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KMS STATES, ENTROPY, AND A VARIATIONAL PRINCIPLE FOR PRESSURE

Abstract

We relate the concepts of entropy and pressure to that of KMS states for C^*-algebras. Several different definitions of entropy are known in our days: the one we present here is quite natural, extending the usual one for Dynamical Systems in Thermodynamic Formalism Theory, being basically obtained from transfer operators (also called Ruelle operators) and having the advantage of being very easily introduced. We also present a concept of pressure as a min-max principle.

Later on, we consider the concept of a KMS state as an equilibrium state for a potential, in the context of C^*-algebras, and we show that there is a relation between equilibrium measures and KMS states for certain algebras arising from a continuous transformation.

1 Introduction and Main Result.

We want to relate equilibrium measures from the theory of Thermodynamic Formalism with KMS states, their analogues in the C^*-algebras theory. Nowa-

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days, several different definitions of entropy are known (see [19]). The definition we present here is quite natural and extends the usual one for Dynamical Systems in Thermodynamic Formalism Theory (see [14]), being basically obtained from transfer operators, and having the advantage of being very easily introduced. Later on (Section 3), we interpret the concepts of entropy and pressure in the setting of commutative C^*-algebras.

Finally, we consider the concept of a KMS state as an equilibrium state for a potential (in the context of C^*-algebras) and show that they are related to the equilibrium measures of the Thermodynamic Formalism Theory. This problem, in a similar context, was also considered in [22], and [11].

In the next section we describe briefly the main prerequisites for the statement of our main result, which is stated and proved in the last section.

2 Review of Thermodynamic Formalism and C^*-algebras.

First we present the main concepts of the theory of Thermodynamic Formalism, a mathematical theory initially developed by D. Ruelle and Y. Sinai and inspired by problems borrowed from Statistical Mechanics.

We denote by $C(X)$ the space of continuous real functions of X, where (X, d) is a compact metric space and consider the Borel sigma-algebra over X. Given a continuous transformation $T : X \to X$, we denote by $\mathcal{M}(T)$ the set of invariant probabilities ν for T, that is, those probabilities satisfying $\int f \circ T \, d\nu = \int f \, d\nu$, for every $f \in C(X)$.

From now on, suppose that T is an expanding map (see definition in [26]). We refer the reader to [24], [26], and [25] for general definitions and properties of Thermodynamic Formalism, as well as expanding maps; for these maps there are many nice results (see [26]).

Typical examples of such expanding maps are the shift transformation T of the Bernoulli space $\Omega = \{1, 2, \ldots, d\}^{\mathbb{N}}$, as well as the $C^{1+\alpha}$-transformations of the unit circle satisfying $|T'(x)| > c > 1$, for some constant c, where $| \cdot |$ denotes the usual norm (associating the unit circle to the interval $[0, 1)$ in a standard way).

Our results also apply to the following type of expanding maps (see [1], and [16]): the geodesic flow of a compact constant negative curvature surface induces a Markov transformation G of the boundary of the Poincaré disk such that, for some n, the iterate $G^n = T$ is a continuous expanding transformation acting on the unit circle.

For each $\nu \in \mathcal{M}(T)$, let $h(\nu)$ denote the Shannon-Kolmogorov entropy of ν (see [20]). This entropy measures the dynamic complexity of the action of the transformation T on sets of measure one. One interesting problem is to consider the maximal entropy among invariant probabilities, that is, $h(T) = \sup\{h(\nu) \,|\, \nu \in \mathcal{M}(T)\}$, called the topological entropy of T. A probability that attains such a supremum value is called a measure of maximal entropy.

This way, we are looking for the probability with the largest complexity. In the case of the Bernoulli space $\Omega = \{1, 2, \ldots, d\}^{\mathbb{N}}$, the maximal value of the entropy of invariant probabilities is $\log d$, and there is a unique probability μ that attains such a value. In this case, the maximal entropy measure μ is the independent probability with weights $1/d$.

One of the main principles of physics is that nature maximizes entropy. In statistical mechanics, the maximal entropy measure μ corresponds to what is to be expected at infinite temperature (see [24], and [26]). When the temperature is finite, nature maximizes pressure. A probability that maximizes pressure is called a Gibbs probability (or state). In fact, there exists an external potential $A : \Omega \to \mathbb{R}$ which describes the interaction $A(w) = A(w_0, w_1, w_2, \ldots)$, with $w = (w_0, w_1, w_2, \ldots) \in \Omega$, around neighborhoods in the lattice $\Omega = \{1, 2, \ldots, d\}^{\mathbb{N}}$.

The simplest case occurs with this potential being a function $A(w)$ which depends only on a finite number of coordinates; for example, on two coordinates, $A(w) = A(w_0, w_1)$, when we get a finite range iteration potential. These types of potentials are more easily dealt with, but the more important potentials, for mathematical applications, are those that depend on entire sequences $w = (w_0, w_1, w_2, \ldots) \in \{1, 2, \ldots, d\}^{\mathbb{N}}$.

A common example in statistical mechanics occurs with $d = 2$, when we associate 1 to the spin $+$ and 2 to the spin $-$. An element w in this Bernoulli space could be, for instance, $w = (+ - - + - + - + + \ldots)$, that is, an element with spin up or down in different positions of a lattice over the set $\mathbb{N}$. The Gibbs probability for A describes the probabilities of the Borel sets of the space $\{+, -\}^{\mathbb{N}}$ that are determined by the interactions given by A.

As in statistical mechanics, we may also consider an extra real parameter $\beta = 1/\tau$, where τ represents the temperature.

Definition 2.1. Given a potential A, the pressure of A at temperature $\tau = 1/\beta$ is given by

$$P(\beta\, A) = \sup \left\{ h(\nu) \, + \, \beta \int A \, d\nu \,\middle|\, \nu \in \mathcal{M}(T) \right\}.$$

A measure $\mu = \mu_A$ satisfying such a supremum is called a Gibbs state for A at temperature τ. This probability μ_A, also called an equilibrium state for A, describes what is physically observable, in probabilistic terms (see [24]). When $\beta = 1$, we simply write $P(A)$. When $\beta = 0$, which corresponds to $A = 0$, we get the case of infinite temperature, and the Gibbs state is the independent probability mentioned earlier.

If the potential A is Hölder, which corresponds to a fast decay of interaction between neighborhoods, the Gibbs state μ_A is unique (see [25], and [20]).

For a differentiable transformation $T : S^1 \to S^1$ of the unit circle S^1 (or an interval), a very important potential to consider is $A(x) = -\log T'(x)$, when $-\int \log T'(x)\, d\mu(x)$ measures the μ-mean sensibility with respect to initial conditions. In this case, the measure μ that maximizes pressure is called the Bowen-Ruelle-Sinai probability (see [18]).

For more general transformations of the unit circle we may consider an extra parameter β (which now has nothing to do with temperature) and the potentials $\beta(-\log T')$. A special value of β, namely the one for which $P(-\beta \log T') = 0$, is associated to the Hausdorff dimension of sets that are important for the dynamical viewpoint (see [26], and [17]).

Applications to geometry, the dimension of fractals, zeta functions, as well as others, may be found in [3], [1], [17], [20], and [16].

The main tool for obtaining the Gibbs probability is the Ruelle operator, which is called the transfer operator in statistical physics. Let us consider the general setting.

Definition 2.2. Given a potential $A : X \to \mathbb{R}$, the Ruelle operator, or transfer operator, $\mathcal{L}_A : C(X) \to C(X)$ is given by

$$\mathcal{L}_A(f)(x) = \sum_{T(z)=x} e^{A(z)}\, f(z),$$

for each continuous $f : X \to \mathbb{R}$, and any $x \in X$.

We may also consider the dual Ruelle operator $\mathcal{L}_A^*$, acting on measures over the Borel sigma-algebra of X. When A is such that $\mathcal{L}_A(1) = 1$, we say that the potential A is normalized; in this case, if ν is a probability, $\mathcal{L}_A^*(\nu)$ is also a probability. If A is Hölder, then the Ruelle operator also acts on the space $\mathcal{H} = \mathcal{H}_\alpha$ of α-Hölder functions taking complex values (with fixed $0 < \alpha \leq 1$).

We will now state a main result of this theory in the particular setting of Bernoulli spaces (see [25], and [20]), which is a more advanced version of the Perron Theorem for positive matrices.

Theorem 2.1 (Ruelle)**.** *If $A : \{1, 2, \ldots, d\}^{\mathbb{N}} \to \mathbb{R}$ is Hölder, then there exist a maximal eigenvalue λ and an associated Hölder eigenfunction ϕ for $\mathcal{L}_A$, that is, $\mathcal{L}_A(\phi) = \lambda\, \phi$. Moreover, λ is isolated in the spectrum of the operator $\mathcal{L}_A$, and there exists an eigen-probability ν such that $\mathcal{L}_A^*(\nu) = \lambda\, \nu$. Finally, the Gibbs state probability μ_A for A is given by $\mu_A = \phi\, \nu$ (after suitable normalization).*

We point out that when A depends only on two coordinates, Ruelle's Theorem is a consequence of Perron's Theorem for positive matrices (see [20]).

There is a different way to compute the entropy, via the Perron operator acting on different potentials.

Theorem 2.2 ([14]). *Let $\mathbb{B}^+$ denote the set of Borel positive functions of Ω. Given $\mu \in \mathcal{M}(T)$ and a Hölder potential A, the entropy of μ is given by*

$$h(\mu) = \inf_{f \in \mathbb{B}^+} \int \log\left(\frac{P_A f}{A\, f}\right) d\mu.$$

This result shows that we may avoid the dynamical viewpoint of entropy (considering partitions of the Bernoulli space, refinements of the partition by iterations, and so on) and address all the computations to the action of the Ruelle operator. This turns out to be quite useful for the generalization to C^*-algebras, where there is no natural dynamics involved, and where a dynamics based definition would be quite complicated.

There is also a different way to compute pressure, via a min-max principle.

Theorem 2.3 ([14]). *Let $\mathbb{B}^+$ denote the set of Borel positive functions on Ω. Given a Hölder potential A, the topological pressure is given by*

$$P(A) = \sup_{\mu \in \mathcal{M}(T)} \inf_{f \in \mathbb{B}^+} \int \log\left(\frac{P_A\, f}{f}\right) d\mu.$$

Now we briefly describe some basic results concerning the theory of C^*-algebras, which was initially developed by I. M. Gelfand and J. von Neumann, and is presented, quite elegantly, in [21], and [2]. We refer the reader to [12], [23], [6], [7], [9], [8] and [10] for a more thorough description of the relation between Thermodynamic Formalism and C^*-algebras.

The role that KMS states play in quantum statistical mechanics is very important, being, as we will see, that of equilibrium states in C^*-algebras. In quantum mechanics, the potential $A : X \to \mathbb{C}$, also called an observable, is replaced by an operator acting on the complex Hilbert space $\mathcal{L}^2(\mu)$. Thus, the commutative algebra of functions (with the usual complex product structure) gives place to the non-commutative algebra of bounded operators B of $\mathcal{L}^2(\mu)$ (where the product structure is the composition of operators). The norm of the algebra is the operator norm and, for the operation $*$ of the algebra, we consider the adjoint operator B^* of each operator B.

We refer the reader to [2] for more detailed definitions and the main properties of C^*-algebras (see therein, for instance, Definition 2.1.1 and Example

2.1.2). Here we will only recall some terminology. Firstly, a C^*-algebra is a complete normed algebra $\mathcal{A}$ over $\mathbb{C}$ with an involution operation $*$ satisfying

$$\|a\,a^*\| = \|a\|^2,$$

for all $a \in \mathcal{A}$. We say that an element $a \in \mathcal{A}$ is positive, if it is given by $a = b\,b^*$, for some element $b \in \mathcal{A}$.

A state of a C^*-algebra $\mathcal{A}$ with unit 1 is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1) = 1$ and $\phi(a)$ is a positive real number, whenever a is a positive element of $\mathcal{A}$. The states ϕ of a C^*-algebra play the role of the probabilities ν in Thermodynamic Formalism.

A one-parameter group of automorphisms in a C^*-algebra $\mathcal{A}$ is a strongly continuous group homomorphism $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$, which we interpret as a dynamic temporal evolution in the C^*-algebra. We write σ_t for the automorphism $\sigma(t)$ and we say that an element $a \in \mathcal{A}$ is analytic for σ if $\sigma_t(a)$ has an analytic extension from $t \in \mathbb{R}$ to all $z \in \mathbb{C}$. We remark that the set of analytic elements is always dense in $\mathcal{A}$.

Definition 2.3. Let σ be a one-parameter group of automorphisms of $\mathcal{A}$ and let $\beta \in \mathbb{R}$ be given. We say that a state ϕ of $\mathcal{A}$ is a (σ, β)-KMS state if

$$\phi(a\,b) = \phi(b\,\sigma_{\beta i}(a)),$$

for any $a, b \in \mathcal{A}$, with a analytic.

From now on, $C(X)$ denotes the space of continuous functions defined on the compact metric space X and taking values in $\mathbb{C}$. Also, T is an expanding map of X and μ is a Gibbs measure for a fixed potential $\tilde{A}$. For the Bernoulli space $\Omega = \{1, 2, \ldots, d\}^{\mathbb{N}}$ (our main case of interest), this potential may be taken as the constant potential $-\log d$; it follows that μ is the independent probability, with weights $1/d$, over $\{1, 2, \ldots, d\}^{\mathbb{N}}$ and the dual of Ruelle operator of $\tilde{A}$, acting on probabilities, satisfies $\mathcal{L}^*_{\tilde{A}}(\mu) = \mu$ (see [9], and [20]).

An important class of linear operators of $\mathcal{L}^2(\mu)$ is obtained as follows: for any fixed $f \in C(X)$, the operator $M_f : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu)$, sometimes denoted simply by f, is defined by

$$M_f(\eta)(x) = f(x)\eta(x),$$

for any η in $\mathcal{L}^2(\mu)$, $x \in X$. The product operation satisfies $M_f \circ M_g = M_{f \cdot g}$, for $f, g \in C(X)$, where the dot $\cdot$ denotes the complex multiplication, and the involution operation $*$ is given by $M_f^* = M_{\overline{f}}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Thus, $M_{\overline{f}}$ is the adjoint operator of M_f over $\mathcal{L}^2(\mu)$.

In Thermodynamic Formalism it is usual to consider the Koopman operator S acting on $\mathcal{L}^2(\mu)$, that is, the bounded linear operator $S : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu)$ given by $(S\eta)(x) = \eta(T(x))$, for any $\eta \in \mathcal{L}^2(\mu)$, $x \in X$. It is well known that its adjoint S^* over $\mathcal{L}^2(\mu)$ is the operator $\mathcal{L}_{\tilde{A}}$, acting on $\mathcal{L}^2(\mu)$ (which is well defined, as can be seen in [20]). The main point for our choice of μ is precisely the assertion $\mathcal{L}_{\tilde{A}} = S^*$.

Now we have all elements to define our two C^*-subalgebras $\mathcal{U}$ and $\mathcal{V}$ of the C^*-algebra of bounded operators $B : \mathcal{L}^2(\mu) \to \mathcal{L}^2(\mu)$.

Let $\mathcal{V} = \mathcal{V}(\mathcal{T}, \mu)$ denote the C^*-subalgebra generated by S and M_f, for all $f \in C(X)$, and $\mathcal{U} = \mathcal{U}(T, \mu)$ the C^*-subalgebra generated by the elements $M_f S^n (S^*)^n M_g$, for all $n \in \mathbb{N}$ and $f, g \in C(X)$.

The algebra $\mathcal{U}$ is a C^*-subalgebra of $\mathcal{V}$. In fact, it suffices to observe that each element of $\mathcal{V}$ is the limit of finite sums $\sum_i M_{f_i} S^{n_i} (S^*)^{m_i} M_{g_i}$, whereas an element of $\mathcal{U}$ is the limit of finite sums $\sum_i M_{f_i} S^{n_i} (S^*)^{n_i} M_{g_i}$, with identical exponents for S and S^*.

We now consider certain dynamical evolutions in the C^*-algebras $\mathcal{U}$ and $\mathcal{V}$. Given a strictly positive function $H : X \to \mathbb{R}$, we define an associated one-parameter group of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{V})$ as follows: for each fixed $t \in \mathbb{R}$, σ_t is given by $\sigma_t(M_f) = M_f$, for $f \in C(X)$, and $\sigma_t(S) = M_{H^{it}} \circ S$, in the following sense:

$$(\sigma_t(S)(\eta))(x) = H^{i\,t}(x)\eta(T(x)) \in \mathcal{L}^2(\mu),$$

for any $\eta \in \mathcal{L}^2(\mu)$, $x \in X$. Since $\sigma_t(\mathcal{U}) \subset \mathcal{U}$ for each $t \in \mathbb{R}$, we may restrict σ to $\mathcal{U}$.

In terms of C^*-dynamical systems formalism, the shift transformation T here simply plays the role of spatial translation in the lattice, while the positive function H defines the dynamics of the evolution with time $t \in \mathbb{R}$, corresponding to the potential A in Thermodynamic Formalism, via $H = e^A$. If we introduce a parameter β, we will have to consider the potential H^β.

Given H and β, we let $\phi_{H,\beta}$ denote a KMS state, leaving the letter ϕ for a general C^*-dynamical system state. The state $\phi_{H,\beta}$ is what is expected, from the quantum statistical point of view, of a system governed by H, under temperature $\tau = 1/\beta$ (see [2]).

Our purpose is to analyze these KMS states $\phi_{H,\beta}$. Given a pair (H, β), it is easy to see that the condition

$$\phi(a \cdot b) = \phi(b \cdot \sigma_{\beta i}(a)), \quad \text{for all} \quad a, b \in \mathcal{A}$$

is equivalent to

$$\phi(\sigma_\tau(a) \cdot b) = \phi(b \cdot \sigma_{\tau + \beta i}(a)), \quad \text{for all} \quad a, b \in \mathcal{A} \text{ and } \tau \in \mathbb{R}.$$

It follows (see Section 8.12 of [20]) that if ϕ is a KMS state for (H, β), then, for any analytic $a \in \mathcal{U}$, the extension of $\tau \to \phi(\sigma_\tau(a))$ to $z \to \phi(\sigma_z(a))$ is a bounded entire function, and therefore constant. In this sense, ϕ is stationary.

A natural question arises: for given β and H, when does the KMS state $\phi_{H,\beta}$ exist, and when is it unique? This question is considered in [7], for the C^*-algebra $\mathcal{V}$, and in [9], for the C^*-algebra $\mathcal{U}$. Another presentation of the uniqueness part of this question appears in [10].

Notice that the action of the linear functional ϕ on the set of operators M_f (with f ranging over all continuous functions) defines a measure ν over X, via the Riesz Theorem, that is, we have $\phi(M_f) = \int f \, d\nu$, for each $f \in C(X)$. In fact, ν is a probability, by the hypotheses we imposed on the C^*-states ϕ.

One of the main points in [9] is that for the KMS state of $\mathcal{U}$ associated to H and β, this measure ν is the eigen-measure $\nu_{H,\beta}$ for the dual Ruelle operator $\mathcal{L}_A^*$, where $A = -\beta \log H$. Thus, we associate, in a unique way, each KMS state $\phi_{H,\beta}$ to an eigen-measure $\nu_{H,\beta}$, establishing an interesting relation between the Thermodynamic Formalism and C^*-algebras.

For the KMS states in $\mathcal{V}$, there exists the extra condition that the pressure of $H^{-\beta}$ is zero, therefore the KMS states exist for only one value of β ([8], [6]).

3 Statement and Proof of our Results.

In this section we will present our results: a definition of entropy and pressure for states in the C^*-algebra and a proof of the existence of a state with maximum pressure.

Suppose that $\mathcal{A}$ is a commutative C^*-algebra with unit and that $\alpha : \mathcal{A} \to \mathcal{A}$ is an injective unit preserving endomorphism. We say that a state ϕ in $\mathcal{A}$ is α-invariant if $\phi \circ \alpha = \phi$.

In the special case $\mathcal{A} = C(X)$, the Gelfand-Naimark Theorem yields a continuous transformation $T : X \to X$ satisfying $\alpha(a) = a \circ T$, for each $a \in \mathcal{A}$. In the general case, we say that a linear transformation $L : \mathcal{A} \to \mathcal{A}$ is a transfer operator for α if

$$L\big(\alpha(a)b\big) = a\, L(b),$$

for every $a, b \in \mathcal{A}$. Moreover, we say that such a transfer operator is normalized if $L(1) = 1$.

If $\mathcal{A}$ is a commutative algebra, the transfer operator takes the form of a Ruelle operator (see [20], [5], [4], [12], and [6]).

Proposition 3.1 ([13]). *If L is a transfer operator for α, then $L(1)$ is a* *central positive element of* $\mathcal{A}$, *and*

$$L(a\,\alpha(b)) = L(a)\,b,$$

for every $a, b \in \mathcal{A}$.

Proposition 3.2 ([13])**.** *If* T *is a local homeomorphism, then every transfer operator* L *for* α *is of the form* L_ρ, *given, for every* $a \in \mathcal{A}$, *and* $x \in X$, *by*

$$L_\rho(a)(x) = \sum_{T(y)=x} \rho(y)\,a(y),$$

where $\rho : X \to [0,\infty)$ *is some continuous function. Moreover, for any continuous function* $\rho : X \to [0,\infty)$, *the sum on the right side defines a transfer operator. (In the notation of the previous section,* $\rho = e^A$.*)*

From now on, we will assume that T is a local homeomorphism and we will write $\mathcal{A}_+ := \{a \in \mathcal{A} \mid \sigma(a) \in (0,\infty)\}$.

Generalizing the viewpoint of [14], and [15], we now introduce a notion of entropy for a state $\phi : \mathcal{A} \to \mathbb{C}$, using the transfer operator L_ρ defined by ρ.

Definition 3.1. Given a state ϕ in $\mathcal{A}$, we say that

$$h(\phi) = \inf_{a \in \mathcal{A}_+} \phi\left[\log\Big(\frac{L_\rho(a)}{\rho\,a}\Big)\right]$$

is the entropy of ϕ.

Our definition is independent of the choice of ρ. Indeed, if $\rho' : X \to [0,\infty)$ is another continuous function, then $a' = a\,\rho\,(\rho')^{-1} \in \mathcal{A}_+$, for any $a \in \mathcal{A}_+$, implying

$$\frac{L_{\rho'}(a')}{\rho' a'} = \frac{1}{\rho\,a} \sum_{y=T(x)} \rho'(x)a(x)\rho(x)\rho'(x)^{-1} = \frac{L_\rho(a)}{\rho\,a}$$

and showing that the infimum is taken over the same set.

Definition 3.2. Given an element $b \in \mathcal{A}_+$, we say that

$$p(b) = \sup\big\{h(\phi) + \phi(\log b) \mid \alpha\text{-invariant } \phi\big\}$$

is the topological pressure of b. We say that ϕ is a C^*-equilibrium state for b if ϕ is an α-invariant state such that $p(b) = h(\phi) + \phi(\log b)$.

Proposition 3.3. *If* $L_\rho(1) = 1$, *there exists a state* ϕ *such that* $\phi \circ L_\rho = \phi$.

PROOF. Let $\mathcal{S}$ denote the set of all states of $\mathcal{A}$. From $L_\rho(1) = 1$, it follows that $L_\rho^*(\mathcal{S}) \subset \mathcal{S}$. Now the Tychonoff-Schauder Theorem yields a fixed point for $L_\rho^*\big|_{\mathcal{S}}$. □

Proposition 3.4. *If $L_\rho(1) = 1$, then $p(\rho) = 0$. Moreover, every state ϕ that satisfies $\phi \circ L_\rho = \phi$ is an equilibrium state for ρ.*

PROOF. Using our definitions of entropy and pressure, we obtain

$$p(\rho) = \sup_{\phi \text{ inv}} \inf_{a \in \mathcal{A}_+} \phi\left[\log\left(\frac{L_\rho(a)}{a}\right)\right]$$

and, therefore, a choice of $a = 1$ inside the infimum, guarantees that $p(\rho) \le 0$.

On the other hand, $L_\rho \circ \alpha = Id$, because L_ρ is normalized, and $\phi \circ L_\rho = \phi$ implies $\phi \circ \alpha = \phi \circ L_\rho \circ \alpha = \phi$. Since log is concave, $\log(L_\rho(a)) \ge L_\rho(\log a)$ and, therefore,

$$\phi\left[\log\left(\frac{L_\rho(a)}{a}\right)\right] = \phi\big(\log(L_\rho(a)) - \log a\big) \ge \phi\big(L_\rho(\log a) - \log a\big).$$

If $\phi \circ L_\rho = \phi$, the right hand side of this inequality equals zero, and therefore, $\inf\left\{\phi\left[\log\left(\frac{L_\rho(a)}{a}\right)\right] \mid a \in \mathcal{A}_+\right\} = 0$. It follows that $p(\rho) \ge 0$ and, therefore,

$$h(\phi) + \phi(\rho) = \inf_{a \in \mathcal{A}_+} \phi\left[\log\left(\frac{L_\rho(a)}{a}\right)\right] = 0 = p(\rho)$$

holds for eigen-states. □

In the context of an algebra $\mathcal{A}$, an injective unit preserving endomorphism α, and a normalized transfer operator L, we may consider, among others, two different C^*-algebras, namely, the cross-product endomorphism $\mathcal{A} \rtimes_{\alpha,L} \mathbb{N}$ and the C^*-algebra given by approximately proper equivalence relations $C^*(\mathcal{R}, \mathcal{E})$ (see [6], and [9]). The second algebra is related to the equivalence relation $x \sim y \iff$ there exists $n \in \mathbb{N}$ such that $T^n(x) = T^n(y)$, and the first considers the broader equivalence relation $x \sim y \iff$ there exist $n, m \in \mathbb{N}$ such that $T^n(x) = T^m(y)$.

The algebra $C^*(\mathcal{R}, \mathcal{E})$ generalizes the algebra $\mathcal{U}$ of the previous section, whereas $\mathcal{A} \rtimes_{\alpha,L} \mathbb{N}$ generalizes the algebra $\mathcal{V}$. In fact, in the context of the previous section, for each Gibbs measure μ, we find representations of $\mathcal{A} \rtimes_{\alpha,L} \mathbb{N}$ and $C^*(\mathcal{R}, \mathcal{E})$ in the Hilbert space $\mathcal{L}^2(\mu)$ with images $\mathcal{V}$ and $\mathcal{U}$, respectively.

We want to relate the KMS states of $\mathcal{A} \rtimes_{\alpha,L} \mathbb{N}$ and $C^*(\mathcal{R}, \mathcal{E})$ with the equilibrium states (in $\mathcal{A}$) of the potential $h^{-\beta}$, where β, again, represents the reciprocal of temperature.

If the algebra $\mathcal{A}$ is commutative, we have unique conditional expectations $F : \mathcal{A} \rtimes_{\alpha,L} \mathbb{N} \to \mathcal{A}$ and $G : C^*(\mathcal{R}, \mathcal{E}) \to \mathcal{A}$. Moreover, if $E := \alpha \circ L : \mathcal{A} \to \alpha(\mathcal{A})$, is a conditional expectation with finite index, then the KMS states ψ of $\mathcal{A} \rtimes_{\alpha,L} \mathbb{N}$ can be decomposed as $\psi = \phi \circ F$, where ϕ is a state of $\mathcal{A}$ which satisfies

$$\phi(a) = \phi\big(L(\Lambda\, a)\big), \quad \text{for all } a \in \mathcal{A},$$

with $\Lambda = h^{-\beta}\mathrm{ind}(E)$. The KMS state ψ of $C^*(\mathcal{R}, \mathcal{E})$ can be decomposed as $\psi = \phi \circ G$, where ϕ is a state of $\mathcal{A}$ which satisfies

$$\phi(a) = \phi\big(\Lambda^{-[n]} E_n(\Lambda^{[n]} a)\big), \quad \text{for all } a \in \mathcal{A} \text{ and } n \in \mathbb{N},$$

with $E_n = \alpha^n \circ L^n$, and $\Lambda^{[n]} = \prod_{i=0}^{n-1} \alpha^i\big(h^{-\beta}\mathrm{ind}(E)\big)$.

Proposition 3.5. *If $\psi = \phi \circ F$ is an (h, β)-KMS state for $\mathcal{A} \rtimes_{\alpha,L} \mathbb{N}$, and $L(\Lambda 1) = 1$, then ϕ is an equilibrium state (in $\mathcal{A}$) for the potential $h^{-\beta}$.*

PROOF. The condition $L(\Lambda 1) = 1$ implies that $L_{h^{-\beta}}$ is a normalized transfer operator, and, therefore, $p(h^{-\beta}) = 0$. Moreover, the KMS condition says that $\phi(a) = \phi(L_{h^{-\beta}}(a))$, which implies that $\phi(\alpha(a)) = \phi(a)$. By Proposition 3.4, it follows that ϕ is an equilibrium state for $h^{-\beta}$. $\square$

In the constructions of the algebras that interest us, the choice of the normalized transfer operator is arbitrary, since two such operators define isomorphic algebras. If we suppose that $L_\rho(k) = \lambda\, k$, for some $\lambda > 0$ and $k \in \mathcal{A}_+$, and write $\widetilde{\rho} = \frac{\rho\, k}{\lambda \alpha(k)}$, it therefore follows that $L_{\widetilde{\rho}}$ is a normalized transfer operator for α, which can be used to construct $C^*(\mathcal{R}, \mathcal{E})$.

Theorem 3.1. *Let $\psi = \phi \circ G$ be an (h, β)-KMS state of $C^*(\mathcal{R}, \mathcal{E})$. Let $\rho = h^{-\beta}$, suppose that $L_\rho(k) = \lambda\, k$, for some $\lambda > 0$ and $k \in \mathcal{A}_+$, and denote $\widetilde{\rho} = \frac{\rho\, k}{\lambda \alpha(k)}$. Finally, let $\widetilde{\phi}$ be the state of $\mathcal{A}$ given by $\widetilde{\phi}(a) = \phi(ka)$. If, for every $a \in \mathcal{A}$,*

$$\lim_{n \to \infty} \big\| L^n_{\widetilde{\rho}}(a) - \widetilde{\phi}(a) \big\| = 0,$$

then $\widetilde{\phi}$ is an equilibrium state for $\widetilde{\rho}$.

PROOF. Without loss of generality, we may assume that $C^*(\mathcal{R}, \mathcal{E})$ has been obtained from $L_{\widetilde{\rho}}$ in such a way that $\mathrm{ind}(E) = \widetilde{\rho}^{-1}$. Then,

$$\Lambda^{[1]} = \big(\rho \widetilde{\rho}^{-1}\big) = \rho\, \frac{\lambda\, \alpha(k)}{\rho\, k} = \frac{\lambda\, \alpha(k)}{k}\,,$$

and, more generally,

$$\Lambda^{[n]} = \prod_{i=0}^{n-1} \alpha^i\big(\rho\widetilde{\rho}^{\,-1}\big) = \lambda^n \, \frac{\prod_{i=0}^{n-1} \alpha^{i+1}(k)}{\prod_{i=0}^{n-1} \alpha^i(k)} = \frac{\lambda^n \, \alpha^n(k)}{k} \, .$$

The KMS condition implies that, for every $n \in \mathbb{N}$,

$$\phi(a) = \phi\Big[\frac{k}{\lambda^n \alpha^n(k)} \, \alpha^n L_{\widetilde{\rho}}^n\Big(\frac{\lambda^n \alpha^n(k)}{k} a\Big)\Big] = \phi\Big[k \, \alpha^n L_{\widetilde{\rho}}^n\Big(\frac{a}{k}\Big)\Big] \, ,$$

and it follows that, for every $n \in \mathbb{N}$,

$$\widetilde{\phi}(a) = \phi(a\,k) = \phi\Big[k \, \alpha^n L_{\widetilde{\rho}}^n\Big(\frac{a}{k} k\Big)\Big] = \widetilde{\phi}\Big[\alpha^n L_{\widetilde{\rho}}^n(a)\Big] \, .$$

Now,

$$\begin{aligned}
\Big|\widetilde{\phi}\big(L_{\widetilde{\rho}}(a) - a\big)\Big| &= \Big|\widetilde{\phi}\big[\alpha^n\big(L_{\widetilde{\rho}}^{n+1}(a) - L_{\widetilde{\rho}}^n(a)\big)\big]\Big| \\
&\leq \widetilde{\phi}\Big(\big\|\alpha^n\big(L_{\widetilde{\rho}}^{n+1}(a) - L_{\widetilde{\rho}}^n(a)\big)\big\|\Big) \\
&\leq \widetilde{\phi}\Big(\big\|L_{\widetilde{\rho}}^{n+1}(a) - L_{\widetilde{\rho}}^n(a)\big\|\Big) \\
&\leq \widetilde{\phi}\Big(\big\|L_{\widetilde{\rho}}^{n+1}(a) - \widetilde{\phi}(a)\big\|\Big) - \widetilde{\phi}\Big(\big\|\widetilde{\phi}(a) - L_{\widetilde{\rho}}^n(a)\big\|\Big) \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

and, therefore $\widetilde{\phi} \circ L_{\widetilde{\rho}} = \widetilde{\phi}$. The claim follows by Proposition 3.4. ☐

Notice that our main hypothesis, namely, the convergence of $L_{\widetilde{\rho}}^n$, is one of the conclusions of the Ruelle-Perron-Frobenius Theorem (see [20], [9], and [4]), which means that the classical setting satisfies the hypotheses of our result.

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References

[1] R. Bowen and C. Series, *Markov maps associated with Fuchsian groups*, Inst. Hautes Études Sci. Publ. Math., **50** (1979), 153–170.

[2] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, **2**, Equilibrium states. Models in quantum statistical mechanics. Second edition, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

[3] J.-P. Conze, J.-P. and A. Raugi, *Fonctions harmoniques pour un operateur de transition e applications*, Bull. Soc. Math. France, **118(4)** (1990), 273–310.

[4] D. E. Dutkay and P. E. T. Jorgensen, *Iterated function systems, Ruelle operators, and invariant projective measures*, Math. Comp., **75(256)** (2006), 1931–1970.

[5] D. E. Dutkay and P. E. T. Jorgensen, *Disintegration of projective measures*, Proc. Amer. Math. Soc., **135(1)** (2007), 169–179.

[6] R. Exel, *A new look at the crossed-product of a C^*-algebra by an endomorphism*, Ergodic Theory Dynam. Systems, **23(6)** (2003), 1733–1750.

[7] R. Exel, *Crossed-products by finite index endomorphisms and KMS states*, J. Funct. Anal., **199(1)** (2003), 153–188.

[8] R. Exel, *KMS states for generalized Gauge actions on Cuntz-Krieger Algebras*, Bull. Braz. Math. Soc., **35(1)** (2004), 1–12.

[9] R. Exel and A. O. Lopes, *C^*-algebras, approximately proper equivalence relations, and Thermodynamic Formalism*, Ergodic Theory Dynam. Systems, **24** (2004), 1051–1082.

[10] R. Exel and A. O. Lopes, *C^*-algebras and thermodynamic formalism*, São Paulo Journal of Mathematical Sciences, **2** (2008), 285–307.

[11] D. Kerr and C. Pinzari, *Noncommutative pressure and the variational principle in Cuntz-Krieger-type C^*-algebras*, J. Funct. Anal., **188** (2002), 156–215.

[12] A. Kumjian and J. Renault, *KMS states on C^*-algebras associated to expansive maps*, Proc. Amer. Math. Soc., **134(7)** (2006), 2067–2078.

[13] B. K. Kwaśniewski, *On transfer operators for C^*-dynamical systems*, arXiv:math/0703798.

[14] A. O. Lopes, *An analogy of the charge distribution on Julia sets with the Brownian motion*, J. Math. Phys., **30(9)** (1989), 2120–2124.

[15] A. O. Lopes and E. R. Oliveira, *Entropy and variational principles for holonomic probabilities of IFS*, Discrete Contin. Dyn. Syst. Ser. A, **23(3)** (2009), 937–955.

[16] A. O. Lopes and P. Thieullen, *Mather measures and the Bowen-Series transformation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **23** (2006), 663–682.

[17] D. Mauldin and M. Urbanski, *Graph Directed Markov Systems. Geometry and Dynamics of Limit Sets*, Cambridge Tracts in Mathematics, **148**, Cambridge University Press, Cambridge, 2003.

[18] W. de Melo and S. Van Strien, *One-Dimensional Dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **25**, Springer-Verlag, Berlin, 1993.

[19] S. Neshveyev and E. Størmer, *Dynamical Entropy in Operator Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Folge, A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics], **50**, Springer-Verlag, Berlin, 2006.

[20] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, (French Summary), Astérisque, **187-188** (1990).

[21] G. K. Pedersen, *C^*-algebras and their Automorphism Groups*, London Mathematical Society Monographs, **14**, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1979.

[22] C. Pinzari, Y. Watatani, and K. Yonetani, *KMS states, entropy and the variational principle in full C^*-dynamical systems*, Comm. Math. Phys., **213** (2000), 331–379.

[23] J. Renault, *KMS states on C^*-algebras associated to expansive maps*, Proc. Amer. Math. Soc., **134(7)** (2006), 2067–2078.

[24] D. Ruelle, *Statistical Mechanics of a one-dimensional lattice gas*, Comm. Math. Phys., **9** (1968), 267–278.

[25] D. Ruelle, *Thermodynamic Formalism. The Mathematical Structures of Classical Equilibrium Statistical Mechanics*, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota, Encyclopedia of Mathematics and its Applications, **5**, Addison-Wesley, Reading, Mass., 1978.

[26] D. Ruelle, *The Thermodynamic Formalism for expanding maps*, Comm. Math. Phys., **125** (1989), 239–262.