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CONTINUITY OF SUBADDITIVE PRESSURE FOR SELF-AFFINE SETS

Abstract

A 'pressure' functional $\Phi^s(T_1, \ldots, T_N)$, defined as the limit of sums of singular value functions of products of linear mappings $(T_1, \ldots, T_N)$, is central in analysing fractal dimensions of self-affine sets. We investigate the continuity of Φ^s with respect to the linear mappings $(T_1, \ldots, T_N)$ which underlie the self-affine sets.

1 Introduction and Background.

Recall that the *singular values* $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$ of a linear mapping $T \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ are the positive square roots of the eigenvalues of T^*T or equivalently the lengths of the semi-axes of the ellipsoid $T(B)$ where B is the unit ball in $\mathbf{R}^n$. For $0 \leq s \leq n$ the *singular value function* is given by $\phi^s(T) = \alpha_1\alpha_2 \ldots \alpha_m \alpha_{m+1}^{s-m}$, where m is the integer such that $m < s \leq m+1$, so that

$$\phi^s(T) = \phi^m(T)^{m+1-s}\phi^{m+1}(T)^{s-m}. \tag{1.1}$$

Note that $\phi^1(T)$ is the operator norm of T induced by the Euclidean norm and $\phi^n(T)$ is the determinant of T. It is well-known that the singular value functions are submultiplicative, that is for each s

$$\phi^s(TU) \leq \phi^s(T)\phi^s(U), \quad T, U \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n), \tag{1.2}$$

with $\phi^s(T)$ decreasing in s if T is a contraction, see [2].

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Let $T_1, \ldots, T_N \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$. We code compositions of the T_i by words formed by the symbols $\{1, \ldots, N\}$ in the usual way. Thus we write $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ where $i_j \in \{1, \ldots, N\}$, with $|\mathbf{i}| = k$ for the length of the word. We write J for the set of all finite words and write $\mathbf{ij}$ for the concatenation of $\mathbf{i}$ and $\mathbf{j}$. We abbreviate $T_{\mathbf{i}} = T_{i_1} \ldots T_{i_k}$.

Taking sums over words of length k we define

$$\Phi_k^s \equiv \Phi_k^s(T_1, \ldots, T_N) = \sum_{|\mathbf{i}|=k} \phi^s(T_{\mathbf{i}}), \tag{1.3}$$

so for each s the sequence Φ_k^s is submultiplicative, that is $\Phi_{k+l}^s \leq \Phi_k^s \Phi_l^s$. We define

$$\Phi^s \equiv \Phi^s(T_1, \ldots, T_N) = \lim_{k \to \infty} (\Phi_k^s(T_1, \ldots, T_N))^{1/k} = \inf_k (\Phi_k^s(T_1, \ldots, T_N))^{1/k}, \tag{1.4}$$

with the limit existing and equalling the infimum by the standard properties of submultiplicative sequences.

The $\Phi^s(T_1, \ldots, T_N)$ occur naturally in connection with the dimension of self-affine fractals. Given contracting linear mappings $\{T_1, \ldots, T_N\}$ and translations $\mathbf{a} \equiv (a_1, \ldots, a_N)$ on $\mathbf{R}^n$, the theory of iterated function systems gives that there is a unique non-empty compact subset $F(\mathbf{a}) \equiv F$ of $\mathbf{R}^n$ that satisfies $F = \bigcup_{i=1}^N (T_i(F) + a_i)$, termed a self-affine set, see [1, 7]. The following is the basic result on the dimension of self-affine sets.

Theorem 1.1. *Assume that $\|T_i\| < \frac{1}{2}$ for $i = 1, \ldots, N$. For almost all $\mathbf{a} \in \mathbf{R}^{nN}$ (in the sense of nN-dimensional Lebesgue measure) $\dim F(\mathbf{a}) = \min\{n, s\}$, where s is the unique number such that $\Phi^s(T_1, \ldots, T_N) = 1$ and* dim *denotes either Hausdorff or box dimension.*

This theorem was proved by Falconer [2] with a norm bound of $\frac{1}{3}$, and Solomyak [11] strengthened this to $\frac{1}{2}$. The singular value functions arise from estimating the numbers of balls of small radii needed to cover the ellipsoids $T_{\mathbf{i}}(B)$. Indeed, the dimension s of a self-affine set might be expected to satisfy $\Phi^s(T_1, \ldots, T_N) = 1$ rather more generally and various other conditions have been obtained for this to be so, see [3, 6, 9, 10]. In the parlance of thermodynamic formalism, Φ^s may be thought of as (the exponential of) a subadditive pressure expression.

Provided that the $\{T_i\}$ are non-singular, $\Phi^s(T_1, \ldots, T_N)$ is continuous in s and is strictly decreasing in s if the T_i are non-singular contractions. Clearly each $\Phi_k^s(T_1, \ldots, T_N)$ is continuous in $(T_1, \ldots, T_N)$ and one would certainly expect $\Phi^s(T_1, \ldots, T_N)$ to be continuous, but in general this seems far from obvious even in the 'norm' case of $s = 1$. The question of continuity was raised by

Käenmäki and Shmerkin [8] where continuity was proved for a class of transformations for which the $\{T_i\}$ all map a certain cone into itself. For non-singular contractions T_i it is easy to see that the value of s satisfying $\Phi^s(T_1, \ldots, T_N) = 1$ will vary continuously with $(T_1, \ldots, T_N)$ wherever $\Phi^s(T_1, \ldots, T_N)$ is continuous for all $s > 0$.

Upper semicontinuity is straightforward and the following proposition summarises the most basic properties.

Proposition 1.2. *For each $0 < s \leq n$ we have $\Phi^s(T_1, \ldots, T_N)$ upper semicontinuous at all $(T_1, \ldots, T_N)$. Moreover, $\Phi^s(T_1, \ldots, T_N)$ is continuous for $(T_1, \ldots, T_N) \in K$ where K is a compact set if and only if $(\Phi^s_k(T_1, \ldots, T_N))^{1/k}$ converges uniformly on K.*

PROOF. For each k, $\Phi^s_k(T_1, \ldots, T_N)$ is a finite sum involving finite products of the T_i, so is continuous in $(T_1, \ldots, T_N)$. Then, noting (1.4), $\Phi^s(T_1, \ldots, T_N)$ is upper semicontinuous as the infimum of a set of continuous functions.

If $\Phi^s(T_1, \ldots, T_N)$ is the uniform limit of the sequence of continuous functions $(\Phi^s_k(T_1, \ldots, T_N))^{1/k}$ on K, then it is continuous.

On the other hand, if Φ^s is continuous on K, then since $(\Phi^s_{2^k})^{1/2^k}$ is monotonic decreasing, this subsequence converges uniformly to Φ^s by Dini's theorem. By submultiplicativity, $\Phi^s_{qm+r} \leq (\Phi^S_N)^q \Phi^s_r$ so

$$\left(\Phi^s_{qm+r}\right)^{1/(qm+r)} \leq \left((\Phi^s_m)^{1/m}\right)^{qm/(qm+r)} \left(\Phi^s_r\right)^{1/(qm+r)}.$$

By fixing $m = 2^k$ and taking large values of $qm + r$ where $0 \leq r < m$ uniform convergence extends to the full sequence Φ^s_k. $\square$

We present two approaches which provide partial answers to the question of lower semicontinuity. In Section 2 we introduce a rather technical condition $C(s)$ that implies that Φ^s is quasimultiplicative, see (2.1). The following immediate corollary of Theorem 2.5 gives a flavour of the main result.

Corollary 1.3. *Consider the family of all N-tuples $(T_1, \ldots, T_N)$ of linear contractions $T_i \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ as an open subset O of $\mathbf{R}^{n^2 N}$. There is an open and dense subset $V \subset O$ such that Φ^s is continuous on V for all $s > 0$. The set V is defined by a finite number of irreducibility conditions.*

In Section 3 we consider the complementary case where the T_i can be represented by upper triangular matrices. Some special cases, in particular in $\mathbf{R}^2$, are highlighted in Section 4.

2 Quasimultiplicativity.

In this section we show that a quasimultiplicative condition, namely that there is a constant c such that

$$\begin{aligned} c\Phi^s_p(T_1,\ldots,T_N)\Phi^s_q(T_1,\ldots,T_N) &\leq \Phi^s_{p+q}(T_1,\ldots,T_N) \\ &\leq \Phi^s_p(T_1,\ldots,T_N)\Phi^s_q(T_1,\ldots,T_N) \quad (p,q\geq 0) \end{aligned} \tag{2.1}$$

holds in a neighbourhood of 'most' $(T_1,\ldots,T_N)$, from which continuity at $(T_1,\ldots,T_N)$ will follow,

We write Λ^m for the m-th exterior power of $\mathbf{R}^n$ with $\mathbf{v}\in\Lambda^m$ a typical m-vector. An m-vector is decomposable if it can be written $\mathbf{v}=v_1\wedge\ldots\wedge v_m$ and we write Λ^m_0 for the set of decomposable m-vectors.

Given a linear mapping $T\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^n)$ by slight abuse of notation we also write T for the induced map $T\equiv\Lambda^m T:\Lambda^m\to\Lambda^m$ given by

$$T\mathbf{v}\equiv T(v_1\wedge\ldots\wedge v_m)=Tv_1\wedge\ldots\wedge Tv_m$$

for $\mathbf{v}\in\Lambda^m_0$ and extended to Λ^m by linearity.

For each integer $0\leq m\leq n$, denote by $\|\mathbf{v}\|_m$ the m-dimensional volume of a parallelepiped defined by $\mathbf{v}\in\Lambda^m_0$, that is $\|\mathbf{v}\|_m=|v_1|\ldots|v_m|$ where a representation of $\mathbf{v}$ is chosen with the v_i mutually orthogonal. (We take $\|\mathbf{v}\|_0=1$ for $\mathbf{v}\neq 0$.) Indeed, $\|\mathbf{v}\|_m$ is the norm induced by the inner product on Λ^m given by

$$\langle\mathbf{v},\mathbf{w}\rangle\omega=\mathbf{v}\wedge *\mathbf{w}$$

where ω is the normalised volume form on $\mathbf{R}^n$ and $*:\Lambda^m\to\lambda^{n-m}$ is the Hodge star operator defined by the requirement that $*(e_1\wedge\ldots\wedge e_m)=e_{m+1}\wedge\ldots\wedge e_n$, where $e_1,\ldots,e_n$ is any oriented orthonormal basis of $\mathbf{R}^n$.

For $0<s<n$ non-integral we write $\Lambda^m_0\times\mathbf{R}^n\equiv\Lambda^s_0$, where m is the integer such that $m<s<m+1$, and consider pairs $\underline{\mathbf{v}}\equiv(\mathbf{v},v)\in\Lambda^s_0$. For such $\underline{\mathbf{v}}$ we will work with both $\mathbf{v}\in\Lambda^m_0$ and $\mathbf{v}\wedge v\in\Lambda^{m+1}_0$ and we define $\underline{\mathbf{v}}=0$ if $\mathbf{v}\wedge v=0$ (which is certainly the case if $\mathbf{v}=0$). For $T\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^n)$ we also write T for the induced map on Λ^s_0 given by $T\underline{\mathbf{v}}=T(\mathbf{v},v)=((\Lambda^m T)\mathbf{v},Tv)$.

Write

$$\|\underline{\mathbf{v}}\|_s=\|\mathbf{v}\|_m^{m+1-s}\|\mathbf{v}\wedge v\|_{m+1}^{s-m} \text{ for } \underline{\mathbf{v}}\in\Lambda^s_0. \tag{2.2}$$

Note that Λ^s_0 is not a vector space nor is $\|\,\|_s$ a norm for non-integral s. When $s=m$ is an integer we simply take $\Lambda^s_0=\Lambda^m_0$ and $\underline{\mathbf{v}}=\mathbf{v}$. For $0\leq s\leq n$ the singular value function is given by

$$\phi^s(T)=\sup\left\{\frac{\|T\underline{\mathbf{v}}\|_s}{\|\underline{\mathbf{v}}\|_s}:0\neq\underline{\mathbf{v}}\in\Lambda^s_0\right\}; \tag{2.3}$$

this is standard for s an integer and follows from (2.2) for non-integral s.

We will need certain conditions on a family of mappings $\{S_i\}$ of $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$. For m an integer, let $C(m)$ be the condition:

$$\text{For all } 0 \neq \mathbf{v}, \mathbf{w} \in \Lambda_0^m, \quad \langle S_i \mathbf{v}, \mathbf{w} \rangle \neq 0 \text{ for some } i. \tag{$C(m)$}$$

It is easy to verify that this is equivalent to

$$\text{For all } V \in G_m, W \in G_{n-m}, \quad S_i(V) \cap W = \{0\} \text{ for some } i \tag{$C'(m)$}$$

where G_m denotes the Grassmanian of m-dimensional subspaces of $\mathbf{R}^n$.

For non-integral s the condition depends on the integers on either side of s. We write $C(s)$ for the condition:

$$\begin{aligned} &\text{For all } 0 \neq \underline{\mathbf{v}}, \underline{\mathbf{w}} \in \Lambda_0^s \text{ there exists } i \text{ such that} \\ &\text{both } \langle S_i \mathbf{v}, \mathbf{w} \rangle \neq 0 \text{ and } \langle S_i(\mathbf{v} \wedge v), \mathbf{w} \wedge w \rangle \neq 0. \end{aligned} \tag{$C(s)$}$$

Note that for $0 < s \leq 1$ the condition $C(s)$ reduces to $C(1)$ and for $n-1 \leq s \leq n$ the condition $C(s)$ is just $C(n-1)$ since $S\mathbf{v}$ is just multiplication by the determinant of S for $\mathbf{v} \in \Lambda^n$.

Proposition 2.1. *Let $S_1, \ldots, S_k \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and $0 < s \leq n$. Let m be the integer such that $m < s \leq m+1$. Then there is a number $c > 0$ such that*

$$\phi^s(U) \|\underline{\mathbf{v}}\|_s \leq c \sum_{j=1}^{k} \|U S_j \underline{\mathbf{v}}\|_s \quad \text{for all } \underline{\mathbf{v}} \in \Lambda_0^s \text{ and } U \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n) \tag{2.4}$$

if and only if $C(s)$ holds for $\{S_i\}_{i=1}^k$. (Recall that the conditions differ depending on whether or not s is an integer, as above).

We can take $c \equiv c(S_1, \ldots, S_k)$ to be given by

$$\begin{aligned} c^{-1} = \min \Big\{ &\max_j \{ |\langle S_j \mathbf{v}, \mathbf{w} \rangle|^{m+1-s} |\langle S_j(\mathbf{v} \wedge v), \mathbf{w} \wedge w \rangle|^{s-m} \} \\ &: \|\mathbf{v}\|_m = \|\mathbf{w}\|_m = \|\mathbf{v} \wedge v\|_{m+1} = \|\mathbf{w} \wedge w\|_{m+1} = 1 \Big\}. \end{aligned} \tag{2.5}$$

Proof. We give the proof for non-integral s, the integer case is similar but simpler.

Assume that $C(s)$ holds. Let $U \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and let $u_1, \ldots, u_n$ be an orthonormal family of unit eigenvectors of the self adjoint mapping U^*U in order of decreasing eigenvalues $\alpha_1^2 \geq \ldots \geq \alpha_n^2 \geq 0$ where the α_i are the singular values of U. The set of m-vectors $\{u_{i_1} \wedge \ldots \wedge u_{i_m} : 1 \leq i_1 < \ldots < i_m \leq n\}$ gives an orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ of Λ^m where $p = \binom{n}{m}$. In particular, choosing $\mathbf{w}_1 = u_1 \wedge \ldots \wedge U_N$ we have $\|U\mathbf{w}_1\|_m = \phi^m(U)$. In the same way, taking wedge products of $m+1$ of the u_i gives an orthonormal basis $\{\mathbf{w}'_1, \ldots, \mathbf{w}'_{p'}\}$ of Λ^{m+1}, where $p' = \binom{n}{m+1}$ and we choose $\mathbf{w}'_1 = u_1 \wedge \ldots \wedge U_N \wedge u_{m+1}$ so that $\|U\mathbf{w}'_1\|_{m+1} = \phi^{m+1}(U)$.

For each $\underline{\mathbf{v}} = (\mathbf{v}, v) \in \Lambda_0^s$ with $\|\mathbf{v}\|_m = \|\mathbf{v} \wedge v\|_{m+1} = 1$ we have the orthogonal expansion for each j

$$S_j\mathbf{v} = \sum_{i=1}^{p} \langle S_j\mathbf{v}, \mathbf{w}_i\rangle \mathbf{w}_i$$

so that

$$US_j\mathbf{v} = \sum_{i=1}^{p} \langle S_j\mathbf{v}, \mathbf{w}_i\rangle U\mathbf{w}_i.$$

The u_i are orthonormal eigenvectors of U^*U so the $Uu_1, \ldots, Uu_n$ are orthogonal and thus $\{U\mathbf{w}_1, \ldots, U\mathbf{w}_p\}$ are orthogonal m-vectors (some of which may be zero). Thus

$$\|US_j\mathbf{v}\|_m \geq |\langle S_j\mathbf{v}, \mathbf{w}_1\rangle| \|U\mathbf{w}_1\|_m = |\langle S_j\mathbf{v}, \mathbf{w}_1\rangle| \phi^m(U).$$

In exactly the same way

$$\|US_j(\mathbf{v} \wedge v)\|_{m+1} \geq |\langle S_j(\mathbf{v} \wedge v), \mathbf{w}'_1\rangle| \phi^{m+1}(U).$$

By $C(s)$ we may choose j such that both $\langle S_j\mathbf{v}, \mathbf{w}_1\rangle \neq 0$ and $\langle S_j(\mathbf{v} \wedge v), \mathbf{w}'_1\rangle \neq 0$, so using (1.1) and (2.2) gives

$$\|US_j(\underline{\mathbf{v}})\|_s \geq |\langle S_j\mathbf{v}, \mathbf{w}_1\rangle|^{m+1-s} |\langle S_j(\mathbf{v} \wedge v), \mathbf{w}'_1\rangle|^{s-m} \phi^s(U). \tag{2.6}$$

Extending this to general $\underline{\mathbf{v}} \in \Lambda_0^s$ by homogeneity, inequality (2.4) follows. The value of c is a consequence of (2.6); it is well-defined and strictly positive by continuity and compactness.

For the converse, assume that $C(s)$ fails, so there exist $0 \neq \underline{\mathbf{v}}, \underline{\mathbf{w}} \in \Lambda_0^s$ such that for each j either $\langle S_j\mathbf{v}, \mathbf{w}\rangle = 0$ or $\langle S_j(\mathbf{v} \wedge v), \mathbf{w} \wedge w\rangle = 0$. By normalising, we may assume that there is an orthonormal family $u_1, \ldots, u_n$ such that $\mathbf{w} = u_1 \wedge \ldots \wedge U_N$ and $w = u_{m+1}$.

As before, the m-vectors $\{u_{i_1} \wedge \ldots \wedge u_{i_m}\}$ provide an orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$ of Λ^m where $p = \binom{n}{m}$ and we take $\mathbf{w}_1 = \mathbf{w}$. Similarly, the $(m+1)$-vectors $\{u_{i_1} \wedge \ldots \wedge u_{i_{m+1}}\}$ give an orthonormal basis $\{\mathbf{w}'_1, \ldots, \mathbf{w}'_{p'}\}$ of Λ^{m+1}, where $p' = \binom{n}{m+1}$, with $\mathbf{w}'_1 = \mathbf{w}_1 \wedge w = \mathbf{w} \wedge u_{m+1}$.

Let $\epsilon > 0$ and define $U_\epsilon : \mathbf{R}^n \to \mathbf{R}^n$ in terms of basis elements by

$$U_\epsilon(u_i) = \begin{cases} u_i & (1 \le i \le m) \\ \epsilon u_i & (i = m+1) \\ \epsilon^2 u_i & (m+2 \le i \le n) \end{cases} .$$

Then

$$\phi^m(U_\epsilon) = 1, \quad \phi^{m+1}(U_\epsilon) = \epsilon \quad \text{so} \quad \phi^s(U_\epsilon) = \epsilon^{s-m}. \tag{2.7}$$

If $\langle S_j \mathbf{v}, \mathbf{w} \rangle = 0$ we have an expansion

$$S_j \mathbf{v} = \sum_{i=1}^{p} \langle S_j \mathbf{v}, \mathbf{w}_i \rangle \mathbf{w}_i = \sum_{i=2}^{p} \langle S_j \mathbf{v}, \mathbf{w}_i \rangle \mathbf{w}_i$$

so that

$$\|U S_j \mathbf{v}\|_m \le \sum_{i=2}^{p} |\langle S_j \mathbf{v}, \mathbf{w}_i \rangle| \|U \mathbf{w}_i\|_m \le c_1 \epsilon,$$

where c_1 is independent of ϵ, noting that for $i \ge 2$ each $\mathbf{w}_i$ has a component u_l where $l \ge m+1$. We also have the expansion

$$S_j(\mathbf{v} \wedge v) = \sum_{i=1}^{p'} \langle S_j(\mathbf{v} \wedge v), \mathbf{w}'_i \rangle \mathbf{w}'_i,$$

so

$$\|U S_j(\mathbf{v} \wedge v)\|_{m+1} \le \sum_{i=1}^{p'} |\langle S_j(\mathbf{v} \wedge v), \mathbf{w}'_i \rangle| \|U \mathbf{w}'_i\|_{m+1} \le c_2 \epsilon,$$

since each $\mathbf{w}'_i$ has a component u_l where $l \ge m+1$. We conclude that

$$\|U_\epsilon S_j \mathbf{v}\|_m \le c_1 \epsilon, \quad \|U_\epsilon S_j(\mathbf{v} \wedge v)\|_{m+1} \le c_2 \epsilon, \quad \text{so } \|U_\epsilon S_j(\underline{\mathbf{v}})\|_s \le c_3 \epsilon, \tag{2.8}$$

where c_3 is independent of U.

In a very similar way we see that if $\langle S_j(\mathbf{v} \wedge v), \mathbf{w} \wedge w \rangle = 0$ then

$$\|U_\epsilon S_j \mathbf{v}\|_m \le c_1, \quad \|U_\epsilon S_j(\mathbf{v} \wedge v)\|_{m+1} \le c_2 \epsilon^2, \quad \text{so } \|U_\epsilon S_j(\underline{\mathbf{v}})\|_s \le c_3 \epsilon^{2(s-m)}. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9) and noting that ϵ may be taken arbitrarily small, we see that (2.4) cannot hold uniformly in U. □

Corollary 2.2. *Let $S_1, \ldots, S_k \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and $0 < s \leq n$. Suppose that $C(s)$ holds for $\{S_i\}_{i=1}^k$. Then there is a number c, given by (2.5), such that*

$$\phi^s(U)\phi^s(T) \leq c\sum_{j=1}^{k}\phi^s(US_jT) \quad \textit{for all } U, T \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n). \tag{2.10}$$

Proof. Again we give the proof in the case when s is not an integer. Let m be the integer such that $m < s < m+1$. Given T we may choose $\underline{\mathbf{t}} = (\mathbf{t}, t) \in \Lambda_0^s$ such that $\|T\mathbf{t}\|_m = \phi^m(T)$ and $\|T(\mathbf{t} \wedge t)\|_{m+1} = \phi^{m+1}(T)$ with $\|\mathbf{t}\|_m = \|\mathbf{t} \wedge t\|_{m+1} = 1$. Taking $\mathbf{v} = T\mathbf{t}$ and $v = Tt$ in Proposition 2.1

$$\phi^s(U)\phi^s(T) = \phi^s(U)\|T\mathbf{t}\|_s \leq c\sum_{j=1}^{k}\|US_jT\mathbf{t}\|_s$$

$$\leq c\sum_{j=1}^{k}\phi^s(US_jT)\|\mathbf{t}\|_s = c\sum_{j=1}^{k}\phi^s(US_jT).$$

□

We now apply this result to the situation described in the introduction. We consider the semigroup of linear mappings $\mathcal{S} \equiv \{T_\mathbf{i} : \mathbf{i} \in J\}$ generated by a given set $T_1, \ldots, T_N \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$. A compactness argument establishes that the family of mappings $\mathcal{S}$ satisfies $C(s)$ if and only if there is an integer $r \geq 0$ such that $\{T_\mathbf{i} : |\mathbf{i}| \leq r\}$ satisfies $C(s)$; thus we need only consider such finite sets of mappings.

Note that $\phi(T_\mathbf{i})$ satisfying an inequality such as (2.11) might be termed *nearly quasimultiplicative*. Such properties have been utilised in studying the multifractal behaviour of norms of matrix products and of measures of overlapping construction, see [5, 12].

Corollary 2.3. *Let $0 \leq s \leq n$ and let $\{T_\mathbf{i} : |\mathbf{i}| \leq r\}$ satisfy $C(s)$. Then there is a number $c \equiv c(\{T_\mathbf{i} : |\mathbf{i}| \leq r\})$ given by (2.5), such that*

$$\phi^s(T_{\mathbf{ij}}) \leq \phi^s(T_\mathbf{i})\phi^s(T_\mathbf{j}) \leq c\sum_{|\mathbf{l}| \leq r}\phi^s(T_\mathbf{i}T_\mathbf{l}T_\mathbf{j}) \quad (\mathbf{i}, \mathbf{j} \in J). \tag{2.11}$$

In particular

$$\begin{aligned}\Phi^s_{p+q}(T_1, \ldots, T_N) &\leq \Phi^s_p(T_1, \ldots, T_N)\Phi^s_q(T_1, \ldots, T_N) \\ &\leq c_1\Phi^s_{p+q}(T_1, \ldots, T_N) \quad (p, q \geq 0)\end{aligned} \tag{2.12}$$

where

$$c_1 \equiv c_1(\{T_{\mathbf{i}} : |\mathbf{i}| \le r\}) = c(N^{r+1}-1)(N-1)^{-1} \max_{|\mathbf{l}|\le r}\{\phi^s(T_{\mathbf{l}})\}. \tag{2.13}$$

Proof. The left hand inequality of (2.11) is just (1.2). The right hand inequality follows from applying Corollary 2.2 to the family $\{T_{\mathbf{i}} : |\mathbf{i}| \le r\}$ taking $U = T_{\mathbf{i}}$ and $T = T_{\mathbf{j}}$.

Summing (2.11) over $|\mathbf{i}| = p$ and $|\mathbf{j}| = q$ gives

$$\Phi^s_{p+q}(T_1,\dots,T_N) \le \Phi^s_p(T_1,\dots,T_N)\Phi^s_q(T_1,\dots,T_N) \le c \sum_{|\mathbf{l}|\le r}\sum_{|\mathbf{i}|=p+q} \phi^s(T_{\mathbf{i}}T_{\mathbf{l}})$$

$$\le c \sum_{|\mathbf{l}|\le r}\sum_{|\mathbf{i}|=p+q} \phi^s(T_{\mathbf{i}})\phi^s(T_{\mathbf{l}}) \le c_1 \Phi^s_{p+q}(T_1,\dots,T_N).$$

□

We require the following simple lemma on rates of convergence of quasimultiplicative sequences.

Lemma 2.4. *Let (a_k) be a quasimultiplicative sequence of positive numbers, that is with*

$$b_1 a_p a_q \le a_{p+q} \le b_2 a_p a_q \quad (p, q \ge 1) \tag{2.14}$$

where $0 < b_1 < b_2$. Then

$$a b_2^{-1/p} \le a_p^{1/p} \le a b_1^{-1/p} \quad (p \ge 1) \tag{2.15}$$

where $a = \lim_{j\to\infty} a_j^{1/j}$.

Proof. Iterating (2.14) $r-1$ times gives

$$b_1^{r-1} a_p^r \le a_{rp} \le b_2^{r-1} a_p^r \ .$$

Setting $r = p^k$ and raising to the power $1/p^{k+1}$ gives

$$b_1^{1/p-1/p^{k+1}} a_p^{1/p} \le (a_{p^{k+1}})^{1/p^{k+1}} \le b_2^{1/p-1/p^{k+1}} a_p^{1/p}.$$

Letting $k \to \infty$ gives (2.15). □

We now obtain an estimate of the rate of convergence of $(\Phi^s_k)^{1/k}$ and establish the continuity of $\Phi^s(T_1,\dots,T_N)$.

Theorem 2.5. *Let $0 < s \le n$ and let $\{T_{\mathbf{i}} : \mathbf{i} \in J\}$ satisfy $C(s)$. Then*

$$\Phi^s(T_1, \dots, T_N) \le \Phi^s_k(T_1, \dots, T_N)^{1/k} \le c_1^{1/k} \Phi^s(T_1, \dots, T_N) \tag{2.16}$$

for some c_1 independent of k. Moreover, Φ^s is continuous at $(T_1, \dots, T_N)$.

Proof. As remarked above, there is an integer r such that $\{T_{\mathbf{i}} : |\mathbf{i}| \le r\}$ which satisfies $C(s)$. Applying Lemma 2.4 to (2.12) gives (2.16) with $c_1 = c_1(\{T_{\mathbf{i}} : |\mathbf{i}| \le r\})$ given by (2.13).

If $\Phi^s(T_1, \dots, T_N) = 0$ then lower semicontinuity is automatic since $\Phi^s \ge 0$, so assume that $\Phi^s(T_1, \dots, T_N) > 0$. Since $\{T_{\mathbf{i}} : |\mathbf{i}| \le r\}$ satisfies $C(s)$, a continuity argument using the definition of $C(s)$ gives that the family $\{U_{\mathbf{i}} : |\mathbf{i}| \le r\}$ satisfies $C(s)$ for $(U_1, \dots, U_N)$ in a closed neighbourhood V of $(T_1, \dots, T_N)$. Since $c_1(\{U_{\mathbf{i}} : |\mathbf{i}| \le r\})$ of (2.13) varies continuously with $(U_1, \dots, U_N)$ there is a constant c_0 such that

$$\Phi^s(U_1, \dots, U_N) \le \Phi^s_k(U_1, \dots, U_N)^{1/k} \le c_0^{1/k} \Phi^s(U_1, \dots, U_N) \tag{2.17}$$

for $(U_1, \dots, U_N)$ in V and all k. Setting $k = 1$ and using the continuity of Φ^s_1 it follows that $\Phi^s_k(U_1, \dots, U_N)^{1/k}$ is uniformly bounded away from 0 in V. Thus by (2.17) $(\Phi^s_k)^{1/k}$ converges uniformly to Φ^s in a neighbourhood of $(T_1, \dots, T_N)$ so Φ^s is continuous at $(T_1, \dots, T_N)$ by Proposition 1.2. $\square$

There is a convenient criterion for the hypothesis of Theorem 2.5 to hold.

Lemma 2.6. *Let m be an integer. Then $\{T_{\mathbf{i}} : \mathbf{i} \in J\}$ satisfies $C(m)$ if and only if there is no non-trivial subspace V of Λ^m that is invariant under T_i (i.e. satisfies $(\Lambda^m T_i)(V) \le V$) for all i. In particular, if $0 < s \le 1$, then $C(s) \equiv C(1)$ is satisfied if and only if there is no non-trivial subspace V of $\mathbf{R}^n$ such that $T_i(V) \le V$ for all i; and if $n-1 \le s \le n$, then $C(s) \equiv C(n)$ is satisfied if and only if there is no non-trivial subspace V of Λ^{n-1} such that $(\Lambda^{n-1} T_i)(V) \le V$ for all i.*

Proof. Suppose that $C(m)$ does not hold. Then there exist $0 \ne \mathbf{v}, \mathbf{w} \in \Lambda^m$ such that (writing $T_{\mathbf{i}}$ for $\Lambda^m T_{\mathbf{i}}$ as usual) $\langle T_{\mathbf{i}} \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{i} \in J$. Let $V = \operatorname{span}\{T_{\mathbf{i}} \mathbf{v} : \mathbf{i} \in J\}$. Then $\{0\} \ne V \le (\operatorname{span}\{\mathbf{w}\})^\perp$ and $T_i(V) \le V$ for $i = 1, \dots, m$ so V is a proper common invariant subspace.

Conversely let V be a proper common invariant subspace for the $\{T_i\}$. Taking non-zero $\mathbf{v} \in V$ and $\mathbf{w} \in V^\perp$ we have $T_{\mathbf{i}} \mathbf{v} \in V$ and so $\langle T_{\mathbf{i}} \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{i} \in J$.

The other cases follow since $C(s) = C(1)$ or $C(n)$ if $0 < s \le 1$ or $n - 1 \le s \le n$ respectively. $\square$

3 Upper Triangular Representations.

In this section we consider a situation that is in a sense complementary to that of Section 2, namely when there is a basis of $\mathbf{R}^n$ with respect to which all the transformations $(T_1, \ldots, T_N)$ have upper triangular form.

Recall that the spectral radius $\rho(T)$ of $T \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ is

$$\rho(T) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\} = \lim_{k\to\infty} \|T^k\|^{1/k}, \tag{3.1}$$

which is independent of the norm chosen on $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$.

Lemma 3.1. *For all families of linear transformations* $(T_1, \ldots, T_N)$ *on* $\mathbf{R}^n$,

$$\rho(T_1 + \ldots + T_N) \leq \Phi^1(T_1, \ldots, T_N). \tag{3.2}$$

PROOF. We have

$$\rho(T_1 + \ldots + T_N) = \lim_{k\to\infty} \|(T_1 + \ldots + T_N)^k\|^{1/k} = \lim_{k\to\infty} \Big\| \sum_{|\mathbf{i}|=k} T_{\mathbf{i}} \Big\|^{1/k}$$

$$\leq \lim_{k\to\infty} \Big[\sum_{|\mathbf{i}|=k} \|T_{\mathbf{i}}\| \Big]^{1/k} = \Phi^1(T_1, \ldots, T_N).$$

$\square$

We now fix a basis of $\mathbf{R}^n$ and identify linear mappings with their matrices with respect to this basis. It is convenient to use the the matrix norm $\|T\| = \sum_{p,q} |[T]_{p,q}|$ where $[T]_{p,q}$ is the p,q-th entry of the matrix T.

For a set of matrices $(T_1, \ldots, T_N)$ we define an associated set $(\widetilde{T}_1, \ldots, \widetilde{T}_N)$ where $\widetilde{T}_i = \pm T_i$ as follows. Choose (the least) $p \, (1 \leq p \leq n)$ such that $\sum_{i=1}^N |[T_i]_{p,p}| \geqslant \sum_{i=1}^N |[T_i]_{q,q}|$ for all $1 \leq q \leq n$. For all $i = 1, \ldots, N$ we set

$$\widetilde{T}_i = \begin{cases} T_i & \text{if } [T_i]_{p,p} \geq 0 \\ -T_i & \text{if } [T_i]_{p,p} < 0 \end{cases} . \tag{3.3}$$

Thus $\widetilde{T}_i = \pm T_i$ and $[\widetilde{T}_i]_{p,p} \geq 0$ for all i.

The following lemma complements Lemma 3.1 in the case of upper triangular matrices.

Lemma 3.2. *Suppose that a family of linear mappings are simultaneously represented by upper triangular matrices* $(T_1, \ldots, T_N)$ *with respect to some basis. Then*

$$\rho(\widetilde{T}_1 + \ldots + \widetilde{T}_N) \geq \Phi^1(T_1, \ldots, T_N). \tag{3.4}$$

PROOF. For given k write $\mathcal{R}$ for the set of sequences $\{(r_1, r_2, \dots, r_k, r_{k+1}) : 1 \leq r_1 \leq r_2 \leq \dots \leq r_{k+1} \leq n\}$. Since the matrices are upper triangular $[T_i]_{p,q} = 0$ unless $p \leq q$. Thus expanding the matrix products,

$$\begin{aligned}
\sum_{|\mathbf{i}|=k} \|T_{\mathbf{i}}\| = \sum_{|\mathbf{i}|=k} &\|T_{i_1} T_{i_2} \dots T_{i_k}\| \\
&= \sum_{i_1 \dots i_k} \sum_{r_1 \dots r_{k+1} \in \mathcal{R}} |[T_{i_1}]_{r_1 r_2}| \dots |[T_{i_k}]_{r_k r_{k+1}}| \\
&= \sum_{r_1 \dots r_{k+1} \in \mathcal{R}} \Big(\sum_{i=1}^{N} |[T_i]_{r_1 r_2}|\Big) \dots \Big(\sum_{i=1}^{N} |[T_i]_{r_k r_{k+1}}|\Big) \\
&\leq c^n \sum_{r_1 \dots r_{k+1} \in \mathcal{R}} \Big(\sum_{i=1}^{N} |[\widetilde{T}_i]_{p,p}|\Big)^{k-n},
\end{aligned}$$

where $c = \max_{p,q}\{\sum_{i=1}^{N} |[T_i]_{p,q}|\}$, noting that for $(r_1, \dots, r_{k+1}) \in \mathcal{R}$ we have $r_j = r_{j+1}$ for all but at most n of the indices j. The number of words in $\mathcal{R}$ is at most $(k+2)^n$ so we conclude that

$$\sum_{|\mathbf{i}|=k} \|T_{\mathbf{i}}\| \leq c^n (k+2)^n \Big(\sum_{i=1}^{N} |[\widetilde{T}_i]_{p,p}|\Big)^{k-n} \leq c^n (k+2)^n \Big\| \sum_{i=1}^{N} \widetilde{T}_i \Big\|^{k-n}.$$

Inequality (3.4) follows on taking the kth root and letting $k \to \infty$. $\square$

These lemmas lead to the following continuity property.

Theorem 3.3. *Suppose that the linear mappings $(T_1, \dots, T_N)$ on $\mathbf{R}^2$ may be simultaneously represented by upper triangular matrices with respect to some basis. Then Φ^1 is continuous at $(T_1, \dots, T_N)$.*

PROOF. Suppose that $(S_1, \dots, S_N) \to (T_1, \dots, T_N)$. Set

$$\widetilde{S}_i = \begin{cases} S_i & \text{if } [T_i]_{p,p} \geq 0 \\ -S_i & \text{if } [T_i]_{p,p} < 0 \end{cases}$$

where p is chosen as for (3.3); thus $\widetilde{S}_i$ of opposite sign to S_i if and only if $\widetilde{T}_i$ of opposite sign to T_i. Then $(\widetilde{S}_1, \dots, \widetilde{S}_N) \to (\widetilde{T}_1, \dots, \widetilde{T}_N)$ so

$$\rho(\widetilde{S}_1 + \dots + \widetilde{S}_N) \to \rho(\widetilde{T}_1 + \dots + \widetilde{T}_N).$$

By Lemmas 3.1 and 3.2

$$\Phi^1(S_1,\ldots,S_N) = \Phi^1(\widetilde{S}_1,\ldots,\widetilde{S}_N) \geq \rho(\widetilde{S}_1 + \ldots + \widetilde{S}_N)$$

and

$$\rho(\widetilde{T}_1 + \ldots + \widetilde{T}_N) \geq \Phi^1(T_1,\ldots,T_N)$$

It follows that

$$\liminf \Phi^1(S_1,\ldots,S_N) \geqslant \Phi^1(T_1,\ldots,T_N)$$

giving lower semi-continuity at $(T_1,\ldots,T_N)$. Upper semi-continuity follows from Proposition 1.1. $\square$

4 Summary and Conclusions.

We summarise some cases where we have continuity. Firstly, we can say rather more for mappings of $\mathbf{R}^2$.

Proposition 4.1. *Let $T_1,\ldots,T_N \in \mathcal{L}(\mathbf{R}^2,\mathbf{R}^2)$.*

(a) If $T_1,\ldots,T_N$ have no common real eigenvector then $\Phi^s(T_1,\ldots,T_N)$ is continuous at $(T_1,\ldots,T_N)$ for all $0 < s \leq 2$.

(b) If there is a cone $C \equiv \{0 \neq x \in \mathbf{R}^2 : |x.\theta|/|x| < c\}$ for some unit vector c and $0 < c < 1$ such that for all i either $T_i(\overline{C}) \subset C \cup \{0\}$ or $T_i(\overline{C}) \subset -C \cup \{0\}$ (where $\overline{C}$ is the closure of C) then $\Phi^s(T_1,\ldots,T_N)$ is continuous at $(T_1,\ldots,T_N)$ for all $0 < s \leq 2$.

(c) $\Phi^1(T_1,\ldots,T_N)$ is continuous at all $(T_1,\ldots,T_N)$.

PROOF. (a) Recall that in $\mathbf{R}^2$ the condition $C(s)$ reduces to $C(1)$ for all $0 < s \leq 2$. Given $v \neq 0$ we can choose T_i for which v is not an eigenvector so $\mathrm{span}\{v, T_i v\} = \mathbf{R}^2$. The conclusion follows on taking $\mathcal{S} = \{T_{\mathbf{i}}\}$ in Proposition 2.5.

(b) This is the case considered by Käenmäki and Shmerkin [8]. By changing the sign of some of the T_i if necessary we can assume that $T_i(\overline{C}) \subset C \cup \{0\}$ for all i and by choosing an appropriate basis we can assume that the matrices representing the T_i have strictly positive entries. The observations that $\|T_{\mathbf{i}}\| \asymp [T_{\mathbf{i}}]_{p,q}$ for all $\mathbf{i} \in J$ and $1 \leq p,q \leq 2$ and that the determinant is multiplicative then lead to the conclusion.

(c) If the $T_1, \ldots, T_N$ have no common eigenvector this is just part (a). Otherwise, if $v \neq 0$ is a common eigenvector, the T_i may be simultaneously represented by upper triangular matrices with respect to a basis which includes v so continuity follows from Proposition 3.3. $\square$

For general n and $0 < s \leq 1$, Lemma 2.6 gives another simple condition for Theorem 2.5 to hold.

Proposition 4.2. *Let $0 < s \leq 1$. If $T_1, \ldots, T_N \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ have no non-trivial common invariant subspace, then Φ^s is continuous at $(T_1, \ldots, T_N)$.*

Proof. Since condition $C(s)$ is just $C(1)$ for $0 < s \leq 1$ this follows from Theorem 2.5 and Lemma 2.6. $\square$

It seems awkward to prove continuity of Φ^s at all $T_1, \ldots, T_N \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ even when $n = 2$, although by Proposition 4.1 the possible points of discontinuity are very limited. Moreover results of Falconer and Miao [4] giving an explicit expression for $\Phi^s(T_1, \ldots, T_N)$ when the T_i are simultaneously upper triangular imply that Φ^s is continuous when restricted to upper triangular matrices. When seeking counterexamples in these cases one would require very subtle cancellation of terms when perturbing upper triangular mappings to ones that were not simultaneously upper triangular.

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