. L. Montgomery were "deterministic" while those of J.-P. Kahane used probabilistic methods from [15].)

Finally, in [4], together with Roger Jones and Bahman Saffari, I achieved the lower bound $\max_{x>0} \frac{\sin x}{\sqrt{\pi x}} = .4802...$for $\sqrt{c_2}$: which, in [12], was proved to be best possible, thus $c_2 = \max_{x>0} \frac{\sin^2 x}{\pi x} = .2306....$ (See [13] for a more detailed exposition of the contents of [12].)

The next step was to try to generalize by demonstrating L^p concentration for values of p other than 2. Unfortunately, it appears that there is no longer any simple connection with functional analysis when $p \neq 2$. Nevertheless a group of five of us found that L^p concentration was true when $p > 1$. The main idea of our approach will be sketched in the next section. We delayed the publication of [1] and [2] for some years because we felt that we ought to resolve the case of $p = 1$. We finally gave up and finished publishing in 2007. We ended by conjecturing that concentration fails when $p = 1$.

In 2008, Bonami and Révész overturned our conjecture for $p = 1$ in a very nice paper which contains Theorem 1 as well as excellent results for the more general problem of concentrating idempotents onto sets of positive measure [8]. The main idea of their approach is sketched in Section 6. Some remaining open questions are raised in the last section.

A lot of things have happened in the last year or two that I have not been able to touch on here. The best source for keeping up to date on this quickly evolving field is probably on the website of Szilárd Révész[1] I will briefly comment on two of the items to be found there.

First, Bonami and Révész extend the results of [8] in a companion paper [9] wherein they demonstrate that L^1 concentration for sets of positive measure is quite large, in particular it is much larger than L^2 concentration for intervals. On the other hand, they prove a discrete version of our original L^1 conjecture involving the groups $\mathbb{Z}/q\mathbb{Z}$, with q a prime number. As they remark, this is in a way a positive answer to the original conjecture.

Second, the very clearly written and comprehensive Chapter 3 of Révész's dissertation for the "Doctor of the Academy" degree [19], mentions further

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[1]http://www.renyi.hu/~revesz/preprints.html.

applications of concentration results and techniques to other areas of classical harmonic analysis. Another thing in Chapter 3 is Révész's telling of the story of how Terry Tao suggested the crucial step that enabled Bonami and Révész to push past the seemingly impenetrable concentration barrier between the cases of $p > 1$ and $p \leq 1$. I heartily recommend Chapter 3 of [19] to any reader interested in pursuing this subject beyond what I have written in this survey.

4   How to Show L^p Concentration for $p > 1$.

The Dirichlet kernel, $D_n(x)$, is the only idempotent important enough to have its own name; it is the one corresponding to the set $S = \{0, 1, 2, 3, \ldots, n-1\}$. (This notation is nonstandard in that S is not symmetrical about 0. In this subject only the absolute values of functions are ever considered so the distinction is harmless since $\left| \sum_{\mu=-m}^{m} e(\mu x) \right| = |e(-mx)| \left| \sum_{\mu=0}^{2m} e(\mu x) \right| = \left| \sum_{\mu=0}^{2m} e(\mu x) \right|$.) Here are some of its properties. First, note D_n is a geometric sum with ratio $e^{2\pi i x}$, apply the geometric sum formula, and use the identity $\left| e^{2i\theta} - 1 \right| = \left| e^{i\theta} \right| \left| e^{i\theta} - e^{-i\theta} \right| = \left| e^{i\theta} - e^{-i\theta} \right|$ to derive

$$|D_n(x)| = \left| \frac{\sin \pi n x}{\sin \pi x} \right|. \tag{4.1}$$

Here is the graph of $|D_{10}(x)|$
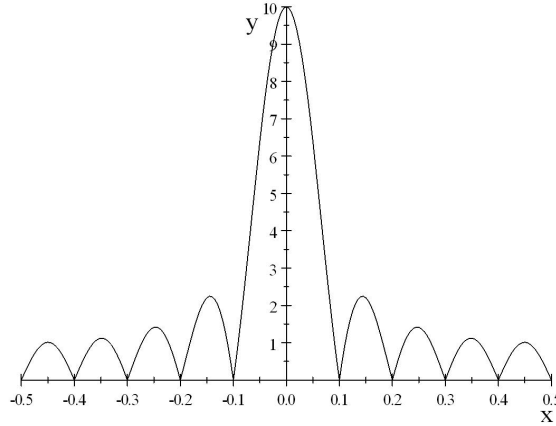
And here is the graph of $|D_{100}(x)|$.
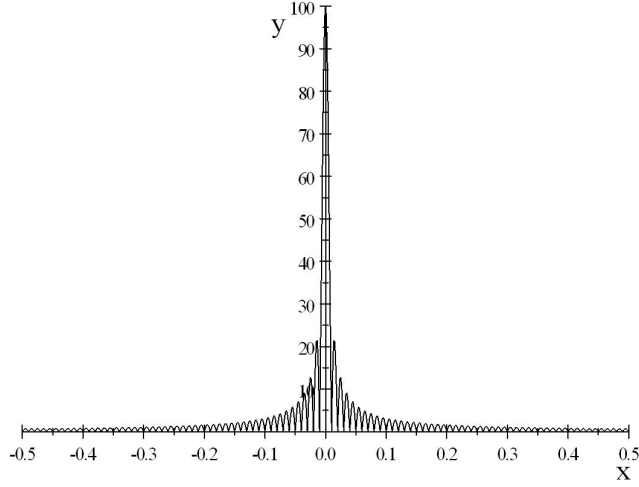
We have

$$|D_n(x)| \leq \sum_{k=0}^{n-1} |e(kx)| = \sum 1 = n, \tag{4.2}$$

and from $2x \leq \sin \pi x$ for $x \in \left[0, \frac{1}{2}\right]$, we have

$$|D_n(x)| \leq \frac{1}{|2x|} \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{4.3}$$

Somewhat more sophisticated standard estimates give constants c_p so that

$$\text{as } n \to \infty, \int_{-1/2}^{1/2} |D_n(x)|^p \, dx \simeq \begin{cases} c_p n^{p-1} & \text{if } p > 1 \\ \\ c_1 \ln n & \text{if } p = 1 \end{cases}. \tag{4.4}$$

Fix $p > 1$. We will show how to concentrate onto any open interval $I \subset [-1/2, 1/2)$. We will consider three cases: Case 0 when $0 \in I$, Case 1 when I contains an interval centered at $\frac{1}{q}$ and having length at least $\frac{2}{q^2}$ for some (large) prime q, and Case 2 when I contains a small interval centered at $\frac{a}{q}$ and having length at least $\frac{2}{q^2}$ for some prime q and some integer a, $0 < |a| < \frac{q}{2}$. Notice that picking the prime q very large assures that the grid of points

$\{j/q : j \in \mathbb{Z}, |j| < q/2\}$ is so fine that 3 consecutive elements, say $\frac{a-1}{q}, \frac{a}{q}, \frac{a+1}{q}$, with $a \neq 0$ are all in I. This means that cases 0, 1, and 2 cover all possibilities. In fact, Case 0 is extraneous. Even if I is a small interval symmetric about $x = 0$, I is still an instance of Case 1 for q big enough. We only do Case 0 because it is so simple.

Case 0. Simply pick the concentrated function to be $D_n(x)$ for a very large value of n. For $p > 1$, the L^p mass of D_n is concentrated near 0 more and more as n increases, as one might guess from our graph of $|D_{100}|$. (This is not quite obvious when $p \leq 1$; in fact it is not even true when $p < 1$.)

From estimates (4.4), $\int_{-1/2}^{1/2} |D_n(x)|^p\, dx \to \infty$ as $n \to \infty$ for each $p \geq 1$; but for each fixed $\epsilon > 0$, $\int\limits_{|x| \geq \epsilon} |D_n(x)|^p\, dx \leq 2 \int_\epsilon^{1/2} \frac{1}{(2x)^p}\, dx = C(\epsilon)$. Thus for each $\epsilon > 0$,

$$\frac{\int_{|x| < \epsilon} |D_n|^p}{\int_{-1/2}^{1/2} |D_n|^p} \geq \frac{\int_{-1/2}^{1/2} |D_n|^p - C(\epsilon)}{\int_{-1/2}^{1/2} |D_n|^p} \to 1 \text{ as } n \to \infty.$$

Case 1. We concentrate L^p mass on the interval J_1, where $J_a = \left[\frac{a}{q} - \frac{1}{q^2}, \frac{a}{q} + \frac{1}{q^2} \right]$, by considering the idempotent

$$i(x) = D_{q^2}(qx)\, D_{\frac{q-1}{2}}(x). \tag{4.5}$$

First we check that i is an idempotent. The general term of the first factor is $e(mqx)$ where $0 \leq m < q^2$, the general term of the second factor is $e(nx)$ where $0 \leq n < q/2$. Since $e(mqx)\, e(nx) = e((mq + n)x)$, all nonzero coefficients are positive integers. Further they are all 1 since $mq + n = m'q + n'$ implies q divides $n - n'$ implies $n = n'$ implies $m = m'$.

We have to understand why

$$\frac{\int_{J_1} fg\, dx}{\int_{-1/2}^{1/2} fg\, dx}, \text{ where } f = \left| D_{q^2}(qx) \right|^p \text{ and } g = \left| D_{\frac{q-1}{2}}(x) \right|^p$$

is bounded below with a bound that is independent of q. To see this, we will first consider as a model for $|D_n(x)|$ a narrow, tall flat pulse centered at the origin and having period 1. In what follows, we will often repeat this process of replacing a function by a simpler function which shares the most immediately relevant property of the original function.
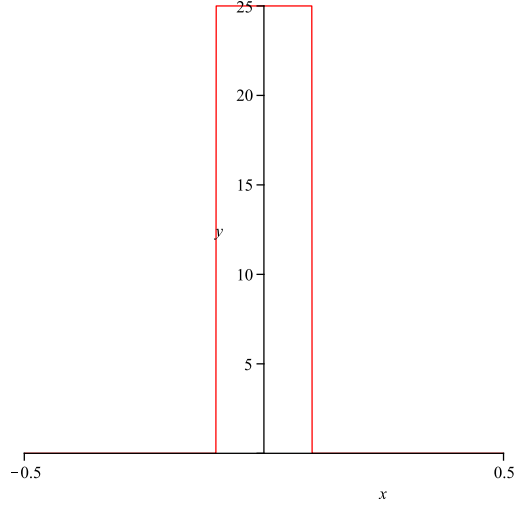
Since $|D_n(qx)|$ has period $1/q$, we approximate f as
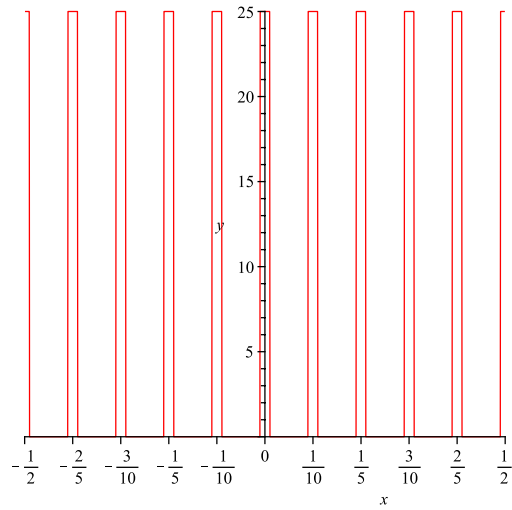
$$\sum_{a=-(q-1)/2}^{(q-1)/2} c\chi_{J_a}(x)$$

where for each a, χ_{J_a} is the characteristic function of the interval J_a,

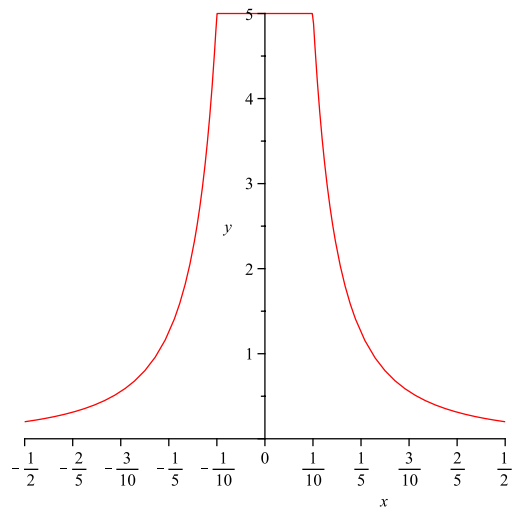$$\chi_{J_a} = \begin{cases} 1 & \text{if } x \in J_a \\ 0 & \text{if } x \notin J_a \end{cases}.$$

The first factor of $i(x)$ (see (4.5)) is concentrated near $x = 0$ and has period $1/q$ so we think of it as being roughly

$$c\sum_{j=-(q-1)/2}^{(q-1)/2} \chi_{\left[\frac{j}{q}-\frac{1}{q^2},\frac{j}{q}+\frac{1}{q^2}\right]}(x),$$
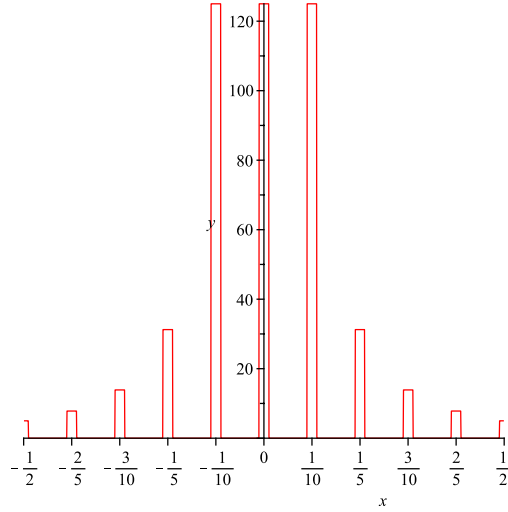
in other words as a series of q equal pulses.

We think of the last factor of $i(x)$ (see (4.5) as being even and having decay like $1/x$ on $[1/q, 1/2]$.

Multiplying the above two functions together gives this model for $i(x)$:

Now L^p concentration follows, since

$$\frac{\int_{\frac{1}{q}-\frac{1}{q^2}}^{\frac{1}{q}+\frac{1}{q^2}}|i(x)|^p\,dx}{\int_{-1/2}^{1/2}|i(x)|^p\,dx} \approx \frac{\sum_{j=1}^{1}\frac{1}{j^p}}{1+2\sum_{j=1}^{(q-1)/2}\frac{1}{j^p}} \geq \frac{1}{1+2\sum_{j=1}^{\infty}\frac{1}{j^p}} > 0.$$

This completes our heuristic discussion for Case 1.

Case 2. Our goal is to find an idempotent concentrated on J_a. We have assumed that the integer $a \neq 0$ so that $1 \leq |a| \leq (q-1)/2$. Start with the idempotent $i(x)$ of Case 1 which is concentrated on J_1. The set $\mathbb{Z}_q^* = \{n \in \mathbb{Z} : 0 < |n| \leq (q-1)/2\}$ is a group under multiplication modulo q. Thus a has a unique inverse $b \in \mathbb{Z}_q^*$ so that $ab \equiv 1\,(\mathrm{mod}\,q)$, that is $ab = 1 + rq$ for some integer r. Form $j(x) = i(bx)$ where $i(x)$ is the idempotent that was concentrated at $1/q$ to solve Case 1. Then j is concentrated near a/q since for small x, in other words for x much smaller than $1/q$ we have

$$\begin{aligned}
j(a/q+x) &= i(ab/q+bx)\\
&= i((1+rq)/q+bx)\\
&= i(1/q+r+bx)\\
&= i(1/q+bx).
\end{aligned}$$

In other words, the behavior of j near a/q is the same as that of i near $1/q$. A similar argument shows that the amounts of concentration of j at the $q-2$ points of the set $\{c/q : c \in \mathbb{Z}_q^* \setminus \{a\}\}$ are the same as those of i at the $q-2$ points of the set $\{c/q : c \in \mathbb{Z}_q^* \setminus \{1\}\}$. Thus Case 2 has been reduced to Case 1.

5   How to Get p Down to 1.

Let $p = 1$. Recall that we only have to deal with Case 1 and Case 2. The deduction of Case 2 from Case 1 is treated as before. So we are faced with trying to concentrate at $1/q$. We can't do what we did before, since when $p = 1$ we are faced with

$$\frac{\sum_{j=1}^{1} \frac{1}{j}}{1 + 2 \sum_{j=1}^{(q-1)/2} \frac{1}{j}} \approx \frac{1}{\ln q}$$

which is not bounded away from 0. A natural candidate seems to be

$$D_{q^2}(qx) D_{\frac{q-1}{2}}(x)^2.$$

Now the concentration is fine since where we had a single $1/x$ factor we now have two $1/x$ factors and the ratio becomes

$$\frac{\sum_{j=1}^{1} \frac{1}{j^2}}{1 + 2 \sum_{j=1}^{(q-1)/2} \frac{1}{j^2}}.$$

What goes wrong is that we do not have an idempotent. In fact,

$$\begin{aligned} D_{\frac{q-1}{2}}(x)^2 &= (1 + e(x) + e(2x) + \dots)(1 + e(x) + e(2x) + \dots) \\ &= 1 + 2e(x) + 3e(2x) + \dots \end{aligned}$$

A quick fix for this is to replace this function by $D_{\frac{q-1}{2}}(x) D_{\frac{q-1}{2}}((q+1)x)$. As in checking that $i(x)$ was an idempotent, we have to check that if $m + n(q+1) = m' + n'(q+1)$, where m, n, m', n' all are in $\left[0, \frac{q-1}{2}\right]$, then $m = m'$ and $n = n'$. But this is immediate from thinking of $m + n(q+1)$ as a two digit base $q+1$ integer.

Note that if x is a grid point $\frac{j}{q}$ where $|j| \leq \frac{q-1}{2}$,

$$
\begin{aligned}
D_{\frac{q-1}{2}}\left((q+1)x\right) &= D_{\frac{q-1}{2}}\left((q+1)\left(\frac{j}{q}\right)\right) \\
&= D_{\frac{q-1}{2}}\left(j+\frac{j}{q}\right) \\
&= D_{\frac{q-1}{2}}\left(\frac{j}{q}\right) \\
&= D_{\frac{q-1}{2}}(x),
\end{aligned}
$$

so that we may still think of each factor as being approximately $1/x$ and we still have the right concentration, namely

$$
\frac{\sum_{j=1}^{1}\frac{1}{j^2}}{1+2\sum_{j=1}^{(q-1)/2}\frac{1}{j^2}}. \tag{5.1}
$$

But when we multiply by $D_{q^2}(qx)$ there is lots of repetition and we no longer have an idempotent. No obvious manipulation of Dirichlet kernels seems to do any good. A new idea is needed. We need an idempotent $t(x)$ so that on the one hand, $t(qx)$ can produce the pulses near $0, \pm 1/q, \pm 2/q, \ldots$ that $D_{q^2}(qx)$ did, while on the other hand $t(qx)D_{\frac{q-1}{2}}(x)D_{\frac{q-1}{2}}(qx)$ remains an idempotent. So we want

1. t is an idempotent,

2. t is L^1 concentrated near 0,

3. t has very large gaps between frequencies.

Actually, we will do a little better by replacing the second property with L^1 concentration near $1/2$. This is better because with $D_{q^2}(qx)$ the three most central pulses were centered at $-1/q, 0$, and $1/q$. Consequently, in attempting to concentrate at $1/q$, we were forced to accept some wasted concentration at 0. With $t(x)$ being concentrated at $1/2$, the two most central pulses of $t(qx)$ will be centered at $-\frac{1}{2q}$ and $\frac{1}{2q}$ with the next two pulses centered at $\pm\frac{3}{2q}$, a neater situation.

We now replace $D_{\frac{q-1}{2}}\left((q+1)x\right)$ by $D_{\frac{q-1}{2}}\left((2q+1)x\right)$ since we are now concentrating on the grid $\ldots, -\frac{3}{2q}, -\frac{1}{2q}, \frac{1}{2q}, \frac{3}{2q}, \ldots$ and $D_{\frac{q-1}{2}}\left((2q+1)\frac{j}{2q}\right) = D_{\frac{q-1}{2}}\left(\frac{j}{2q}\right)$. Thus our winning idempotent will be

$$
t(x)D_{\frac{q-1}{2}}(x)D_{\frac{q-1}{2}}\left((2q+1)x\right).
$$

The sum of a frequency appearing in $D_{\frac{q-1}{2}}(x)$ and one appearing in $D_{\frac{q-1}{2}}((2q+1)x)$ is at most $\left(\frac{q-1}{2}-1\right)(1+(2q+1))$ so adjusting the gaps of t to be larger than that will keep our final function an idempotent.

Here is the required idempotent:

$$t(x) = [e(0)+e(Rx)+e((2R+1)x)]\left[e(0)+e\left(R^2x\right)+e\left(\left(2R^2+1\right)x\right)\right]\cdots$$
$$\left[e(0)+e\left(R^{J-1}x\right)+e\left(\left(2R^{J-1}+1\right)x\right)\right]\left[e(0)+e\left(R^Jx\right)\right.$$
$$\left.+e\left(\left(2R^J+1\right)x\right)\right]$$

Whatever the value of J, t is an idempotent with gap size at least R provided $R > \max\{J,3\}$. Fully expand $t(x)$ into 3^J terms. A typical term has the form

$$e((\alpha_1+\cdots+\alpha_J)x)$$

where each $\alpha_j = 0, R^j$, or $2R^j+1$. Write $\alpha_j = \beta_j R^j + \delta_j$ where β_j is $0,1$, or 2 and δ_j is 0 or 1. Idempotency requires that distinct terms in the expansion of $t(x)$ have distinct frequencies. This is clear since

$$\alpha_1+\cdots+\alpha_J = \left(\sum_{j=1}^{J}\delta_j\right)R^0 + \sum_{j=1}^{J}\beta_j R^j.$$

Also from this representation it is not hard to verify that all gaps are at least R.

It remains to show that picking J large will L^1 concentrate $t(x)$ near $\frac{1}{2}$. Let $I = \left[-\frac{1}{2},-\frac{1}{2}+\epsilon\right]\cup\left[\frac{1}{2}-\epsilon,\frac{1}{2}\right]$ be a small interval centered at $\frac{1}{2}$. Then

$$\int_I \left|1+e\left(R^1x\right)+e\left(\left(2R^1+1\right)x\right)\right|\cdots\left|1+e\left(R^Jx\right)+e\left(\left(2R^J+1\right)x\right)\right|\ dx \approx$$

$$(5.2)$$

$$\int_I\left\{\int_T\left|1+e(y)+e(2y+x)\right|\ dy\right\}\cdots\left\{\int_T\left|1+e(y)+e(2y+x)\right|\ dy\right\}\ dx =$$
$$\int_I\left\{F(x)\right\}^J\ dx$$

where

$$F(x) = \int_T\left|1+e(y)+e(2y+x)\right|dy.$$

It turns out that the continuous even function $F(x)$ is monotone and increasing on $\left[0,\frac{1}{2}\right]$. (I can give no motivation for this fact, but a straightforward proof of it is found in [8].) Thus $F(x)$ has a unique maximum at $\frac{1}{2}$. From this it

is immediate that sufficiently large J will make F^J as concentrated as desired at $\frac{1}{2}$.

So we will have seen that $t(x)$ has both large gaps and excellent L^1 concentration at $\frac{1}{2}$ as soon as we tie up one last loose end: Why is the approximation (5.2) true? We start by observing that for $\varphi(x,y)$ continuous and I an interval, as R becomes large

$$\int_I \varphi(x, Rx)\, dx \approx \int_I \left(\int_{\mathbb{T}} \varphi(x,y)\, dy \right) dx.$$

To see this, first expand $\varphi(x,y)$ into a Fourier series in y with coefficients in x and then use a density argument to truncate that series. Thus we may assume that

$$\varphi(x,y) = \sum_{|m| \leq M} \varphi_m(x)\, e(my).$$

Then

$$\int_{\mathbb{T}} \varphi(x,y)\, dy = \sum \varphi_m(x) \int_{\mathbb{T}} e(my)\, dy = \varphi_0(x),$$

$$\int_I \left(\int_{\mathbb{T}} \varphi(x,y)\, dy \right) dx = \int_I \varphi_0(x)\, dx,$$

while

$$\varphi(x, Rx) = \sum_n \varphi_n(x)\, e(nRx)$$

$$= \varphi_0(x) + \sum_{|n| \geq 1} \varphi_n(x)\, e(nRx).$$

Integrate over I and observe that for each fixed n, if R is very large,

$$\int_I \varphi_n(x)\, e(nRx)\, dx$$

is the integral of a very rapidly oscillating function over an interval and hence is very small. (In fact, we have here the $-nR$ Fourier coefficient of the function $\chi_I \varphi_n$, which tends to zero as $R \to \infty$ by the Riemann-Lebesgue Theorem.)

A very similar argument shows that

$$\int_I \varphi(x, Rx)\, \varphi(x, R^2 x)\, dx \approx \int_I \left(\int_{\mathbb{T}} \varphi(x,y)\, dy \right)^2 dx.$$

Use the same expansion for φ. The right side becomes $\int_I \varphi_0\,(x)^2\,dx$, while the left side becomes

$$\int_I \left(\varphi_0\,(x)^2 + \sum_{\substack{|m|,|n|\le M \\ (m,n)\ne(0,0)}} \varphi_m\,(x)\,\varphi_n\,(x)\,e\left(\left(mR+nR^2\right)x\right) \right) dx.$$

Integrate over I and observe that for each fixed $(m,n)\ne(0,0)$, if R is very large,

$$\int_I \varphi_m\,(x)\,\varphi_n\,(x)\,e\left(\left(mR+nR^2\right)x\right) dx$$

is small, being the $-\left(mR+nR^2\right)$ Fourier coefficient of the function $\chi_I \varphi_m \varphi_n$. Notice that we have just done the $J=1$ and $J=2$ cases of the approximation (5.2) when

$$\varphi\,(x,y) = |1 + e\,(y) + e\,(2y+x)|\,.$$

The proof for larger J is no different. This completes the argument for L^1 concentration.

The motivation for picking $1 + e\,(y) + e\,(x)\,e\,(2y)$ is produced by the argument just given. Everything flows quite naturally once F maximizes at $x = \frac{1}{2}$. If one were to study very simple sums of the form

$$\sum e\,(mx)\,e\,(ny)$$

trying to find one where

$$F\,(x) = \int_{\mathbb{T}} \left| \sum e\,(mx)\,e\,(ny) \right| dx$$

is maximal at 0 or $\frac{1}{2}$, $1 + e\,(y) + e\,(x)\,e\,(2y)$ would show up early in the search.

Notice that the amount of concentration of t at $x = 1/2$ actually approaches 1 as J increases. For an even function, concentration exceeding $1/2$ is possible only at $x = 0$ and $x = 1/2$, the latter because $-1/2$ and $1/2$ are identified in the definition of the torus $\mathbb{T}$. In other words, 0 and $1/2$ are the only self-symmetric points of $\mathbb{T}$ since every point $a \in (-1/2, 1/2] \setminus \{0, 1/2\}$ satisfies $a \ne -a$.

6   How to Get p Down to All Small Positive Values.

It turns out that for $0 < p < 2$,

$$F_p\,(x) = \int_{\mathbb{T}} |1 + e\,(y) + e\,(2y+x)|^p\,dy$$

is also maximized at $x = \frac{1}{2}$. (For $p > 2$, F_p is maximized at $x = 0$; Parseval's equality gives $F_2(x) \equiv 1^2 + 1^2 + |e(x)|^2 = 3$.) This allows a similar argument to be concocted for any $p \in (0, 2)$. The idempotent

$$t(qx) D_{\frac{q-1}{2}}(x) D_{\frac{q-1}{2}}((2q+1)x) \tag{6.1}$$

not only concentrates near $\frac{1}{q}$ in the L^1 sense, it also concentrates there nicely in the L^p sense when $p \in \left(\frac{1}{2}, 2\right)$ since the p-analogue of estimate (5.1) is

$$\frac{\sum_{j=1}^{1} \frac{1}{j^{2p}}}{1 + 2 \sum_{j=1}^{(q-1)/2} \frac{1}{j^{2p}}}$$

which is bounded below, uniformly in q, by $\left(1 + 2 \sum_{j=1}^{\infty} j^{-2p}\right)^{-1}$ so long as $p > \frac{1}{2}$. To push p down as close to zero as you like, just keep adjoining more factors on the right side of definition (6.1) while widening the gaps of t appropriately. For example,

$$t(qx) D_{\frac{q-1}{2}}(x) D_{\frac{q-1}{2}}((2q+1)x) D_{\frac{q-1}{2}}\left(\left((2q)^2 + 1\right)x\right)$$

will concentrate at $\frac{1}{q}$ in the L^p sense so long as $p > \frac{1}{3}$.

7 An Extension and Remaining Questions.

Although we have not tracked the amount of concentration carefully, as we mentioned above, the method of the last two sections can produce full concentration of .5 whenever $p > 0$ is not an even integer. Furthermore the maximum amount of concentration when $p = 2$ is exactly $\max_{x>0} \frac{\sin^2 x}{\pi x} = .2306....$ In particular, since $.2306... < .5$, this means that the existence of a function similar to t, that is, for a given $\epsilon > 0$, an idempotent $t_\epsilon(x)$ with large gaps and concentration $> 1 - \epsilon$ in the L^2 sense at $x = 1/2$, is impossible. Otherwise a simple extension of the argument given above would produce L^2 concentrations close to .5 on all open intervals, contradicting $.2306... < .5$.

The amount of concentrations for $p = 4, 6, 8, \ldots$ is fairly well estimated in [8] but remains unknown. This is the first open question.

We have asked about concentrating an idempotent on an interval. A more general question involves replacing "interval" by "set of positive measure." Now less is known. All the results found for intervals still hold here (with the same concentrations) when $p > 1$. For $p \in (1/2, 1]$, concentration is known to

be positive, but whether concentration of .5 can be achieved is unknown. For $p \in (0, 1/2]$, whether even positive concentration holds is unknown.

In all the extensions to sets of positive measure achieved so far, metric number theory has played an interesting role. For example, in [2], we start with a subset $E \subset \mathbb{T}$ of positive measure. We pick a point of density $\xi \in E$. Next we pick a large prime q and look at the grid of mesh $\frac{1}{q}$ where $\frac{a}{q}$ is the grid point closest to ξ. Finally we concentrate a lot of L^p mass near $\frac{a}{q}$, say within $I = \left[\frac{a}{q} - \frac{1}{q^2}, \frac{a}{q} + \frac{1}{q^2}\right]$. The mass must be fairly evenly distributed within I and this turns out to be the case for the constructions just described in Section 4, provided $\frac{a}{q}$ is a very good approximation of ξ, say

$$\left|\xi - \frac{a}{q}\right| < \frac{1}{q^2}. \tag{7.1}$$

So it is very convenient that there is a theorem of metric number theory stating that for almost every ξ, there are infinitely many primes q for which inequality (7.1) holds for some integer a. (See [14].) Choosing q to be prime is somewhat convenient, but not at all crucial. If one takes that path, earlier work done by Szüsz ([21]) and generalized by Schmidt ([20]) is applicable. I would also like to mention that extending concentration from intervals to sets of positive measure in [2] also required other ideas suggested to us by Fedja Nazarov. One of these suggestions is worked out in [6]. A more general version of the result of [6] can be found in [3].

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