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Szymon Głąb,* Mathematical Institute, Polish Academy of Science, Śniadeckich 8, 00-956 Warszawa, Poland; Current Address: Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland. email: szymon_glab@yahoo.com

ON THE COMPLEXITY OF CONTINUOUS FUNCTIONS DIFFERENTIABLE ON COCOUNTABLE SETS

Abstract

We prove that the set of all functions in $C[0,1]$, with countably many points at which the derivative does not exist, is $\mathbf{\Pi}_1^1$–complete, in particular non–Borel. We obtain the classical Mazurkiewicz's theorem and the recent result of Sofronidis as corollaries from our result.

Let $C[0,1]$ stand for the Banach space of all real valued continuous functions on $[0,1]$, with the supremum norm. The classical result of Mazurkiewicz [3, 33.9] states that the set $DIFF$ of all functions in the space $C[0,1]$ which are differentiable everywhere is $\mathbf{\Pi}_1^1$–complete, in particular non–Borel. In the recent paper [6] Sofronidis showed that the set of piecewise differentiable functions forms a $\mathbf{\Pi}_1^1$–complete set. By definition, a piecewise differentiable function has finitely many points at which derivative does not exist. In this note we study what will happen if we change in Sofronidis' theorem the statement "finitely" into "countable". Namely, Corollary 3 (iv) states that the set of all functions from $C[0,1]$, with countably many points at which the derivative does not exist, is $\mathbf{\Pi}_1^1$–complete. Our basic construction which leads to Theorem 1 mimics a technique contained in the proof of the Mazurkiewicz theorem presented in Kechris' monograph [3]. As corollaries, we will obtain the results of Mazurkiewicz and Sofronidis. The modification of the construction from [3]

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consists in using an additional parameter $d \in \{0,1\}^{\mathbb{N}}$. Thanks to this we can generate appropriate perfect sets. By a perfect set in a metric space we mean a non-empty, closed and dense-itself set.

We use standard notation. For the descriptive set–theoretical background we refer the reader to [3]. Let X be a Polish space. A subset A of X is called analytic if it is the projection of a Borel subset B of $X \times X$ onto the first factor. A subset D of X is called coanalytic if $X \setminus D$ is analytic. The pointclasses of analytic and coanalytic sets are denoted by $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$, respectively. A set $D \subset X$ is said to be $\mathbf{\Pi}_1^1$–hard if for every zero–dimensional Polish space Y and every coanalytic set $B \subset Y$ there is a continuous function $f : Y \to X$ such that $f^{-1}(D) = B$. A set is called $\mathbf{\Pi}_1^1$–complete if it is $\mathbf{\Pi}_1^1$–hard and coanalytic.

For a non-empty set A, by $A^{<\mathbb{N}}$ we denote the set of all finite sequences of elements of A together with the empty sequence $\emptyset$. For $s = (a_0, ..., a_{n-1}) \in A^{<\mathbb{N}}$ and $m \in \mathbb{N}$ such that $m < n$, let $s|m = (a_0, ..., a_{m-1})$ and $|s| = n$ (additionally $s|0 = \emptyset$ and $|\emptyset| = 0$). Analogously for an infinite sequence $\alpha \in A^{\mathbb{N}}$ let $\alpha|m = (\alpha(0), ..., \alpha(m-1))$. A set $T \subset A^{<\mathbb{N}}$ is called a tree on A if

$$\forall s \in T \forall m \in \mathbb{N}(m < |s| \Rightarrow s|m \in T).$$

For a tree T on A let $[T] = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \forall m \in \mathbb{N}(\alpha|m \in T)\}$. We say that T is well-founded if $[T] = \emptyset$. By Tr we denote the space of all trees on $\mathbb{N}$, and by WF we denote the set of all well founded trees in Tr. Identifying trees on $\mathbb{N}$ with their characteristic functions we may treat Tr as a subspace of $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ (this space is homeomorphic to the Cantor space $\{0,1\}^{\mathbb{N}}$). It is known that Tr is a closed subset of $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ (cf. [3, 4.32]). Hence Tr is a Polish space. In the sequel we will use the fact that WF is $\mathbf{\Pi}_1^1$–complete (cf. [3, 32.B]); to prove the $\mathbf{\Pi}_1^1$–hardness of a set $A \subset X$ we will define a continuous map $f : Tr \to X$ such that $f^{-1}(A) = WF$. This is the most common way to prove $\mathbf{\Pi}_1^1$–hardness; a nontrivial part of such a proof is to find a suitable continuous map.

Basic construction (cf. [3, pp. 248–251])

For an interval $K = [u, v]$, by $K^{(L)}$ and $K^{(R)}$ we denote the left half and the right half of K, respectively (i.e. $K^{(L)} = [u, \frac{1}{2}(u+v)]$ and $K^{(R)} = [\frac{1}{2}(u+v), v]$); $|J|$ is the length of the interval J; if s is a finite sequence, denote by $|s|$ the length of s. Let $Z = \{(s,d) \in \mathbb{N}^{<\mathbb{N}} \times \{0,1\}^{<\mathbb{N}} : |s| = |d|\}$ and fix a bijection $(s,d) \mapsto \langle (s,d) \rangle$ between Z and $\mathbb{N}$. For $T \in Tr$ let $Z(T) = \{(s,d) \in Z : s \in T\}$. For $(s,d) \in Z$ by $|(s,d)|$ we denote a common value of $|s|$ and $|d|$. For $f \in C[0,1]$ let $ND(f) = \{x \in [0,1] : f'(x) \text{ does not exist}\}$ (here "$f'(x)$ does not exist" means that $\lim_{y \to x} \frac{f(x)-f(y)}{x-y}$ does not exist or is infinite).

Given a closed interval $I = [a,b] \subset [0,1]$, define $\varphi(x, I) : [0,1] \to \mathbb{R}$ by the formula

$$\varphi(x,I)=\begin{cases}\frac{16(x-a)^2(x-b)^2}{(b-a)^3}, & \text{if } x\in I,\\ 0, & \text{otherwise.}\end{cases}$$

Note that $\varphi(x,I)$ is differentiable on $[0,1]$, and $\varphi(x,I)\le(b-a)=|I|$, for every $x\in[0,1]$.

Now, for each $(s,d)\in Z$, define closed intervals $J_{(s,d)}$ and $K_{(s,d)}$ as follows:

i) $K_{(s,d)}\subset J_{(s,d)}$ is concentric in $J_{(s,d)}$, and $|K_{(s,d)}|\le 2^{-\langle(s,d)\rangle-1}(|J_{(s,d)}|-|K_{(s,d)}|)$;

ii) $J_{(s\hat{\ }n,d\hat{\ }i)}\subset K^{(L)}_{(s,d)}$ for each $n\in\mathbb{N}$ and $i\in\{0,1\}$;

iii) $J_{(s\hat{\ }n,d\hat{\ }i)}\cap J_{(s\hat{\ }m,d\hat{\ }j)}=\emptyset$, if $(n,i)\ne(m,j)$.

Let $J_{(\emptyset,\emptyset)}=[0,1]$ and the further construction of the above intervals is easy to obtain by induction with respect to the length $|(s,d)|$. Given a tree T on $\mathbb{N}$, let

$$F_T(x)=\sum_{(s,d)\in Z(T)}\varphi(x,K^{(R)}_{(s,d)}),x\in[0,1].$$

Since $0\le\varphi(x,K^{(R)}_{(s,d)})\le|K^{(R)}_{(s,d)}|\le 2^{-\langle(s,d)\rangle}$, then $F_T\in C[0,1]$. We will show that $T\mapsto F_T$ is a continuous mapping from Tr to $C[0,1]$. Let $\varepsilon>0$ and let $N\in\mathbb{N}$ be such that $2^{-(N-2)}<\varepsilon$. Fix $T\in Tr$ and let $S\in Tr$ be any tree such that

$$T\cap\{s\in\mathbb{N}^{<\mathbb{N}}:\forall d\in\{0,1\}^{<\mathbb{N}}(|d|=|s|\Rightarrow\langle(s,d)\rangle<N)\}=$$
$$S\cap\{s\in\mathbb{N}^{<\mathbb{N}}:\forall d\in\{0,1\}^{<\mathbb{N}}(|d|=|s|\Rightarrow\langle(s,d)\rangle<N\}.$$

Then for any $x\in[0,1]$ we have

$$\begin{aligned}|F_T(x)-F_S(x)|\le&\sum_{(s,d)\in Z(T),\langle(s,d)\rangle\ge N}\varphi(x,K^{(R)}_{(s,d)})\\&+\sum_{(s,d)\in Z(S),\langle(s,d)\rangle\ge N}\varphi(x,K^{(R)}_{(s,d)})\\&\le\sum_{i\ge N}(2^{-i}+2^{-i})=\frac{1}{2^{N-2}}<\ \varepsilon.\end{aligned}$$

Let

$$G_T=\bigcup_{y\in[T]}\bigcap_n\bigcup_{d\in\{0,1\}^n}J_{(y|n,d)},T\in Tr.$$

Note that for every $y\in\mathbb{N}^{\mathbb{N}}$ the set $\bigcap_n\bigcup_{d\in 2^n}J_{(y|n,d)}$ is a homeomorphic image of the Cantor set. Hence for every $T\in Tr$ we have

(∗) ($T \in WF \iff G_T = \emptyset$) and ($T \notin WF \iff G_T$ contains a perfect set).

Theorem 1. *The function $T \mapsto F_T$ has the following properties:*

1) $T \in WF \iff ND(F_T) = \emptyset$;

2) $T \notin WF \iff ND(F_T)$ contains a nonempty perfect set.

PROOF. By (∗) it suffices to prove that for each $x \in [0,1]$ we have

$$x \notin G_T \iff F_T'(x) \text{ exists.}$$

If $x \in G_T$, then there are $y \in [T]$ and $z \in \{0,1\}^\omega$ such that $x \in K^{(L)}_{(y|n,d|n)}$ for all $n \in \mathbb{N}$. Let c_n be the centre of $K^{(R)}_{(y|n,d|n)}$ and let $l_n = |K^{(R)}_{(y|n,d|n)}|/2$. Then $F_T(x) = 0$ and $F_T(c_n + l_n) = 0$ for every $n \in \mathbb{N}$, so

$$\forall n \in \mathbb{N} \frac{F_T(c_n + l_n) - F_T(x)}{c_n + l_n - x} = 0.$$

On the other hand,

$$\forall n \in \mathbb{N} \frac{F_T(c_n) - F_T(x)}{c_n - x} \geq \frac{2l_n}{3l_n} = \frac{2}{3}.$$

Since $c_n \to x$, $c_n + l_n \to x$, then $F_T'(x)$ does not exist.

If $x \notin G_T$, then x is an element of at most finitely many intervals of type $J_{(s,d)}$, so there is $N \in \mathbb{N}$ such that

$$\forall (s,d) \in Z(T)(\langle (s,d) \rangle \geq N \Rightarrow x \notin J_{(s,d)}).$$

Let a pair $(s,d) \in Z(T)$ be such that $\langle (s,d) \rangle \geq N$ and let $h \in \mathbb{R} \setminus \{0\}$. Since $x \notin J_{(s,d)}$, then $\varphi(x, K^{(R)}_{(s,d)}) = 0$. If $|h| < \frac{1}{2}\left(|J_{(s,d)}| - |K_{(s,d)}|\right)$, then $x + h \notin K^{(R)}_{(s,d)}$, so $\varphi(x+h, K^{(R)}_{(s,d)}) = 0$. If $|h| \geq \frac{1}{2}\left(|J_{(s,d)}| - |K_{(s,d)}|\right)$, then

$$\begin{aligned}\left|\frac{\varphi(x+h, K^{(R)}_{(s,d)}) - \varphi(x, K^{(R)}_{(s,d)})}{h}\right| &= \frac{\varphi(x+h, K^{(R)}_{(s,d)})}{|h|} \\ &\leq \frac{2|K^{(R)}_{(s,d)}|}{|J_{(s,d)}| - |K_{(s,d)}|} \leq 2^{-\langle (s,d) \rangle}.\end{aligned}$$

For $n \geq N$ let

$$F_T^{(n)}(x) = \sum_{(s,d)\in Z(T), \langle (s,d)\rangle \leq n} \varphi(x, K^{(R)}_{(s,d)}).$$

We show that $F'_T(x)$ exists. Let $\varepsilon > 0$ and let $n \geq N$ be such that $2^{-n} < \varepsilon/2$. Let $k = \min\{|(s,d)| : (s,d) \in Z(T) \text{ and } \langle (s,d)\rangle \geq n\}$ and fix a pair $(\overline{s}, \overline{d}) \in Z(T)$ such that $|(\overline{s}, \overline{d})| = k$. Put $\overline{\delta} = |J_{(\overline{s},\overline{d})}| - |K_{(\overline{s},\overline{d})}|$. Let $|h| \in (0, \overline{\delta})$. Then we have

$$\left| \frac{F_T(x+h) - F_T(x)}{h} - \frac{F_T^{(n)}(x+h) - F_T^{(n)}(x)}{h} \right| \leq$$
$$\sum_{(s,d)\in Z(T), \langle (s,d)\rangle > n} \left| \frac{\varphi(x+h, K^{(R)}_{(s,d)}) - \varphi(x, K^{(R)}_{(s,d)})}{h} \right| \leq \sum_{j=n+1}^{\infty} 2^{-j} = 2^{-n} < \frac{\varepsilon}{2}.$$

Since $F_T^{(n)}$ is differentiable, there is $\delta \in (0, \overline{\delta}]$ such that

$$\left| \frac{F_T^{(n)}(x+h) - F_T^{(n)}(x)}{h} - \frac{F_T^{(n)}(x+h') - F_T^{(n)}(x)}{h'} \right| < \frac{\varepsilon}{2}$$

for every h, h' such that $|h|, |h'| \in (0, \delta)$. From this and the previous estimations we obtain

$$\left| \frac{F_T(x+h) - F_T(x)}{h} - \frac{F_T(x+h') - F_T(x)}{h'} \right| < \varepsilon$$

for every h, h' such that $|h|, |h'| \in (0, \delta)$. Hence $F'_T(x)$ exists and is finite. $\square$

Corollary 2. *Let $\mathcal{R}$ be a family of countable subsets of $[0,1]$ such that $\emptyset \in \mathcal{R}$. Then a set $\{f \in C[0,1] : ND(f) \in \mathcal{R}\}$ is $\mathbf{\Pi}^1_1$–hard. In particular, if this set is coanalytic, then it is $\mathbf{\Pi}^1_1$–complete.*

Corollary 3. *(i) $\{f \in C[0,1] : ND(f) = \emptyset\}$ is $\mathbf{\Pi}^1_1$–complete (Mazurkiewicz [3, 33.9]);*

(ii) $\{f \in C[0,1] : ND(f)$ is finite$\}$ is $\mathbf{\Pi}^1_1$–complete (Sofronidis [6]);

(iii) $\{f \in C[0,1] : ND(f)$ is countable$\}$ is $\mathbf{\Pi}^1_1$–complete;

(iv) $\{f \in C[0,1] : ND(f)$ is countable $G_\delta\}$ is $\mathbf{\Pi}^1_1$–complete.

PROOF. By Corollary 2 the sets in (i)–(iv) are $\mathbf{\Pi}_1^1$–hard. It remains to prove that they are coanalytic.

Let $E = \{(f, x) \in C[0,1] \times [0,1] : f'(x) \text{ does not exist}\}$. It is known that E is $\mathbf{\Sigma}_3^0$ ([3, 23.23]). The set in (i) is the complement of the projection of E onto the first axis.

The set in (ii) is the complement of the projection of a Borel set

$$\{(f, (x_n)) \in C[0,1] \times [0,1]^\omega : (\forall i \neq j) x_i \neq x_j \text{ and } \forall n(f'(x_n) \text{ does not exist})\}$$

onto the first axis.

Let $E_f = \{x \in [0,1] : (x, f) \in E\}$. We have $\{f \in C[0,1] : ND(f) \text{ is countable}\} = \{f \in C[0,1] : E_f \text{ is countable}\}$. By the Mazurkiewicz–Sierpiński theorem [3, 29.19], the set in (iii) is coanalytic.

To prove (iv) note that a countable set $A \subset [0,1]$ is G_δ if and only if it does not contain a non-empty and dense-in-itself set (see [4, pages 78, 252, 259, 417]). Moreover for every $A \subset [0,1]$ we have

$$A \text{ contains a non-empty and dense-in-itself set } \Leftrightarrow$$
$$\exists \{a_n : n \in \mathbb{N}\} \subset A \forall n, r \in \mathbb{N} \exists k \in \mathbb{N}(0 < |a_n - a_k| < \frac{1}{r+1}).$$

See that the set

$$\{(f, (x_n)) \in C[0,1] \times [0,1]^{\mathbb{N}} :$$
$$\forall n, r \in \mathbb{N} \exists k \in \mathbb{N}(0 < |x_n - x_k| < \frac{1}{r+1} \wedge (f, x_n) \notin E)\}$$

is Borel. Hence the set

$$\{f \in C[0,1] : ND(f) \text{ contains a non-empty and dense-in-itself set } \} =$$
$$\{f \in C[0,1] : \exists (x_n) \in [0,1]^{\mathbb{N}} :$$
$$\forall n, r \in \mathbb{N}\ \exists k \in \mathbb{N}(0 < |x_n - x_k| < \frac{1}{r+1} \wedge (f, x_n) \notin E)\}$$

is analytic. From this and (iii) we obtain that the set in (iv) is coanalytic. $\square$

Now we will describe the idea of another proof of Theorem 1. To do this we define a special class of trees on $\mathbb{N}$. For $s, t \in \mathbb{N}^{<\mathbb{N}}$ such that $|s| = |t|$ and for $n \in \mathbb{N}$ we define $s+t$ and ns in the following natural way: $(s+t)(k) = s(k)+t(k)$

and $(ns)(k) = ns(k)$ for $k \in \mathbb{N}$, $k < |s|$. Analogously we define $\alpha + \beta$ and $n\alpha$ for infinite sequences $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. Then we define $H : Tr \to Tr$ by

$$H(T) = \{2s + \varepsilon : s \in T \text{ and } \varepsilon \in \{0,1\}^{|s|}\}, T \in Tr.$$

Put $Tr^* = H(Tr)$. Since $T \in Tr^*$ if and only if $\forall s \in \mathbb{N}^{<\mathbb{N}}$ $[2s \in T \Rightarrow \forall \varepsilon \in \{0,1\}^{|s|}$ $(2s + \varepsilon \in T)]$, then Tr^* is a closed subset of Tr. Hence it is a Polish subspace of Tr and the trees from Tr^* have the property

$$[T] \neq \emptyset \iff [T] \text{ contains a perfect set.}$$

The implication "$\Leftarrow$" is obvious. To prove "$\Rightarrow$" suppose that $T \in Tr^*$ is such that $[T] \neq \emptyset$. Then there exists a tree $S \in Tr$ such that $T = H(S)$. Let $x \in [T]$. Then $x|n = 2s^{(n)} + \varepsilon^{(n)}$ where $s^{(n)} \in S$ and $\varepsilon^{(n)} \in \{0,1\}^n$ for every $n \in \mathbb{N}$. Let $y \in \mathbb{N}^{\mathbb{N}}$ be such that $y|n = s^{(n)}$ for each $n \in \mathbb{N}$. Then $y \in [S]$ and for every $z \in \{0,1\}^{\mathbb{N}}$ we have $2y + z \in [T]$. This shows that $[T]$ is a perfect set, since it is closed (see [3, 2.4]) and for every $n \in \mathbb{N}$ the set $[T]$ contains, together with a point x, a point $2x + z$ such that $z|n = \varepsilon^{(n)}$, $z(n) = 1 - \varepsilon^{(n+1)}(n)$. Let $WF^* = WF \cap Tr^*$. Clearly H is a continuous map. Hence WF^* is $\mathbf{\Pi}^1_1$–complete.

Now we modify a little bit the proof of the Mazurkiewicz theorem from [3] to obtain Theorem 1. Let $T \mapsto \Phi_T$ be a continuous map from Tr to $C[0,1]$ described in [3, 33.9] which witnesses that $DIFF$ is $\mathbf{\Pi}^1_1$–complete (this map is similar to our function $T \mapsto F_T$ defined above, but in its construction we do not use a parameter d). Let $T \in Tr$. With every sequence $\alpha \in [T]$ there is attached a point x_α such that there is no finite derivative $\Phi'_T(x_\alpha)$. Moreover, for distinct sequences α, $\beta \in [T]$ we have $x_\alpha \neq x_\beta$. On the other hand if $[T] = \emptyset$, then Φ_T has a derivative at every point. Then for $T \in Tr^*$ we have

$$[T] \neq \emptyset \iff |\{x \in [0,1] : \Phi'_T(x) \text{ does not exist}\}| > \omega.$$

Hence the function $T \mapsto \Phi_T$ on Tr^* has the same properties as the function $T \mapsto F_T$ from Theorem 1.

Many examples of $\mathbf{\Pi}^1_1$–complete sets (included the most of such examples from [3]) have the following form:

$$\{\text{objects with no singularity points}\}$$

(cf [1]). Examples of objects are the following: continuous functions on $[0,1]$, continuous function on $\mathbb{T}$ (where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) or homeomorphisms of a compact space, and singularity points can be respectively: points with no finite derivative (cf. the Mazurkiewicz theorem), points at which Fourier series are not convergent (cf. [3, 33.13]) or points with infinite orbits (cf. [3, 33.20]). The standard way of proving the $\mathbf{\Pi}_1^1$-completeness of coanalytic sets of this type is to find a suitable map G from Tr to a given space with the following properties:

(a) if $[T] = \emptyset$, then $G(T)$ has no singularity points;

(b) if $\alpha \in [T]$, then there is x_α such that it is a singularity point of $G(T)$;

(c) if $\alpha, \beta \in [T]$ are distinct sequences, then x_α and x_β are also distinct.

Note that the condition (c) is not necessary for proving $\mathbf{\Pi}_1^1$–completeness, but if it holds, then G has the property

$$\forall T \in Tr^*([T] \neq \emptyset \Leftrightarrow G(T) \text{ has uncountably many singularity points}).$$

At the end we give one application of this reasoning. The analysis of the proof of Theorem [3, 33.11] gives us the following

Corollary 4. *The set* $\{(f_n) \in (C[0,1])^{\mathbb{N}} : (f_n)$ *converges pointwise on cocountable subset of* $[0,1]\}$ *is* $\mathbf{\Pi}_1^1$*–complete.*

PROOF. It is enough to see that the given set is coanalytic. It is known [3, 23.18] that the set

$$E = \{((f_n), x) \in (C[0,1])^{\mathbb{N}} \times [0,1] : (f_n(x)) \text{ is not pointwise convergent}\}$$

is Borel. Let $E_{(f_n)} = \{x \in [0,1] : ((f_n), x) \in E\}$. Then by the Mazurkiewicz–Sierpiński theorem [3, 29.19], the set

$$\{(f_n) \in (C[0,1])^{\mathbb{N}} : (f_n) \text{ converges pointwise on cocountable subset of } [0,1]\}$$
$$= \{(f_n) \in (C[0,1])^{\mathbb{N}} : E_{(f_n)} \text{ is countable}\}$$

is coanalytic. □

Finally, we want to mention some other remarkable results on $\mathbf{\Pi}_1^1$–complete subsets of $C[0,1]$. In [5], it is shown that the set of all continuous functions which do not have a finite derivative anywhere is a $\mathbf{\Pi}_1^1$–complete set (a different proof due to Kechris can be found in [3]). Mauldin also proved (see [3, Remark on page 255], this is an unpublished note) that the set of all Besicovitch functions, i.e. those continuous functions which have neither finite nor infinite one-sided derivative at any point, is a $\mathbf{\Pi}_1^1$–complete set. It would be interesting to find out if, the set of continuous functions which do not have a finite derivative anywhere except for a countable set, and the set of continuous functions that are Besicovitch ones except for a countable set, are also $\mathbf{\Pi}_1^1$–complete. Unfortunately, the technique of proving the $\mathbf{\Pi}_1^1$–completeness used by Mauldin (and Kechris) is not that of finding a reduction to the set of well-founded trees and we cannot use our general argument.

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