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THE SMOOTHNESS OF FUNCTIONS OF TWO VARIABLES AND DOUBLE TRIGONOMETRIC SERIES

Abstract

The notion of smoothness (according to Riemann) is introduced for functions of two variables and some of their properties are established. As an application we prove the uniform smoothness of an everywhere continuous sum of a double trigonometric series in the complex form which is obtained by twice term-by-term integration, over every variable rectangle $[0, x] \times [0, y] \subset [0, 2\pi] \times [0, 2\pi]$ of a double trigonometric series in the complex form absolutely converging at some point. An analogous consideration is given to a double trigonometric series in the real form, the absolute values of whose coefficients form a converging series.

0 Introduction.

0A. According to Riemann, a function $\varphi(x)$ defined in the neighborhood of a point $x_0 \in \mathbb{R}$ is called smooth at x_0 (the term was introduced by Zygmund [5]) if the equality

$$\lim_{h \to 0} \frac{\varphi(x_0 + h) + \varphi(x_0 - h) - 2\varphi(x_0)}{h} = 0 \tag{0.1}$$

is fulfilled.

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Riemann showed that the twice term-by-term integration of a trigonometric series in real form with coefficients converging to zero gives a function that satisfies equality (0.1) for all $x_0 \in \mathbb{R}$ ([4, p. 245], even uniformly [1, p. 184], [5], [6, p. 320]).

A detailed investigation of Riemann-smooth functions with various applications to different classes of functions and to trigonometric series was carried out by Zygmund [5].

0B. Throughout the paper we will use the notions of angular and strong gradients.

Definition 0.1 ([2, p. 71])**.** A finite function F in a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ has the angular partial derivative $F'_{\widehat{x}}(x_0, y_0)$ at (x_0, y_0) with respect to x if for every constant $c > 0$ there exists the limit

$$F'_{\widehat{x}}(x_0, y_0) = \lim_{\substack{h \to 0 \\ |k| \le c|h|}} \frac{F(x_0 + h, y_0 + k) - F(x_0, y_0 + k)}{h} \tag{0.2}$$

and this limit does not depend on c. Analogously,

$$F'_{\widehat{y}}(x_0, y_0) = \lim_{\substack{k \to 0 \\ |h| \le l|k|}} \frac{F(x_0 + h, y_0 + k) - F(x_0 + h, y_0)}{k}, \quad l > 0. \tag{0.3}$$

The angular gradient of the function F at the point (x_0, y_0) is

$$\operatorname{anggrad} F(x_0, y_0) = \big(F'_{\widehat{x}}(x_0, y_0), F'_{\widehat{y}}(x_0, y_0)\big). \tag{0.4}$$

Definition 0.2 ([2, p. 79])**.** If the relations from (0.2) and (0.3) have the limits as $(h, k) \to (0, 0)$, then we call them the strong partial derivatives of the function F at (x_0, y_0) with respect to x and y and denote them by $F'_{[x]}(x_0, y_0)$ and $F'_{[y]}(x_0, y_0)$, respectively.

The strong gradient of the function F at (x_0, y_0) is

$$\operatorname{strgrad} F(x_0, y_0) = \Big(F'_{[x]}(x_0, y_0), F'_{[y]}(x_0, y_0)\Big). \tag{0.5}$$

1 The Smoothness of Functions of Two Variables.

1A. Let us begin with some preliminary material.

Definition 1.1. A function $f(x,y)$ defined in a neighborhood of a point $(x_0,y_0)\in\mathbb{R}^2$ is called smooth at the point (x_0,y_0) if the following equality is fulfilled:

$$\lim_{(h,k)\to(0,0)}\frac{f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)}{|h|+|k|}=0. \tag{1.1}$$

If $f(x,y)$ is a smooth function at every point from some open set $E\subset\mathbb{R}^2$, then f is called smooth in E. If f is continuous and satisfies condition (1.1) uniformly with respect to all points $(x_0,y_0)\in E$, then f is called uniformly smooth in E.

It is obvious that:

1) if a function $f(x,y)$ is smooth at a point (x_0,y_0), then the partial functions $f(x,y_0)$ and $f(x_0,y)$ of one variable are smooth at the points x_0 and y_0, respectively.

2) If the functions $a(x)$ and $b(y)$ are smooth at the points x_0 and y_0, respectively, then their sum $f(x,y)=a(x)+b(y)$ is smooth at (x_0,y_0).

Theorem 1.2. *A function of two variables which is differentiable at some point is smooth at the same point. The converse statement is not true.*

PROOF. Let a function $f(x,y)$ be differentiable at a point (x_0,y_0). Then we have the equality

$$\lim_{(h,k)\to(0,0)}\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-Ah-Bk}{|h|+|k|}=0, \tag{1.2}$$

where the coefficients A and B of the differential $df(x_0,y_0)=Ah+Bk$ are equal to the angular partial derivatives of the function $f(x,y)$ at the point (x_0,y_0) with respect to the variables x and y ([2, p. 71]):

$$A=f'_{\widehat{x}}(x_0,y_0),\quad B=f'_{\widehat{y}}(x_0,y_0). \tag{1.3}$$

The desired statement follows from the equality

$$\begin{aligned}\frac{f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)}{|h|+|k|}&\\ =\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-Ah-Bk}{|h|+|k|}&\\ +\frac{f(x_0-h,y_0-k)-f(x_0,y_0)-A(-h)-B(-k)}{|-h|+|-k|}&,\end{aligned}$$

because its right-hand part tends to zero as $(h,k)\to(0,0)$.

To verify that the converse statement is not true, it is sufficient to take the continuous and smooth on $(0,2\pi)$ functions $\alpha(x)$ and $\beta(y)$ which are differentiable only on a set of measure zero ([6, p. 48]). Then the function $\varphi(x,y)=\alpha(x)+\beta(y)$ is continuous and smooth on the rectangle $(0,2\pi)\times(0,2\pi)$, but it is nondifferentiable almost everywhere on $(0,2\pi)\times(0,2\pi)$. □

Corollary 1.3. *If the functions $a(x)$ and $b(y)$ are differentiable at the points x_0 and y_0, respectively, then the product $\phi(x,y)=a(x)\cdot b(y)$ is a smooth function at the point (x_0,y_0).*

Corollary 1.4. *The smoothness of a function $f(x,y)$ at a point (x_0,y_0) takes place when one of the following sufficient conditions for f to be differentiable at (x_0,y_0) is fulfilled: the angular gradient is finite, the gradient is continuous, the strong gradient is finite, of the partial derivatives one is finite and the other is continuous* ([2, pp. 71–74 and 79–80]).

1B. In the class of almost everywhere differentiable functions of two variables we distinguish an important subclass of functions, every one of which is everywhere continuous and is smooth almost everywhere on a rectangle. Namely, we have

Theorem 1.5. *Let a function $f(x,y)$ be summable on the rectangle $[a,b]\times[c,d]$. Then the function*

$$F(x,y)=\int_a^x\int_c^y f(t,\tau)\,dt\,d\tau \tag{1.4}$$

is continuous everywhere and is smooth at almost all interior points (x_0,y_0) of this rectangle.

PROOF. We obtain the proof from Theorem 1.2 and the fact that the function F is differentiable at almost all points of the rectangle ([2, p. 102]). □

This statement can be supplemented in the following manner.

Theorem 1.6. *Let (x_0,y_0) be the point at which the function F defined by equality* (1.4) *is differentiable. Then the function*

$$\Phi(x,y)=\int_{x_0}^x\int_{y_0}^y f(t,\tau)\,dt\,d\tau \tag{1.5}$$

is smooth at the point (x_0,y_0).

PROOF. We have the equality

$$\Phi(x_0 + h, y_0 + k) + \Phi(x_0 - h, y_0 - k) - 2\Phi(x_0, y_0) = \int_{x_0}^{x+h} \int_{y_0}^{y+k} f(t,\tau)\, dt\, d\tau + \int_{x_0}^{x-h} \int_{y_0}^{y-k} f(t,\tau)\, dt\, d\tau. \tag{1.6}$$

If both parts of this equality are divided by the sum $|h|+|k|$, then in the right-hand part we obtain the sum of two integrals with coefficients $(|h|+|k|)^{-1}$ and $(|-h|+|-k|)^{-1}$ and this sum tends to zero as $(h,k) \to (0,0)$ ([2, p. 104]). Therefore the function $\Phi(x,y)$ is smooth at (x_0, y_0). □

1C. In case we wish to have a condition on a function in the neighborhood of some points such that this function is smooth at this point, we can use

Theorem 1.7. *Assume that a function $f(x,y)$ in the neighborhood of a point (x_0, y_0) satisfies the condition*

$$|f(x_0 + u, y_0 + v) + f(x_0 - u, y_0 - v)| \le M, \tag{1.7}$$

in particular, if f is bounded near the point (x_0, y_0). Then the function $\Phi(x,y)$ defined by equality (1.5) *is smooth at the point (x_0, y_0).*

PROOF. Let us write the right-hand part of equality (1.6) in the form

$$\int_0^h \int_0^k [f(x_0 + t, y_0 + \tau) + f(x_0 - t, y_0 - \tau)]\, dt\, d\tau.$$

Therefore

$$\frac{|\Phi(x_0 + h, y_0 + k) + \Phi(x_0 - h, y_0 - k) - 2\Phi(x,y)|}{|h| + |k|} \le M \frac{|h|\,|k|}{|h| + |k|} \to 0, \quad (h,k) \to (0,0).$$

Thus the function Φ is smooth at the point (x_0, y_0). □

2 The Differentiability of a Smooth Function of Two Variables at a Point of Extremum.

Though a function of two variables at the point of smoothness may be nondifferentiable (see Section 1), there may nevertheless occur a case where smoothness implies differentiability.

Theorem 2.1. *If a smooth function $f(x,y)$ at a point (x_0,y_0) has a maximum or a minimum at (x_0,y_0), then $f(x,y)$ at (x_0,y_0) has zero angular partial derivatives $f'_{\widehat{x}}(x_0,y_0)=0$, $f'_{\widehat{y}}(x_0,y_0)=0$ and therefore $df(x_0,y_0)=0$.*

PROOF. We have the equality

$$\begin{aligned}\frac{f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)}{|h|+|k|}\\ =\frac{f(x_0+h,y_0+k)-f(x_0,y_0)}{|h|+|k|}+\frac{f(x_0-h,y_0-k)-f(x_0,y_0)}{|h|+|k|},\end{aligned}\tag{2.1}$$

the left-hand part of which tends to zero as $(h,k)\to(0,0)$ because f is smooth at (x_0,y_0). Therefore the sum of two summands in the right-hand part of equality (2.1) tends to zero as $(h,k)\to(0,0)$.

But both summands are of the same sign for h and k sufficiently small. Therefore each of the two summands tends to zero as $(h,k)\to(0,0)$. Hence in particular we obtain the equality $f'_x(x_0,y_0)=0$ for $k=0$ and the equality $f'_y(x_0,y_0)=0$ for $h=0$.

Further, keeping in mind that the first summand tends to zero, using the equality

$$\lim_{(h,k)\to(0,0)}\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)+f(x_0,y_0+k)-f(x_0,y_0)}{|h|+|k|}=0,$$

we obtain by virtue of $f'_y(x_0,y_0)=0$

$$\lim_{(h,k)\to(0,0)}\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{|h|+|k|}=0.\tag{2.2}$$

Let now $c>0$ be an arbitrary constant and assume that $h\to0$ and $|k|\le c|h|$. Then, writing $|h|+|k|=|h|(1+|k|/|h|)\le|h|(1+c)$, from (2.2) we have

$$\lim_{\substack{(h,k)\to(0,0)\\ |k|\le c|h|}}\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h}=0.$$

This means that $f'_{\widehat{x}}(x_0,y_0)=0$. In an analogous manner we obtain $f'_{\widehat{y}}(x_0,y_0)=0$ and therefore $df(x_0,y_0)=0$. □

3 The Smoothness and Symmetrical Differentiability of Functions of Two Variables.

From the differentiability of a function of two variables we have its symmetrical differentiability without the converse statement [3]. Let us now prove that the symmetrical differentiability implies that the considered function is differentiable when it is smooth.

Definition 3.1 ([3]). A function $\varphi(x, y)$ is called symmetrically differentiable at a point (x_0, y_0) if there exist finite constants A and B with the property

$$\lim_{(h,k)\to(0,0)} \frac{\varphi(x_0 + h, y_0 + k) - \varphi(x_0 - h, y_0 - k) - 2Ah - 2Bk}{|h| + |k|} = 0. \qquad (3.1)$$

Proposition 3.2. *Let a function $f(x, y)$ be smooth at a point (x_0, y_0). Then for f to be differentiable at (x_0, y_0) it is necessary and sufficient that f be symmetrically differentiable at (x_0, y_0).*

PROOF. We obtain the proof from the equality

$$\begin{aligned} &\frac{f(x_0 + h, y_0 + k) + f(x_0 - h, y_0 - k) - 2f(x_0, y_0)}{|h| + |k|} \\ &\quad + \frac{f(x_0 + h, y_0 + k) - f(x_0 - h, y_0 - k) - 2Ah - 2Bk}{|h| + |k|} \\ &\quad = 2\,\frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - Ah - Bk}{|h| + |k|}\,. \end{aligned} \qquad (3.2)$$

□

Corollary 3.3. *The everywhere smooth and almost everywhere nondifferentiable function $\varphi(x, y)$ indicated in the proof of Theorem 1.2 is even symmetrically nondifferentiable almost everywhere.*

4 The Smoothness and Unilateral Differentiability of Functions of Two Variables.

Let $U(O)$ and $U^0(O) = U(O) \setminus \{O\}$ denote the neighborhood and the punctured neighborhood of the point $O = (0,0)$. We use the following sets ([2, p. 43]):

$$A_1^+ = \{(h,k) \in U(O) : h > 0\}, \quad A_2^+ = \{(0,k) \in U(O) : k > 0\},$$
$$A_1^- = \{(h,k) \in U(O) : h < 0\}, \quad A_2^- = \{(0,k) \in U(O) : k < 0\},$$
$$A_{12}^+ = A_1^+ \cup A_2^+, \quad A_{12}^- = A_1^- \cup A_2^-.$$

It is obvious that $A_{12}^+ \cap A_{12}^- = \varnothing$ and $A_{12}^+ \cup A_{12}^- = U^0(O)$.

Let us introduce the following two definitions.

Definition 4.1. A function $f(x,y)$ is called right-differentiable at the point $p_0 = (x_0, y_0)$ if the equality

$$\lim_{\substack{(h,k)\to(0,0)\\ (h,k)\in A_{12}^+}} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - A^+h - B^+k}{|h|+|k|} = 0 \tag{4.1}$$

is fulfilled for some finite numbers A^+ and B^+, and the linear function $A^+h + B^+k$ for $(h,k) \in A_{12}^+$ is called a right-differential of f at the point p_0, denoted by $d^+f(p_0)$ and we write

$$d^+f(p_0) = A^+h + B^+k. \tag{4.2}$$

Definition 4.2. A function $f(x,y)$ is called left-differentiable at the point $p_0 = (x_0, y_0)$ if there exist finite numbers A^- and B^- such that the equality

$$\lim_{\substack{(h,k)\to(0,0)\\ (h,k)\in A_{12}^-}} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - A^-h - B^-k}{|h|+|k|} = 0 \tag{4.3}$$

is fulfilled, and the linear function $A^-h + B^-k$ is called a left-differential of f at the point p_0, denoted by $d^-f(p_0)$, for $(h,k) \in A_{12}^-$, and we write

$$d^-f(p_0) = A^-h + B^-k. \tag{4.4}$$

The next two propositions are obvious.

Proposition 4.3. *A differentiable at a point p_0 function $f(x,y)$ is bilaterally differentiable at p_0 and the equalities $d^+f(p_0) = df(p_0)$, $d^-f(p_0) = df(p_0)$,*

$$A^+ = A^- = f'_{\widehat{x}}(p_0), \quad B^+ = B^- = f'_{\widehat{y}}(p_0) \tag{4.5}$$

are fulfilled.

Proposition 4.4. *If a function $f(x,y)$ is bilaterally differentiable at a point p_0 and the equalities $A^+ = A^-$ and $B^+ = B^-$ are fulfilled, then f is differentiable at p_0 and*

$$A^+ = f'_{\widehat{x}}(p_0) = A^-, \quad B^+ = f'_{\widehat{y}}(p_0) = B^-. \tag{4.6}$$

We have

Theorem 4.5. *A smooth at some point $p_0 = (x_0, y_0)$ function $f(x,y)$ is differentiable at p_0 if and only if it is unilaterally differentiable at the point p_0.*

PROOF. For every finite constants A and B we have

$$\begin{aligned}
&\frac{f(x_0+h, y_0+k) + f(x_0-h, y_0-k) - 2f(x_0,y_0)}{|h|+|k|} \\
&\quad = \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - Ah - Bk}{|h|+|k|} \\
&\qquad + \frac{f(x_0-h, y_0-k) - f(x_0,y_0) - A(-h) - B(-k)}{|-h|+|-k|}.
\end{aligned} \tag{4.7}$$

The left-hand part of equality (4.7) tends to zero as $(h,k) \to (0,0)$, in particular for $(h,k) \in A^+_{12}$, since the function f is smooth at (x_0, y_0). Furthermore, the first summand in the right-hand part of (4.7) tends to zero as $(h,k) \to (0,0)$ under the condition $(h,k) \in A^+_{12}$ in the case of right-hand differentiability of f at (x_0, y_0). Therefore the second summand in the right-hand part of equality (4.7) also tends to zero as $(h,k) \to (0,0)$ provided that $(h,k) \in A^+_{12}$. This means that f is left-hand differentiable at the point (x_0, y_0) since $(-h,-k) \in A^-_{12}$. Thus we obtain the equalities $d^+ f(p_0) = Ah + Bk$ and $d^- f(p_0) = Ah + Bk$. Hence it follows that the function f is differentiable at the point p_0 by virtue of Proposition 4.4 and we have the equality $df(p_0) = f'_{\widehat{x}}(p_0)\,dx + f'_{\widehat{y}}(p_0)\,dy$. As a result we obtain

$$d^+ f(p_0) = f'_{\widehat{x}}(p_0)\,dx + f'_{\widehat{y}}(p_0)\,dy, \tag{4.8}$$

$$d^- f(p_0) = f'_{\widehat{x}}(p_0)\,dx + f'_{\widehat{y}}(p_0)\,dy. \tag{4.9}$$

The theorem is proved. □

5 Smoothness and Separately Smoothness in an Angular Sense.

The differentiability of a function of two variables, generally speaking, of a function of many variables is equivalent to the finiteness of its angular partial derivatives ([2, pp. 62, 65, 71]). Therefore it is natural to investigate the smoothness of a function of two variable by an analogous method. To this end, we introduce some definitions.

Definition 5.1. A function $f(x,y)$ is called smooth in an angular sense with respect to the variable x at a point $p_0=(x_0,y_0)$ if the equality

$$\lim_{\substack{h\to 0\\ |k|\le c|h|}} \frac{f(x_0+h,y_0+k)+f(x_0-h,y_0+k)-2f(x_0,y_0)}{h}=0 \tag{5.1}$$

is fulfilled for all constants $c>0$.

Definition 5.2. A function $f(x,y)$ is called smooth in an angular sense with respect to a variable y at the point $p_0(x_0,y_0)$ if for all constants $\ell>0$ we have

$$\lim_{\substack{k\to 0\\ |k|\ge \ell|h|}} \frac{f(x_0+h,y_0+k)+f(x_0+h,y_0-k)-2f(x_0,y_0)}{k}=0. \tag{5.2}$$

If a function $f(x,y)$ satisfies the equalities (5.1) and (5.2) simultaneously, then f is called smooth in an angular sense at the point p_0 with respect to each variable or, which is the same, in an angular sense separately smooth at p_0.

Definition 5.3 ([3, p. 109])**.** The limit

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0-h,y_0+k)}{2h}=f^{(\prime)}_{[x]}(p_0) \tag{5.3}$$

(respectively the limit

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0+h,y_0-k)}{2k}=f^{(\prime)}_{[y]}(p_0)\ \Big) \tag{5.4}$$

is called a symmetrical strong partial derivative of a function $f(x,y)$ with respect to x (with respect to y) at the point $p_0=(x_0,y_0)$, denoted by $f^{(\prime)}_{[x]}(p_0)$ (respectively by $f^{(\prime)}_{[y]}(p_0)$).

Remark 5.4. 1) The finiteness of the symmetrical strong partial derivatives (5.3) and (5.4) implies the symmetrical differentiability (see Definition 3.1) of a function $f(x, y)$ at the point p_0 without the converse statement ([3, p. 110]); 2) a function $f(x, y)$ has, at the point p_0, the following, noncomparable in general, properties: 2a) the differentiability (see equalities (1.2) and (1.3)) and finiteness of the symmetrical strong partial derivatives (5.3) and (5.4) ([3, p. 111]), 2b) the symmetrical differentiability and finiteness of symmetrical angular partial derivatives ([3, p. 113]).

Theorem 5.5. *The smoothness of a function $f(x, y)$ at a point $p_0 = (x_0, y_0)$ implies its smoothness in an angular sense at p_0 with respect to the variable x provided that*

$$f^{(\prime)}_{[y]}(p_0) = 0. \tag{5.5}$$

Proof. We have the equality

$$\begin{aligned} &f(x_0 + h, y_0 + k) + f(x_0 - h, y_0 + k) - 2f(x_0, y_0) \\ &\quad = [f(x_0 + h, y_0 + k) + f(x_0 - h, y_0 - k) - 2f(x_0, y_0)] \\ &\qquad + [f(x_0 - h, y_0 + k) - f(x_0 - h, y_0 - k)]\,. \end{aligned} \tag{5.6}$$

The smoothness of the function $f(x, y)$ at the point p_0 and the fulfilment of equality (5.5) imply that for every number $\varepsilon^* > 0$ there exists a number $\delta^* = \delta^*(p_0, \varepsilon) > 0$ with the properties

$$|f(x_0 + h, y_0 + k) - f(x_0 - h, y_0 - k) - 2f(x_0, y_0)| < \varepsilon^*(|h| + |k|), \tag{5.7}$$

$$|f(x_0 - h, y_0 + k) - f(x_0 - h, y_0 - k)| < \varepsilon^*|k| \tag{5.8}$$

provided that $|h| < \delta^*$ and $|k| < \delta^*$. Therefore from (5.6) we obtain

$$\begin{aligned} &|f(x_0 + h, y_0 + k) + f(x_0 - h, y_0 + k) - 2f(x_0, y_0)| \\ &\qquad < 2\varepsilon^*(|h| + |k|), \quad |h| < \delta^*, \quad |k| < \delta^*. \end{aligned} \tag{5.9}$$

Hence it follows that

$$\begin{aligned} &\left| \frac{f(x_0 + h, y_0 + k) + f(x_0 - h, y_0 + k) - 2f(x_0, y_0)}{h} \right| \\ &\qquad < 2\varepsilon^* \left(1 + \frac{|k|}{|h|} \right), \quad |h| < \delta^*, \quad |k| < \delta^*. \end{aligned} \tag{5.10}$$

Let $c > 0$ be any number and $\varepsilon > 0$ be an arbitrarily small number. Assume that the point (h, k) tends to the point $(0, 0)$ when the condition $|k| \le c|h|$ is fulfilled. Now putting $\varepsilon^* = \frac{\varepsilon}{2(1+c)}$ and $\delta = \delta(\varepsilon, p_0, c) > 0$ in inequality (5.10), we obtain

$$\left|\frac{f(x_0+h, y_0+k) + f(x_0-h, y_0+k) - 2f(x_0, y_0)}{h}\right| < \varepsilon, \\ |h| < \delta, \quad |k| < \delta, \quad |k| \le c|h|. \tag{5.11}$$

This means the fulfilment of equality (5.1); i.e., the function $f(x, y)$ is smooth in an angular sense at the point p_0 with respect to the variable x. □

The next statement is proved analogously.

Theorem 5.6. *The smoothness of a function $f(x, y)$ at a point $p_0 = (x_0, y_0)$ implies its smoothness in an angular sense at p_0 with respect to the variable y provided that*

$$f_{[x]}^{(\prime)}(p_0) = 0. \tag{5.12}$$

Finally, let us prove the following theorem.

Theorem 5.7. *Let a function $f(x, y)$ defined in a neighborhood of a point $p_0 = (x_0, y_0)$ satisfy conditions* (5.5) *and* (5.12)*. Then for f to be smooth at p_0 it is necessary and sufficient that it is separately smooth in an angular sense at the point p_0.*

PROOF. The necessity follows from Theorems 5.5 and 5.6. We begin proving the sufficiency with the fact that for the particular case $c = 1$, inequality (5.11) takes the form

$$|f(x_0+h, y_0+k) + f(x_0-h, y_0+k)| < \varepsilon|h|, \\ |h| < \delta, \quad |k| < \delta, \quad |k| \le |h|. \tag{5.13}$$

Since $\varepsilon^* < \varepsilon$, from inequality (5.8) it follows that

$$|f(x_0-h, y_0+k) - f(x_0-h, y_0-k)| < \varepsilon|k|, \quad |k| < \delta. \tag{5.14}$$

Inequalities (5.13) and (5.14) obviously imply

$$|f(x_0+h, y_0+k) + f(x_0-h, y_0+k) - 2f(x_0, y_0)| \\ < \varepsilon(|h| + |k|), \quad |h| < \delta, \quad |k| < \delta, \quad |k| \le |h|, \tag{5.15}$$

$$|f(x_0-h, y_0+k) - f(x_0-h, y_0-k)| < \varepsilon(|h| + |k|), \quad |k| < \delta. \tag{5.16}$$

Applying inequalities (5.15) and (5.16) to the equality (5.6) we obtain

$$|f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)|<2\varepsilon(|h|+|k|),\quad |h|<\delta,\ \ |k|<\delta,\ \ |k|\le|h|. \tag{5.17}$$

By a similar reasoning, using equality (5.12) and the fact that the function f is smooth in an angular sense at p_0 with respect to the variable y, we obtain the inequality

$$|f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)|<2\varepsilon(|h|+|k|),\quad |h|<\delta,\ \ |k|<\delta,\ \ |k|\ge|h|. \tag{5.18}$$

It is obvious that the set $\{(h,k):|h|<\delta,\ |k|<\delta,\ |k|\le|h|\}\cup\{(h,k):|h|<\delta,\ |k|<\delta,\ |k|\ge|h|\}$ is the δ-neighborhood of the point p_0. Therefore from inequalities (5.17) and (5.18) it follows that

$$|f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)|<2\varepsilon(|h|+|k|),\quad |h|<\delta,\ \ |k|<\delta. \tag{5.19}$$

Thus equality (1.1) is fulfilled; i.e., the function $f(x,y)$ is smooth at the point p_0. □

6 The Smoothness of Sums of Double Trigonometric Series.

6A. The complex case. Assume that there is a double trigonometric series in the complex form

$$\sum_{m,n=-\infty}^{\infty} c_{mn}e^{i(mx+ny)}, \tag{6.1}$$

which we rewrite as

$$c_{00}+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} c_{m0}e^{imx}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} c_{0n}e^{iny}+\sum_{\substack{m,n=-\infty\\ m\cdot n\neq 0}}^{\infty} c_{mn}e^{i(mx+ny)} \equiv c_{00}+A(x)+B(y)+C(x,y). \tag{6.2}$$

It is assumed that series (6.1) absolutely converges at some point of the square $I = [0, 2\pi] \times [0, 2\pi]$; i.e., it is assumed that there exists the finite limit[1]

$$L = \lim_{\substack{M\to\infty \\ N\to\infty}} \sum_{m=-M}^{M} \sum_{n=-N}^{N} |c_{mn}| = \sum_{m,n=-\infty}^{\infty} |c_{mn}|. \tag{6.3}$$

By twice term-by-term integration of expression (6.2) over every variable rectangle $[0, x] \times [0, y] \subset I$ we obtain the expressions

$$\frac{c_{00}}{4}\, x^2 y^2 \ \text{ from } \ c_{00}, \tag{6.4}$$

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} c_{m0} \left[\frac{y^2}{m^2} - \frac{y^2}{m^2}\, e^{imx} + \frac{i}{m}\, xy^2\right] \ \text{ from } \ A(x), \tag{6.5}$$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_{0n} \left[\frac{x^2}{n^2} - \frac{x^2}{n^2}\, e^{iny} + \frac{i}{n}\, x^2 y\right] \ \text{ from } \ B(y), \tag{6.6}$$

$$\begin{aligned} &\sum_{\substack{m,n=-\infty \\ m\cdot n\neq 0}}^{\infty} c_{mn} \left[\frac{e^{i(mx+ny)}}{m^2n^2} - \frac{e^{imx}}{m^2n^2} - \frac{e^{iny}}{m^2n^2} - \frac{iye^{imx}}{m^2 n}\right. \\ &\left. -\frac{ixe^{iny}}{mn^2} + \frac{ix}{mn^2} + \frac{iy}{m^2 n} - \frac{xy}{mn} + \frac{1}{m^2n^2}\right] \ \text{ from } \ C(x,y). \end{aligned} \tag{6.7}$$

It is obvious that in these expressions, in the square brackets we have everywhere differentiable functions of one or two variables as summands.

It can be verified that these functions satisfy equalities (0.1) or (1.1) uniformly with respect to $(x, y) \in I$ and also uniformly with respect to m and n.

Let us see whether this property is true for some typical functions.

1) For the function $\nu_{mn}(x, y) = e^{i(mx+ny)}$ we have

$$\begin{aligned} &\nu_{mn}(x+h, y+k) + \nu_{mn}(x-h, y-k) - 2\nu_{mn}(x, y) \\ &\qquad = 2e^{i(mx+ny)} [\cos(mh+nk) - 1] = -4e^{i(mx+ny)} \sin^2 \frac{1}{2}(mh+nk). \end{aligned}$$

Therefore for the function

$$\Phi(x, y) = \sum_{\substack{m,n=-\infty \\ m\cdot n\neq 0}}^{\infty} \frac{c_{mn}}{m^2n^2}\, e^{i(mx+ny)} \tag{6.8}$$

[1] In the author's opinion, the assumption of absolute converging is caused by those radical difficulties which in the past 40 years had been revealed in several very interesting works on multiple trigonometric series.

we have the equality

$$\begin{aligned}\Phi(x+h,y+k)&+\Phi(x-h,y-k)-2\Phi(x,y)\\&=-4\sum_{\substack{m,n=-\infty\\ m\cdot n\neq 0}}\frac{c_{mn}}{m^2n^2}\,e^{i(mx+ny)}\sin^2\frac{1}{2}(mh+nk).\end{aligned}\tag{6.9}$$

Now we will prove the inequality

$$\frac{|\Phi(x+h,y+k)+\Phi(x-h,y-k)-2\Phi(x,y)|}{|h|+|k|}\leq 2L(|h|+|k|),\tag{6.10}$$

where the finite constant L is defined by equality (6.3). This inequality follows obviously from equality (6.9) taking into account that

$$\begin{aligned}\frac{4}{|h|+|k|}\cdot\frac{\sin^2\frac{1}{2}(mh+nk)}{m^2n^2}&\leq\frac{2(m^2h^2+n^2k^2)}{m^2n^2(|h|+|k|)}\\&\leq\frac{2m^2h^2}{m^2n^2|h|}+\frac{2n^2k^2}{m^2n^2|k|}=\frac{2|h|}{n^2}+\frac{2|k|}{m^2}\leq 2(|h|+|k|).\end{aligned}\tag{6.11}$$

2) For the 5th summand in the square brackets, from (6.7) we have

$$\begin{aligned}&\frac{1}{|h|+|k|}\left|\frac{-i}{mn^2}\left[(x+h)e^{in(y+k)}+(x-h)e^{in(y-k)}-2xe^{iny}\right]\right|\\&=\frac{1}{|h|+|k|}\cdot\frac{1}{|m|n^2}\left|2e^{iny}\left(-2x\sin^2\frac{1}{2}\,nk+ih\sin nk\right)\right|\\&\leq\frac{1}{|h|+|k|}\cdot\frac{1}{|m|n^2}\cdot 2\left(2\cdot 2\pi\cdot\frac{1}{4}\,n^2k^2+|h|\,|n|\,|k|\right)\\&\leq\frac{2\pi}{|h|+|k|}\left(\frac{k^2}{|m|}+\frac{|h|\cdot|k|}{|mn|}\right)=\frac{2\pi|k|^2}{|m|(|h|+|k|)}+\frac{2\pi|h|\cdot|k|}{|mn|(|h|+|k|)}\\&\leq\frac{2\pi|k|^2}{|m||k|}+\frac{2\pi|h|\cdot|k|}{|mn|k|}=\frac{2\pi}{|m|}\,|k|+\frac{2\pi}{|mn|}\,|h|\leq 2\pi(|h|+|k|).\end{aligned}\tag{6.12}$$

3) For the numerator of the second summand in the square brackets, from

(6.5) we have

$$\begin{aligned}
&\left|e^{im(x+h)}(y+k)^2+e^{im(x-h)}(y-k)^2-2y^2e^{imx}\right| \\
&\quad=\left|y^2(e^{imh}+e^{-imh})+2ky(e^{imh}-e^{-imh})+k^2(e^{imh}+e^{-imh})-2y^2\right| \\
&\quad=\left|2y^2(\cos mh-1)+4iky\sin mh+2k^2\cos mh\right| \\
&\quad=\left|-4y^2\sin^2\frac{1}{2}\,mh+4iky\sin mh+2k^2\cos mh\right| \\
&\quad\le 4(2\pi)^2\,\frac{m^2h^2}{4}+4\cdot 2\pi|m|\,|h|\,|k|+2k^2.
\end{aligned}$$

If all the terms of the last relations are divided by $m^2(|h|+|k|)$, then the first ratio turns out to be less than $10\pi^2(|h|+|k|)$.

We have thereby proved the following:

Theorem 6.1. *Let series* (6.1) *converge absolutely at some point from the square I. Then by twice term-by-term integration of series* (6.1) *over every variable rectangle $[0,x]\times[0,y]\subset I$ we obtain functions* (6.4)–(6.7)*, the sum $\Omega(x,y)$ of which is an everywhere continuous and uniformly smooth function on the square I. Moreover, $\Omega(x,y)$ satisfies inequality* (6.10) *for some constant M instead of $2L$.*

6B. The real case. Let us consider a double trigonometric series in the real form

$$\begin{aligned}
\frac{1}{4}&+\frac{1}{2}\sum_{m=1}^{\infty}(a_{m0}\cos mx+d_{m0}\sin mx)+\frac{1}{2}\sum_{n=1}^{\infty}(a_{0n}\cos ny+c_{0n}\sin ny) \\
&+\sum_{m,n=1}^{\infty}(a_{mn}\cos mx\cos ny+b_{mn}\sin mx\sin ny+c_{mn}\cos mx\sin ny \\
&\qquad+d_{mn}\sin mx\cos ny)\equiv\frac{1}{4}+\frac{1}{2}\,A(x)+\frac{1}{2}\,B(y)+C(x,y). \qquad (6.13)
\end{aligned}$$

It is assumed that

$$\sum_{m=1}^{\infty}(|a_{m0}|+|d_{m0}|)<\infty,\quad \sum_{n=1}^{\infty}(|a_{0n}|+|c_{0n}|))<\infty, \qquad (6.14)$$

$$\sum_{m,n=1}^{\infty}(|a_{mn}|+|b_{mn}|+|c_{mn}|+|d_{mn}|))<\infty. \qquad (6.15)$$

The twice integration of series (6.13) over $[0, x] \times [0, y] \subset I$ gives

$$\frac{1}{16}\, x^2 y^2 \ \text{ from } \ \frac{1}{4}, \tag{6.16}$$

$$\begin{aligned} -\frac{1}{4}\, y^2 \sum_{m=1}^{\infty} \frac{a_{m0} \cos mx + d_{m0} \sin mx}{m^2} &+ \frac{1}{4} y^2 \sum_{m=1}^{\infty} \frac{a_{m0}}{m^2} \\ + \frac{1}{4}\, xy^2 \sum_{m=1}^{\infty} \frac{d_{m0}}{m} \ &\text{ from } A(x), \end{aligned} \tag{6.17}$$

$$\begin{aligned} -\frac{1}{4}\, x^2 \sum_{n=1}^{\infty} \frac{a_{0n} \cos ny + c_{0n} \sin ny}{n^2} &+ \frac{1}{4} x^2 \sum_{m=1}^{\infty} \frac{a_{0n}}{n^2} \\ + \frac{1}{4}\, x^2 y \sum_{n=1}^{\infty} \frac{c_{0n}}{n} \ &\text{ from } \ B(y), \end{aligned} \tag{6.18}$$

$$\begin{aligned} \sum_{m,n=1}^{\infty} \Bigg[& \frac{a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny}{m^2 n^2} \\ & + \frac{c_{mn} \cos mx \sin ny + d_{mn} \sin mx \cos ny}{m^2 n^2} \\ & - \frac{a_{mn} \cos mx + c_{mn} \sin ny + d_{mn} \sin mx}{m^2 n^2} - \frac{a_{mn} \cos ny}{m^2 n^2} \\ & - y\, \frac{b_{mn} \sin mx + c_{mn} \cos mx}{m^2 n} - x\, \frac{b_{mn} \sin ny + d_{mn} \cos ny}{m n^2} \\ & + x\, \frac{d_{mn}}{m n^2} + y\, \frac{c_{mn}}{m^2 n} + xy\, \frac{b_{mn}}{mn} + \frac{a_{mn}}{m^2 n^2} \Bigg] \ \text{ from } \ C(x,y). \end{aligned} \tag{6.19}$$

By virtue of Riemann's theorem ([1], [5], [6]), functions (6.16)–(6.18) and the sums of one-dimensional series from (6) are continuous and uniformly smooth functions.

Let us now find the expression

$$\Delta_F^2(x, y; h, k) = F(x+h, y+k) + F(x-h, y-k) - 2F(x,y) \tag{6.20}$$

for the basic function

$$\begin{aligned} F(x,y) = \sum_{m,n=1}^{\infty} & \frac{a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny}{m^2 n^2} \\ & + \frac{c_{mn} \cos mx \sin ny + d_{mn} \sin mx \cos ny}{m^2 n^2} . \end{aligned} \tag{6.21}$$

For this we will consider the following cases:

1) For $\alpha_{mn}(x,y)=\cos mx\cos ny$ we have

$$\begin{aligned}\alpha_{mn}&(x+h,y+k)+\alpha_{mn}(x-h,y-k)-2\alpha_{mn}(x,y)\\&=2\cos mx\cos ny\cos mh\cos nk\\&\quad+2\sin mx\sin ny\sin mh\sin nk-2\cos mx\cos ny\\&=2\cos mx\cos ny(\cos mh\cos nk-1)\\&\quad+2\sin mh\sin nk\sin mx\sin ny.\end{aligned}\tag{6.22}$$

But $\cos mh\cos nk=\frac{1}{2}\cos(mh-nk)+\frac{1}{2}\cos(mh+nk)$ and therefore

$$\cos mh\cos nk-1=-\sin^2\frac{1}{2}(mh-nk)-\sin^2\frac{1}{2}(mh+nk).\tag{6.23}$$

Thus

$$\begin{aligned}\alpha_{mn}&(x+h,y+k)+\alpha_{mn}(x-h,y-k)-2\alpha_{mn}(x,y)\\&=-2\cos mx\cos ny\left[\sin^2\frac{1}{2}(mh+nk)+\sin^2\frac{1}{2}(mh-nk)\right]\\&\quad+2\sin mh\sin nk\sin mx\sin ny.\end{aligned}\tag{6.24}$$

2) For the function $\beta_{mn}(x,y)=\sin mx\sin ny$ we obtain

$$\begin{aligned}\beta_{mn}&(x+h,y+k)+\beta_{mn}(x-h,y-k)-2\beta_{mn}(x,y)\\&=-2\sin mx\sin ny\left[\sin^2\frac{1}{2}(mh+nk)+\sin^2\frac{1}{2}(mh-nk)\right]\\&\quad+2\sin mh\sin nk\cos mx\cos ny.\end{aligned}\tag{6.25}$$

3) For the function $\gamma_{mn}=\cos mx\sin ny$ we have

$$\begin{aligned}\gamma_{mn}&(x+h,y+k)+\gamma_{mn}(x-h,y-k)-2\gamma_{mn}(x,y)\\&=-2\cos mx\sin ny\left[\sin^2\frac{1}{2}(mh+nk)+\sin^2\frac{1}{2}(mh-nk)\right]\\&\quad-2\sin mh\sin nk\sin mx\cos ny.\end{aligned}\tag{6.26}$$

4) For the function $\delta_{mn}=\sin mx\cos ny$ we obtain

$$\begin{aligned}\delta_{mn}&(x+h,y+k)+\delta_{mn}(x-h,y-k)-2\delta_{mn}(x,y)\\&=-2\sin mx\cos ny\left[\sin^2\frac{1}{2}(mh+nk)+\sin^2\frac{1}{2}(mh-nk)\right]\\&\quad-2\sin mh\sin nk\cos mx\sin ny.\end{aligned}\tag{6.27}$$

Therefore

$$\Delta_F^2(x,y;h,k) = -2\sum_{m,n=1}^{\infty} A_{mn}(x,y)\,\frac{1}{m^2n^2}\left[\sin^2\frac{1}{2}(mh+nk)\right.$$
$$\left. + \sin^2\frac{1}{2}(mh-nk)\right] - 2\sum_{m,n=1}^{\infty}\overline{A}_{mn}(x,y)\,\frac{1}{m^2n^2}\,\sin mh\sin nk \tag{6.28}$$
$$\equiv F_1(x,y;h,k) + F_2(x,y;h,k),$$

where

$$A_{mn} = a_{mn}\cos mx\cos ny + b_{mn}\sin mx\sin ny + c_{mn}\cos mx\sin ny + d_{mn}\sin mx\cos ny,$$
$$\overline{A}_{mn} = a_{mn}\sin mx\sin ny + b_{mn}\cos mx\cos ny - c_{mn}\sin mx\cos ny - d_{mn}\cos mx\sin ny.$$

Further,

$$\sin^2\frac{1}{2}(mh+nk) \le \left(\frac{mh+nk}{2}\right)^2 = \frac{m^2h^2+n^2k^2+2mhnk}{4},$$
$$\sin^2\frac{1}{2}(mh-nk) \le \frac{m^2h^2+n^2k^2-2mhnk}{4}, \tag{6.29}$$
$$\sin^2\frac{1}{2}(mh+nk) + \sin^2\frac{1}{2}(mh-nk) \le \frac{1}{2}(m^2h^2+n^2k^2).$$

Therefore

$$\frac{1}{m^2n^2}\left[\sin^2\frac{1}{2}(mh+nk) + \sin^2\frac{1}{2}(mh-nk)\right] \le \frac{m^2h^2+n^2k^2}{2m^2n^2}$$
$$= \frac{h^2}{2n^2} + \frac{k^2}{2m^2} \le \frac{1}{2}\,(h^2+k^2) \le \frac{1}{2}\,(|h|+|k|)^2. \tag{6.30}$$

Thus uniformly with respect to $(x,y)\in I$ we have

$$|F_1(x,y;h,k)| \le C(|h|+|k|)^2, \tag{6.31}$$

where C is the sum of series (6.15).

Further,

$$\frac{|\sin mh\sin nk|}{m^2n^2} \le \frac{mn|h|\cdot|k|}{m^2n^2} \le |h|\cdot|k| \le \frac{1}{2}\,(|h|+|k|)^2. \tag{6.32}$$

Therefore we have uniformly with respect to $(x,y)\in I$

$$|F_2(x,y;h,k)|\le C(|h|+|k|)^2. \tag{6.33}$$

From (6.31) and (6.33) it follows that the inequality

$$|\Delta^2_F(x,y;h,k)|\le 2C(|h|+|k|)^2 \tag{6.34}$$

holds uniformly with respect to $(x,y)\in I$.

Inequality (6.34) implies in particular that the continuous function $F(x,y)$ defined by equality (6.21) is uniformly smooth on the square I.

The theorem consists of the following.

Theorem 6.2. *Let the coefficients of the double trigonometric series* (6.13) *in the real form satisfy conditions* (6.14) *and* (6.15)*. Then by twice term-by-term integration of series* (6.13) *over every variable rectangle* $[0,x]\times[0,y]\subset I$ *we obtain the double series* (6.21)*, the function* (6.16)*, the one-dimensional series* (6.17)*,* (6.18) *and also the one-dimensional series from expression* (6)*. The sum* $\omega(x,y)$ *of all these functions is an everywhere continuous and uniformly smooth function on the square* $I=[0,2\pi]\times[0,2\pi]$*. Moreover,* $\omega(x,y)$ *satisfies inequality* (6.34) *for some constant* N *instead of* $2C$.

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