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BANACH SPACES FOR THE FEYNMAN INTEGRAL

Abstract

In this paper, we survey progress on the general theory for path integrals as envisioned by Feynman. We introduce a new class of spaces $\mathbf{KS}^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ and $n \in \mathbb{N}$, and their Sobolev counterparts, $\mathbf{KS}^{m,p}(\mathbb{R}^n)$, for $1 \le p \le \infty$, $m \in \mathbb{N}$, which allow us to construct the path integral in the manner originally intended by Feynman. Each space contains all of the standard Lebesgue spaces, $\mathbf{L}^p(\mathbb{R}^n)$ (respectively Sobolev spaces, $\mathbf{W}^{m,p}(\mathbb{R}^n)$), as compact dense embeddings. More importantly, these spaces all provide finite norms for nonabsolutely integrable functions. We show that both the convolution and Fourier transform extend as bounded linear operators. This allows us to construct the path integral of quantum mechanics in exactly the manner intended by Feynman. Finally, we then show how a minor change of view makes it possible to construct Lebesgue measure on (a version of) $\mathbb{R}^\infty$ which is no more difficult than the same construction on $\mathbb{R}^n$. This approach allows us to construct versions of both Lebesgue and Gaussian measure on every separable Banach space, which has a basis.

Contents

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1 Introduction.

Since 1983, a small research group at Howard university has been working on the mathematical foundations for modern physics. Our philosophy is based on the following assumptions:

1. Mathematics can provide a set of tools for constructing faithful representations (models) of the physical world but does not dictate the final outcome.

2. The most conceptually effective and computationally efficient (mathematical) tools also allow the most simple and direct representations.

3. To the extent possible, all definitions should be operational.

4. To the extent possible, all proofs should be constructive.

Objective.

The objective of this paper is to provide a survey of our progress on the general theory for path integrals as envisioned by Feynman (see [17]). For simplicity, the theory is developed using the Henstock-Kurzweil integral. This integral was discovered independently by Henstock [34] and Kurzweil [41] and has been discussed in a number of books. For more detail and different perspectives, see Gordon [29], Henstock [33], Kurzweil [40] or Pfeffer [46].

The important new feature introduced is the construction of a new class of spaces $\mathbf{KS}^p(\mathbb{R}^n)$, for $1 \leq p \leq \infty$ and $n \in \mathbb{N}$ (see [23]), and their Sobolev counterparts which allow us to construct the path integral in the manner originally intended by Feynman. These are separable Banach spaces which contain all of the standard $\mathbf{L}^p$ spaces, as well as the space of finitely additive measures, as dense, continuous, compact embeddings. Equally important is the fact that these spaces provide finite norms for nonabsolutely integrable functions.

Of the many integrals that integrate nonabsolutely integrable functions, the Henstock-Kurzweil (HK) integral is currently the best known. This integral generalizes the Lebesgue, Bochner and Pettis integrals and is equivalent to the (restricted) Denjoy integral. However, it is much easier to understand (and learn) compared to the Denjoy and Lebesgue integrals; and provides useful variants of the same theorems that have made the Lebesgue integral so important. Furthermore, it arises from a simple (transparent) generalization of the Riemann integral that is taught in elementary calculus. Loosely speaking, one uses a Riemann type partition of intervals with the interior points chosen first, while the size of the base interval around any interior point is determined by an arbitrary positive function defined at that point.

The descriptive definition is a nice clear way of relating the integrand, $f = F'$ (in some sense) to the integral or primitive F. For example, if F is a Lebesgue, HK or Denjoy primitive then F is continuous and, with additional conditions, the integral of f is equal to F (see Section 2).

1.1 Summary.

In Section 2, we give a briefly introduction to the various integrals, with some emphasis on the elementary HK-integral, its properties and relationship to the Lebesgue integral (in the one dimensional case). In Section 3, we construct the $\mathbf{KS}^p$, $1 \leq p \leq \infty$, spaces (KS-spaces) and derive some of their important properties. In Section 4, we construct the corresponding Sobolev spaces. In Section 5, we prove that the Fourier transform and convolution operators have bounded extensions to $\mathbf{KS}^2$. These results are applied to show that the weak generator of Markov semigroups on the space of bounded uniformly continuous functions becomes a strong generator on $\mathbf{KS}^p$. They are also applied to give the construction of the elementary path integral in the manner originally intended by Feynman. We then strongly suggest that $\mathbf{KS}^2$ is a more natural Hilbert space for quantum theory when compared to $\mathbf{L}^2$. In Section 7 we show that, contrary to belief, it is possible to construct Lebesgue measure on a version of $\mathbb{R}^\infty$ that is no more difficult than the corresponding construction on $\mathbb{R}^n$. This result is used to provide a construction of both Lebesgue and Gaussian measure on every separable Banach space (that has a basis).

2 Integrals.

In this section, we briefly discuss the various types of integrals. Our main interest is in those integrals that integrate nonabsolutely integrable functions. However, because of its importance, we include the Lebesgue integral. Our main focus is on the HK-integral (in the simplest case). Proofs of all stated results can be found in Gordon [29] (see also Saks [48]). The general case can be found in Henstock [33] or Pfeffer ([46] and [47]). In Section 3 we will show that each integral is in $\mathbf{KS}^p$, $1 \le p \le \infty$.

Background.

Recall that the oscillation $\omega(F,[a,b])$ of a function F on an interval $[a,b]$ is defined by:

$$\omega\left(F,\ [a,b]\right) = \sup\left\{ |F(x) - F(y)| :\ a \leqslant y < x \leqslant b \right\}.$$

Definition 1. *We define the weak variation, $V(F,E)$, and the strong variation, $V_*(F,E)$, by:*

$$V(F,E) = \sup\left\{ \sum\nolimits_{i=1}^{n} |F(b_i) - F(a_i)| \right\},$$
$$V_*(F,E) = \sup\left\{ \sum\nolimits_{i=1}^{n} \omega\left(F,\ [a_i,b_i]\right) \right\},$$

where the supremum is taken over all possible finite collections of non-overlapping intervals that have end points in E.

1. *We say that F is of bounded variation on E, (BV), if $V(F,E) < \infty$.*

2. *We say that F is of bounded variation in the restricted sense on E, (BV_*), if $V_*(F,E) < \infty$.*

3. *We say that F is absolutely continuous on E,* (AC), *if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every collection $\{[a_i, b_i], 1 \leqslant i \leqslant n\}$, of nonoverlapping intervals with end points in E and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then*
$$\sum\nolimits_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

4. *We say that F is absolutely continuous in the restricted sense on E,* $(\mathrm{AC})_*$, *if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every collection $\{[a_i, b_i], 1 \leqslant i \leqslant n\}$, of nonoverlapping intervals with end points in E and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then*
$$\sum\nolimits_{i=1}^{n} \omega\left(F, [a_i, b_i]\right) < \varepsilon.$$

5. *We say that F is generalized absolutely continuous on E,* (ACG), *if $F|_E$ is continuous and E is a countable union of sets $\{E_i\}$ such that F is* (AC) *on each E_i.*

6. *We say that F is generalized absolutely continuous in the restricted sense on E,* $(\mathrm{ACG})_*$, *if $F|_E$ is continuous and E is a countable union of sets $\{E_i\}$ such that F is* $(\mathrm{AC})_*$ *on each E_i.*

We note, for future reference, that the set of functions of bounded variation on $[a, b]$, $\mathcal{BV}([a, b])$, is a Banach space with norm $\|h\|_{BV} = \|h\|_\infty + V(h, [a, b])$.

2.1 Classical Integrals.

Let E be a measurable subset of $\mathbb{R}$ and let $\lambda(E)$ denote the Lebesgue measure of E.

Definition 2. *Let E be a measurable set and let $c \in \mathbb{R}$.*

1. *We say that c is a point of density for E if*
$$d_c E = \lim_{h \to 0^+} \frac{\lambda\left(E \cap (c - h, c + h)\right)}{2h} = 1.$$

2. *We say that c is a point of dispersion for E if*
$$d_c E = \lim_{h \to 0^+} \frac{\lambda\left(E \cap (c - h, c + h)\right)}{2h} = 0.$$

3. *We say that a function $F : [a, b] \to \mathbb{R}$ is approximately continuous at $c \in E \subset [a, b]$, if c is a point of density for E and $F|_E$ is continuous at c.*

4. *We say that a function* $F : [a, b] \to \mathbb{R}$ *is approximately differentiable at* $c \in E \subset [a, b]$*, if* c *is a point of density for* E *and* $F|_E$ *is differentiable at* c*. In this case, we write the derivative as* $F'_{ap}(c)$*.*

Let $\mathbf{C}^k$ denote the set of functions with k continuous derivatives (we let $\mathbf{C}^0 = \mathbf{C}$, the continuous functions).

Theorem 3. *If* E *is a subset of* $[a, b]$*, we have:*

$$\mathbf{C}^1 \subset (AC) \subset (ACG_*) \subset (ACG) \subset \mathbf{C}.$$

In the next theorem, we tie down the left end point for convenience. (This result can be used to provide a descriptive definition of the integrals.)

Theorem 4. *Let* F *be a function defined on* $[a, b]$ *with* $F(a) = 0$*, then the following holds.*

1. *If* F *is (AC) on* $[a, b]$*,* F' *exists (a.e). If* F' *is Lebesgue integrable, then* $\int_a^x F'(y)d(y) = F(x)$*.*

2. *If* F *is* (ACG_*) *on* $[a, b]$*, then* F' *exists (a.e) and* $\int_a^x F'(y)d(y) = F(x)$ *(as a restricted Denjoy, a Perron and a Henstock-Kurzweil integral).*

3. *If* F *is (ACG) on* $[a, b]$*, then* F'_{ap} *exists (a.e) and* $\int_a^x F'(y)d(y) = F(x)$ *(as a wide sense Denjoy or Denjoy-Khintchine integral).*

The above are the most well-known of the possible integrals. Another possibility was introduced by Henstock, which integrates the approximate derivative of an approximately continuous function (see Gordon [29] or Saks [48]).

2.2 HK-Integral.

In this section we discuss the HK-integral (more constructively). It is strong enough for all integrands that have continuous integrals.

Definition 5. *Let* $[a, b] \subset \mathbb{R}$*, let* $\delta(t)$ *map* $[a, b] \to (0, \infty)$*, and let* $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$*, where* $a = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = b$*. We call* $\mathbf{P}$ *an HK-partition for* δ *if, for* $1 \leqslant i \leqslant n$*,* $t_{i-1}, t_i \in (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$*.*

Remark 6. *Gordon writes* $\mathbf{P} = \{(\tau_i, [t_{i-1}, t_i]) : 1 \leqslant i \leqslant n\}$ *and calls* $\{\tau_i\}$ *the tags and* $\{[t_{i-1}, t_i]\}$ *the collection of tagged intervals. Also, the phrase nearly everywhere (n.e.) means "except for a countable set".*

Definition 7. *The function $f(t)$, $t \in [a,b]$, is said to have an HK-integral if there is a number $F[a,b]$ such that, for each $\varepsilon > 0$, there exists a function δ from $[a,b] \to (0,\infty)$ such that, whenever $\mathbf{P}$ is an HK-partition for δ, then (with $\Delta t_i = t_i - t_{i-1}$)*

$$\left|\sum\nolimits_{i=1}^{n} \Delta t_i f(\tau_i) - F[a,b]\right| < \varepsilon.$$

In this case, we write $F[a,b] =$ (HK)-$\int_a^b f(t)dt$.

Theorem 8. *Let $f(t) : [a,b] \to \mathbb{R}$.*

1. *If $f(t)$ is Lebesgue integrable on $[a,b]$, then it is HK-integrable on $[a,b]$ and HK-$\int_a^b f(t)dt =$L-$\int_a^b f(t)dt$.*
2. *If $f(t)$ is HK-integrable and bounded on $[a,b]$, then it is Lebesgue integrable on $[a,b]$.*
3. *If $f(t)$ is HK-integrable and nonnegative on $[a,b]$, then it is Lebesgue integrable on $[a,b]$.*
4. *If $f(t)$ is HK-integrable on every measurable subset of $[a,b]$, then it is Lebesgue integrable on $[a,b]$.*

Corollary 9. *Let $F : [a,b] \to \mathbb{R}$ be continuous. If F is differentiable nearly everywhere on $[a,b]$, then F' is HK-integrable on $[a,b]$ and HK-$\int_a^t F'(s)ds = F(t)$ for each $t \in [a,b]$.*

The last result follows from Theorem 4 (2) and shows in what sense we can think of the HK-integral as the reverse of the derivative. (The result is not true for Lebesgue integrals. The standard example is $F'(t) = 2t sin(\pi/t^2) - (2\pi/t)cos(\pi/t^2)$ for all nonrational numbers on $0 < t < 1$ and equal to 0 at all rational points.)

In his book [47], Pfeffer presents a nice exposition of an invariant multidimensional process of recovering a function from its derivative that extends the HK-integral to Euclidean spaces.

2.3 Distributions.

In the following section, we will define our spaces and their distributional counterparts for functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, so that we develop the general case.

The work of Talvila [52] and others has recently introduced a distributional (or weak) integral. This integral contains all of the above integrals and integrates Radon measures. Although it has been studied in $\mathbb{R}^n$, for convenience and comparison, we provide an introduction to this integral in the one-dimensional case.

Preliminaries.

The following notational conventions are in force:

1. All functions f on $\mathbb{R}^n$ are real-valued (i.e., the range of f is $\mathbb{R}$).
2. All integrals are defined on $\mathbb{R}^n$.
3. A function $u \in \mathbf{L}^1_{loc}(\mathbb{R}^n)$ means that u is Lebesgue integrable on every compact subset of $\mathbb{R}^n$.

Let $\mathcal{D}(\mathbb{R}^n) = \mathbf{C}_c^\infty(\mathbb{R}^n)$ denote the space of infinitely differentiable functions $\phi : \mathbb{R}^n \to \mathbb{R}$ with compact support. We say that a sequence of functions $\{\phi_n\} \subset \mathcal{D}$ converges to $\phi \in \mathcal{D}$ if there is a fixed compact set U such that all functions ϕ_n have their support in U and, for each $k \geq 0$, the sequence of k-derivatives of ϕ_n, $\phi_n^{(k)}$, converges uniformly to $\phi^{(k)}$ on U. We call a function ϕ belonging to $\mathcal{D}(\mathbb{R}^n)$ a test function.

Let $u \in \mathbf{C}^1(\mathbb{R}^n)$. Then, if $\phi \in \mathbf{C}_c^\infty(\mathbb{R}^n)$, integration by parts gives:

$$\int_{\mathbb{R}^n} (u\phi_{y_i})d\lambda = \int_{\partial\mathbb{R}^n} (u\phi)\nu_i d\mathbf{S} - \int_{\mathbb{R}^n} (\phi u_{y_i})d\lambda,\ 1 \leq i \leq n.$$

Since ϕ vanishes on the boundary, we see that the above reduces to:

$$\int_{\mathbb{R}^n} (u\phi_{y_i})d\lambda = -\int_{\mathbb{R}^n} (\phi u_{y_i})\ d\lambda,\ 1 \leq i \leq n.$$

In the general case, for any $u \in \mathbf{C}^m(\mathbb{R}^n)$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n),\ |\alpha| = \sum_{\alpha=1}^n \alpha_i = m$, we have

$$\int_{\mathbb{R}^n} u(D^\alpha\phi)d\lambda = (-1)^m \int_{\mathbb{R}^n} \phi(D^\alpha u)d\lambda.$$

Definition 10. *If α is a multi-index and $u,\ v \in \mathbf{L}^1_{loc}(\mathbb{R}^n)$, we say that v is the α^{th}-weak (or distributional) partial derivative of u and write $D^\alpha u = v$ provided that*

$$\int_{\mathbb{R}^n} u(D^\alpha\phi)d\lambda = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi v\ d\lambda$$

for all functions $\phi \in \mathbf{C}_c^\infty(\mathbb{R}^n)$. Thus, v is in the dual space $\mathcal{D}'(\mathbb{R}^n)$ of $\mathcal{D}(\mathbb{R}^n)$.

The next result is easy.

Lemma 11. *If a weak α^{th}-partial derivative exists for u, then it is unique λ-a.e. (i.e., except on a set of measure zero).*

Definition 12. *If $m \geq 0$ is fixed and $1 \leq p \leq \infty$, we define the Sobolev space $\mathbf{W}^{m,p}(\mathbb{R}^n)$ to be the set of all locally summable functions $u :\ \mathbb{R}^n \to \mathbb{R}$ such that, for each multi-index α with $|\alpha| \leqslant m,\ D^\alpha u$ exists in the weak sense and belongs to $\mathbf{L}^p(\mathbb{R}^n)$.*

Extensions and Decompositions.

We need an extension theorem for functions defined on a domain of $\mathbb{R}^n$ and a result which shows that a domain in $\mathbb{R}^n$ can be written as a union of non-overlapping closed cubes. (Proofs of these results can be found in Evans [16] and Stein [51], respectively.)

Let $\mathbb{D}$ be a bounded open connected set of $\mathbb{R}^n$ (a domain) with boundary $\partial\mathbb{D}$ and closure $\bar{\mathbb{D}}$.

Definition 13. *Let k be a positive integer. We say that $\partial\mathbb{D}$ is of class $\mathbf{C}^k$ if, for every point $\mathbf{x} \in \partial\mathbb{D}$, there is a homeomorphsim ϕ of a neighborhood U of $\mathbf{x}$ into $\mathbb{R}^n$ such that both ϕ and ϕ^{-1} have k continuous derivatives with*

$$\varphi(\mathbb{D} \cap U) \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

and

$$\varphi(\partial\mathbb{D} \cap U) \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}.$$

Theorem 14. *Let $\mathbb{D}$ be a domain in $\mathbb{R}^n$ with $\partial\mathbb{D}$ of class $\mathbf{C}^1$. Let $\mathbb{U}$ be any bounded open set such that $\bar{\mathbb{D}}$, the closure of $\mathbb{D} \subset\subset \mathbb{U}$ (i.e., the closure of $\mathbb{D}$ is a compact subset of $\mathbb{U}$). Then there is a linear operator $\mathfrak{E}$ mapping functions on $\mathbb{D}$ to functions on $\mathbb{R}^n$ such that:*

1. *The operator $\mathfrak{E}$ maps $\mathbf{W}^{1,p}(\mathbb{D})$ continuously into $\mathbf{W}^{1,p}(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.*
2. *$\mathfrak{E}(f)\,|_{\mathbb{D}} = f$ (i.e., $\mathfrak{E}(\cdot)$ is an extension operator).*
3. *$\mathfrak{E}(f)(x) = 0$ for $x \in \mathbb{U}^c$ (i.e., $\mathfrak{E}(f)$ has support inside $\mathbb{U}$).*

Theorem 15. *Let $\mathbb{D}$ be a domain in $\mathbb{R}^n$. Then $\mathbb{D}$ is the union of a sequence of closed cubes $\{\mathbb{D}_k\}$ whose sides are parallel to the coordinate axes and whose interiors are mutually disjoint.*

Thus, if a function f is defined on a domain in $\mathbb{R}^n$, by Theorem 14 it can be extended to the whole space. On the other hand, without loss, by Theorem 15, we can assume that the domain is a cube with sides parallel to the coordinate axes. In either case, the HK-integral can be constructed under these conditions.

2.4 Weak Integral.

Here, we follow Talvila [52] and consider the distributional (or weak) integral on $\mathbb{R}$.

Definition 16. *Let $F' = DF$ be the weak derivative of F. We define*

$$\mathcal{A}_{\mathbf{c}}(\mathbb{R}) = \{ f = DF \mid ,\ F \in \mathcal{B}_{\mathbf{c}}(\mathbb{R}) \},$$

where

$$\mathcal{B}_{\mathbf{c}}(\mathbb{R}) = \left\{ F \in \mathbf{C}(\mathbb{R}) \middle| \lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) \in \mathbb{R} \right\}.$$

If $f \in \mathcal{A}_{\mathbf{c}}(\mathbb{R})$, we say that $F \in \mathcal{B}_{\mathbf{c}}(\mathbb{R})$ is the weak integral of f and write

$$F(x) = (w) \int_{-\infty}^{x} f(y)dy.$$

Alexiewicz [2] has shown that the class $D(\mathbb{R})$, of Denjoy integrable functions (restricted and wide sense), can be normed in the following manner: for $f \in D(\mathbb{R})$, define $\|f\|_D$ by

$$\|f\|_D = \|F\|_\infty = \sup_x \left| \int_{-\infty}^{x} f(y)dy \right|. \tag{2.1}$$

It is clear that this is a norm, and it is known that $D(\mathbb{R})$ is not complete. The following is proved in Talvila [52].

Theorem 17. *With the Alexiewicz norm, the space $\mathcal{A}_{\mathbf{c}}$ has the following properties:*

1. *$\mathcal{A}_{\mathbf{c}}$ is a separable Banach space and a Banach lattice, which contains $\mathbf{L}^1$ and the Denjoy integrable functions (restricted and wide sense) as dense subsets.*
2. *$\mathcal{A}_{\mathbf{c}}$ is isometrically isomorphic to $\mathcal{B}_{\mathbf{c}}$.*
3. *$\mathcal{A}_{\mathbf{c}}$ is the completion of $D(\mathbb{R})$.*
4. *The dual space $\mathcal{A}_{\mathbf{c}}^*$ of $\mathcal{A}_{\mathbf{c}}$ is $\mathcal{BV}(\mathbb{R})$.*

There is also a weak integral in $\mathbb{R}^n$ (see [3] and [45] for details). If $f \in \mathcal{D}'(\mathbb{R}^n)$ then f is integrable if there is a function $F \in \mathbf{C}(\mathbb{R}^{\mathbf{n}})$ such that $DF = f$, where $D = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n}$. Thus,

$$\int_{\mathbb{R}^n} f(x)\varphi(x)dx = \int_{\mathbb{R}^n} DF(x)\varphi(x)dx = (-1)^n \int_{\mathbb{R}^n} F(x)D\varphi(x)dx,$$

for all ϕ in $\mathbf{C}_c^\infty(\mathbb{R}^n)$.

3 KS$^{\mathbf{p}}$ Spaces.

In order to construct the spaces of interest, first recall that the HK-integral is equivalent to the Denjoy integral (see Henstock [33] or Pfeffer [46]). Replacing $\mathbb{R}$ by $\mathbb{R}^n$ in (2.1), for $f \in D(\mathbb{R}^n)$, we have:

$$\|f\|_D = \sup_{r>0} \left| \int_{\mathbf{B}_r} f(\mathbf{x}) d\mathbf{x} \right| = \sup_{r>0} \left| \int_{\mathbf{R}^n} \mathcal{E}_{\mathbf{B}_r}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right| < \infty, \tag{3.1}$$

where $\mathbf{B}_r$ is any closed cube of diagonal r centered at the origin in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and $\mathcal{E}_{\mathbf{B}_r}(\mathbf{x})$ is the characteristic function of $\mathbf{B}_r$.

Now, fix n, and let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n\}$ such that x_i is rational for each i. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots\}$. For each l and i, let $\mathbf{B}_l(\mathbf{x}_i)$ be the closed cube centered at $\mathbf{x}_i$, with sides parallel to the coordinate axes and diagonal $r_l = 2^{-l}, l \in \mathbb{N}$. Now choose the natural order which maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$:

$$\{(1,1),\ (2,1),\ (1,2),\ (1,3),\ (2,2),\ (3,1),\ (3,2),\ (2,3),\ \cdots\}.$$

Let $\{\mathbf{B}_k,\ k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_l(\mathbf{x}_i) \quad |(l,i) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$ and, let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$, so that $\mathcal{E}_k(\mathbf{x})$ is in $\mathbf{L}^p[\mathbb{R}^n] \cap \mathbf{L}^\infty[\mathbb{R}^n]$ for $1 \le p < \infty$. Define $F_k(\ \cdot\ )$ on $\mathbf{L}^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}. \tag{3.2}$$

It is clear that $F_k(\ \cdot\ )$ is a bounded linear functional on $\mathbf{L}^p[\mathbb{R}^n]$ for each k, $\|F_k\|_\infty \le 1$ and, if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $\mathbf{L}^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Fix $t_k > 0$ such that $\sum_{k=1}^\infty t_k = 1$ and define a measure $d\mathbf{P}(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^n \times \mathbb{R}^n$ by:

$$d\mathbf{P}(\mathbf{x}, \mathbf{y}) = \left[\sum\nolimits_{k=1}^\infty t_k \mathcal{E}_k(\mathbf{x}) \mathcal{E}_k(\mathbf{y}) \right] d\mathbf{x} d\mathbf{y}.$$

We first construct our Hilbert space. Define an inner product $(\ \cdot\ )$ on $\mathbf{L}^1[\mathbb{R}^n]$ by

$$\begin{aligned}(f,g) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(\mathbf{x}) g(\mathbf{y})^* d\mathbf{P}(\mathbf{x}, \mathbf{y}) \\ &= \sum\nolimits_{k=1}^\infty t_k \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right] \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) g(\mathbf{y}) d\mathbf{y} \right]^* .\end{aligned} \tag{3.3}$$

We use a particular choice of t_k in Gill and Zachary [23], which is suggested by physical analysis in another context. We call the completion of $\mathbf{L}^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space, $\mathbf{KS}^2[\mathbb{R}^n]$. Following suggestions of Gill and Zachary, Steadman [50] constructed this space by adapting an approach developed by Kuelbs [39] for other purposes. Her interest was in showing that $\mathbf{L}^1[\mathbb{R}^n]$ can be densely and continuously embedded in a Hilbert space which contains the HK-integrable functions. To see that this is the case, let $f \in D[\mathbb{R}^n]$, then:

$$\|f\|_{\mathbf{KS}^2}^2 = \sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^2 \leqslant \sup_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^2 \leqslant \|f\|_D^2 ,$$

so $f \in \mathbf{KS}^2[\mathbb{R}^n]$.

Theorem 18. *The space $\mathbf{KS}^2[\mathbb{R}^n]$ contains $\mathcal{A}_{\mathbf{c}}$ and $\mathbf{L}^p[\mathbb{R}^n]$ (for each p, $1 \leqslant p \leqslant \infty$) as dense subspaces.*

Proof. The first inclusion follows from the above equation and the fact that both $\mathcal{A}_{\mathbf{c}}$ and $\mathbf{KS}^2[\mathbb{R}^n]$ contain $\mathbf{L}^1[\mathbb{R}^n]$ densely. Thus, we need only show that $\mathbf{KS}^2[\mathbb{R}^n] \supset \mathbf{L}^q[\mathbb{R}^n]$ for $q \neq 1$. If $f \in \mathbf{L}^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$\begin{aligned} \|f\|_{\mathbf{KS}^2} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^{\frac{2q}{q}} \right]^{1/2} \\ &\leqslant \left[\sum\nolimits_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left| f(\mathbf{x}) \right|^q d\mathbf{x} \right)^{\frac{2}{q}} \right]^{1/2} \\ &\leqslant \sup_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left| f(\mathbf{x}) \right|^q d\mathbf{x} \right)^{\frac{1}{q}} \leqslant \|f\|_q . \end{aligned}$$

Hence, $f \in \mathbf{KS}^2[\mathbb{R}^n]$. For $q = \infty$, first note that $vol(\mathbf{B}_k)^2 \leq \left[\frac{1}{2\sqrt{n}}\right]^{2n}$, so we have

$$\begin{aligned} \|f\|_{\mathbf{KS}^2} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^2 \right]^{1/2} \\ &\leqslant \left[\left[\sum\nolimits_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^2 \right] \left[ess \sup |f|\right]^2 \right]^{1/2} \leqslant \left[\frac{1}{2\sqrt{n}} \right]^n \|f\|_{\infty} . \end{aligned}$$

Thus $f \in \mathbf{KS}^2[\mathbb{R}^n]$, and $\mathbf{L}^\infty[\mathbb{R}^n] \subset \mathbf{KS}^2[\mathbb{R}^n]$. $\square$

The fact that $\mathbf{L}^\infty[\mathbb{R}^n] \subset \mathbf{KS}^2[\mathbb{R}^n]$, while $\mathbf{KS}^2[\mathbb{R}^n]$ is separable makes it clear in a very forceful manner that separability is not an inherited property. Before proceeding to additional study, we need to construct $\mathbf{KS}^p[\mathbb{R}^n]$.

To construct $\mathbf{KS}^p[\mathbb{R}^n]$ for all p and for $f \in \mathbf{L}^p$, define:

$$\|f\|_{\mathbf{KS}^p} = \begin{cases} \left\{\sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^p\right\}^{1/p}, 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|, p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{\mathbf{KS}^p}$ defines a norm on $\mathbf{L}^p$. If $\mathbf{KS}^p$ is the completion of $\mathbf{L}^p$ with respect to this norm, we have:

Theorem 19. *For each q, $1 \leqslant q \leqslant \infty$, $\mathbf{KS}^p[\mathbb{R}^n] \supset \mathbf{L}^q[\mathbb{R}^n]$ as dense continuous embeddings.*

PROOF. As in the previous theorem, by construction $\mathbf{KS}^p[\mathbb{R}^n]$ contains $\mathbf{L}^p[\mathbb{R}^n]$ densely, so we need only show that $\mathbf{KS}^p[\mathbb{R}^n] \supset \mathbf{L}^q[\mathbb{R}^n]$ for $q \neq p$. First, suppose that $p < \infty$. If $f \in \mathbf{L}^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$\begin{aligned} \|f\|_{\mathbf{KS}^p} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^{\frac{qp}{q}}\right]^{1/p} \\ &\leqslant \left[\sum\nolimits_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{p}{q}}\right]^{1/p} \\ &\leqslant \sup_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^q d\mathbf{x}\right)^{\frac{1}{q}} \leqslant \|f\|_q. \end{aligned}$$

Hence, $f \in \mathbf{KS}^p[\mathbb{R}^n]$. For $q = \infty$, we have

$$\begin{aligned} \|f\|_{\mathbf{KS}^p} &= \left[\sum\nolimits_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^p\right]^{1/p} \\ &\leqslant \left[\left[\sum\nolimits_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^p\right] [ess\sup |f|]^p\right]^{1/p} \leqslant M \|f\|_\infty. \end{aligned}$$

Thus $f \in \mathbf{KS}^p[\mathbb{R}^n]$, and $\mathbf{L}^\infty[\mathbb{R}^n] \subset \mathbf{KS}^p[\mathbb{R}^n]$. The case $p = \infty$ is obvious. $\square$

Theorem 20. *For $\mathbf{KS}^p$, $1 \leq p \leq \infty$, we have:*

1. *If $f, g \in \mathbf{KS}^p$, then $\|f+g\|_{\mathbf{KS}^p} \leqslant \|f\|_{\mathbf{KS}^p} + \|g\|_{\mathbf{KS}^p}$ (Minkowski inequality).*
2. *If K is a weakly compact subset of $\mathbf{L}^p$, it is a compact subset of $\mathbf{KS}^p$.*

3. *If $1 < p < \infty$, then $\mathbf{KS}^p$ is uniformly convex.*

4. *If $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, then the dual space of $\mathbf{KS}^p$ is $\mathbf{KS}^q$.*

5. *$\mathbf{KS}^\infty \subset \mathbf{KS}^p$, for $1 \le p < \infty$.*

PROOF. The proof of (1) follows from the classical case for sums. The proof of (2) follows from the fact that, if $\{f_n\}$ is any weakly convergent sequence in K with limit f, then

$$\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left[f_n(\mathbf{x}) - f(\mathbf{x}) \right] d\mathbf{x} \to 0$$

for each k. It follows that $\{f_n\}$ converges strongly to f in $\mathbf{KS}^p$.

The proof of (3) follows from a modification of the proof of the Clarkson inequalities for l^p norms (see [9]).

In order to prove (4), observe that, for $p \neq 2,\ 1 < p < \infty$, the linear functional

$$L_g(f) = \|g\|_{\mathbf{KS}^p}^{2-p} \sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \right|^{p-2} \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) f(\mathbf{y})^* d\mathbf{y}$$

is a unique duality map on $\mathbf{KS}^q$ for each $g \in \mathbf{KS}^p$ and that $\mathbf{KS}^p$ is reflexive from (3). To prove (5), note that $f \in \mathbf{KS}^\infty$ implies that $\left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|$ is uniformly bounded for all k. It follows that $\left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^p$ is uniformly bounded for each $p,\ 1 \le p < \infty$. It is now clear from the definition of $\mathbf{KS}^\infty$ that:

$$\left[\sum\nolimits_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^p \right]^{1/p} \leqslant \|f\|_{\mathbf{KS}}^{\infty} < \infty.$$

$\square$

Note that, since $\mathbf{L}^1[\mathbf{R}^n] \subset \mathbf{KS}^p[\mathbb{R}^n]$ and $\mathbf{KS}^p[\mathbb{R}^n]$ is reflexive for $1 < p < \infty$, the second dual $\left\{ \mathbf{L}^1[\mathbb{R}^n] \right\}^{**} = \mathfrak{M}[\mathbb{R}^n] \subset \mathbf{KS}^p[\mathbb{R}^n]$. Recall that $\mathfrak{M}[\mathbf{R}^n]$ is the space of bounded finitely additive set functions defined on the Borel sets $\mathfrak{B}[\mathbb{R}^n]$. This space contains the Dirac delta measure and the free-particle Green's function for the Feynman integral. We will return to $\mathfrak{M}[\mathbb{R}^n]$ in the next section.

Remark 21. *There is quite a lot of flexibility in the choice of the family of positive numbers $\{t_k\}$, $\sum\limits_{k=1}^{\infty} t_k = 1$. This is somewhat akin to the standard metric used for $\mathbb{R}^\infty$ (see Section 6.4). Recall that for any two points $X,\ Y \in$*

$\mathbb{R}^\infty,\ d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|X-Y|}{1+|X-Y|}$. The family of numbers $\{\frac{1}{2^n}\}$ can be replaced by any other sequence of positive numbers whose sum is one, without affecting the topology. We have used physical analysis to choose the family $\{t_k\}$ so they are interpreted as probabilities for the occurrence of a particular discrete path.

There is also some ambiguity associated with the order for $\mathbb{Q}_n$ and the order for $\mathbb{N} \times \mathbb{N}$. (We have used simplicity to choose the order for $\mathbb{N} \times \mathbb{N}$.) For our work, the important fact is that, for any combination of orders, the properties of $\mathbf{KS}^2[\mathbb{R}^n]$ are invariant.

4 $\mathbf{KS}^{m,p}$ Spaces.

In many applications, it is convenient to formulate problems on one of the standard Sobolev spaces $\mathbf{W}^{m,p}(\mathbb{R}^n)$. In this section we define the corresponding extension for the $\mathbf{KS}^p$ spaces. (As will be clear, this theory is in its formative stages so that there is much to be done.)

Definition 22. *If $m \geq 0$ is fixed and $1 \leq p \leq \infty$, we define the generalized Sobolev space $\mathbf{KS}^{m,p}(\mathbb{R}^n)$ to be the set of all locally summable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that, for each multi-index α with $|\alpha| \leqslant m$, $D^\alpha u$ exists in the weak sense and belongs to $\mathbf{KS}^p(\mathbb{R}^n)$.*

Definition 23. *If $u \in \mathbf{KS}^{m,p}(\mathbb{R}^n)$, we define the norm by:*

$$\|u\|_{\mathbf{KS}^{m,p}} := \begin{cases} \left\{\sum_{|\alpha| \leqslant m} \sum_{i=1}^{\infty} t_i \left|\int_{\mathbb{R}^n} \mathcal{E}_i(\mathbf{x}) D^\alpha u(\mathbf{x}) d\lambda(\mathbf{x})\right|^p\right\}^{1/p}, \ 1 \leqslant p < \infty, \\ \sum_{|\alpha| \leqslant m} \sup_{i \geqslant 1} \left|\int_{\mathbb{R}^n} \mathcal{E}_i(\mathbf{x}) D^\alpha u(\mathbf{x}) d\lambda(\mathbf{x})\right|, \ p = \infty. \end{cases}$$

Recall that the standard Sobolev space $\mathbf{W}^{k,p}(\mathbb{R}^n)$ is the set of all locally summable functions $u : \mathbb{R}^n \to \mathbb{C}$ such that $D^\alpha u$ exists weakly and belongs to $\mathbf{L}^p(\mathbb{R}^n)$. The next result follows from the definition of the respective norms on $\mathbf{W}^{k,p}(\mathbb{R}^n)$ and $\mathbf{KS}^{k,p}(\mathbb{R}^n)$.

Theorem 24. *The completion of $\mathbf{W}^{k,p}(\mathbb{R}^n)$ relative to the $\mathbf{KS}^p(\mathbb{R}^n)$ norm also defines the space $\mathbf{KS}^{k,p}(\mathbb{R}^n)$, which contains $\mathbf{W}^{k,p}(\mathbb{R}^n)$ as a continuous dense and compact embedding.*

5 Extension of Fourier and Convolution Operators.

Let $L[\mathcal{B}]$, $L[\mathcal{H}]$ denote the bounded linear operators on $\mathcal{B}$, $\mathcal{H}$ respectively, where we assume that the separable Banach space $\mathcal{B}$ is a continuous dense embedding in the separable Hilbert space $\mathcal{H}$. The following is the major result in Gill et al [22]. It generalizes the well-known result of von Neumann [56] for bounded operators on Hilbert spaces.

Theorem 25. *Let $\mathcal{B}$ be a separable Banach space and let A be a bounded linear operator on $\mathcal{B}$. Then A has a well-defined adjoint A^* defined on $\mathcal{B}$ such that:*

1. *the operator $A^*A \geq 0$ (maximal accretive),*
2. *$(A^*A)^* = A^*A$ (selfadjoint), and*
3. *$I + A^*A$ has a bounded inverse.*

The proof depends on the fact that, given a separable Banach space $\mathcal{B}$, there always exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \subset \mathcal{B} \subset \mathcal{H}_2$ as continuous dense embeddings, with $\mathcal{H}_1$ determined by $\mathcal{H}_2$ (see [22]). If A is any bounded linear operator on $\mathcal{B}$, we define A^* by

$$A^*x = J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x), \tag{5.1}$$

where A_1 is A restricted to $\mathcal{H}_1$, $J_2|_{\mathcal{B}}$ maps $\mathcal{B}$ into $\mathcal{H}'_2$ and J_1^{-1} maps $\mathcal{H}'_1$ onto $\mathcal{H}_1$.

It is not clear that A need have a bounded extension to $\mathcal{H}_2$. On the other hand, the theorem by Lax [42] states that:

Theorem 26. *If A is a bounded linear operator on $\mathcal{B}$ such that A is selfadjoint (i.e., $(Ax, y)_{\mathcal{H}_2} = (x, Ay)_{\mathcal{H}_2}$ for all $x, y, \in \mathcal{B}$), then A is bounded on $\mathcal{H}_2$ and $\|A\|_{\mathcal{H}_2} \leq k \|A\|_{\mathcal{B}}$ with k a positive constant.*

Since A^*A is selfadjoint on $\mathcal{B}$, it is natural to expect that the same is true on $\mathcal{H}_2$. However, this need not be the case. To obtain a simple counterexample, recall that, in standard notation, the simplest class of bounded linear operators on $\mathcal{B}$ is $\mathcal{B} \otimes \mathcal{B}'$, in the sense that:

$$\mathcal{B} \otimes \mathcal{B}' : \mathcal{B} \to \mathcal{B}, \text{ by } Ax = (b \otimes b')x = \langle x, b' \rangle\, b.$$

Thus, if b' is in $\mathcal{B}' \backslash \mathcal{H}_2'$, then $J_2\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x)\}$ need not be in $\mathcal{H}_2'$, so that A^*A is not defined as an operator on all of $\mathcal{H}_2$ and thus, cannot have a bounded extension. We can now state the correct extension of Theorem 26.

Theorem 27. *Let A be a bounded linear operator on $\mathcal{B}$. If $\mathcal{B}' \subset \mathcal{H}_2$, then A has a bounded extension to $L[\mathcal{H}_2]$, with $\|A\|_{\mathcal{H}_2} \leq k \|A\|_{\mathcal{B}}$ with k a positive constant.*

PROOF. The proof is now easy if we observe that, with the stated condition, $J_2\{J_1^{-1}[(A_1)']J_2|_{\mathcal{B}}(x)\}$ is in $\mathcal{H}_2^{'}$ for all $x \in \mathcal{B}$. It follows that, for any bounded linear operator A defined on $\mathcal{B}$, the operator $T = A^*A$ is selfadjoint on $\mathcal{H}_2$. Thus, by Lax's theorem, T is bounded on $\mathcal{H}_2$, with $\|A^*A\|_{\mathcal{H}_2} = \|A\|_{\mathcal{H}_2}^2 \leq \|A^*A\|_{\mathcal{B}} \leq k\|A\|_{\mathcal{B}}^2$, where $k = inf\{M| \ \|A^*A\|_{\mathcal{B}} \leq M\|A\|_{\mathcal{B}}^2\}$. □

We can now use Theorem 27 to prove that $\mathfrak{F}$ and $\mathfrak{C}$, the Fourier (transform) operator and the convolution operator respectively, defined on $\mathbf{L}^1[\mathbb{R}^n]$, have bounded extensions to $\mathbf{KS}^2[\mathbf{R}^n]$. It should be noted that this theorem also implies that both operators have bounded extensions to $\mathbf{L}^p[\mathbb{R}^n]$ (for $1 \leq p \leq 2$). This is the first proof based on functional analysis, while the traditional proof is obtained via rather deep methods of (advanced) real analysis. (However this result does not tell us that the restriction of the Fourier transform operator to $\mathbf{L}^p[\mathbb{R}^n]$ maps $\mathbf{L}^p[\mathbb{R}^n]$ to itself, which is not true unless $p = 2$ (see Grafakos [31]).)

Theorem 28. *Both $\mathfrak{F}$ and $\mathfrak{C}$ extend to bounded linear operators on $\mathbf{KS}^2[\mathbb{R}^n]$.*

PROOF. To prove our result, first note that $C_0[\mathbb{R}^n]$, the bounded continuous functions on $\mathbb{R}^n$ which vanish at infinity, is contained in $\mathbf{KS}^2[\mathbb{R}^n]$. Now $\mathfrak{F}$ is a bounded linear operator from $\mathbf{L}^1[\mathbb{R}^n]$ to $C_0[\mathbb{R}^n]$, so we can consider it as a bounded linear operator from $\mathbf{L}^1[\mathbb{R}^n]$ to $\mathbf{KS}^2[\mathbb{R}^n]$. Since $\mathbf{L}^1[\mathbb{R}^n]$ is dense in $\mathbf{KS}^2[\mathbb{R}^n]$ and $\mathbf{L}^\infty[\mathbb{R}^n] \subset \mathbf{KS}^2[\mathbb{R}^n]$, by Theorem 27 $\mathfrak{F}$ extends to a bounded linear operator on $\mathbf{KS}^2[\mathbb{R}^n]$.

To prove that $\mathfrak{C}$ has a bounded extension, fix g in $\mathbf{L}^1[\mathbb{R}^n]$ and define $\mathfrak{C}_g$ on $\mathbf{L}^1[\mathbb{R}^n]$ by:

$$\mathfrak{C}_g(f)(\mathbf{x}) = \int g(\mathbf{y})f(\mathbf{x}-\mathbf{y})d\mathbf{y}.$$

Once again, since $\mathfrak{C}_g$ is bounded on $\mathbf{L}^1[\mathbb{R}^n]$ and $\mathbf{L}^1[\mathbb{R}^n]$ is dense in $\mathbf{KS}^2[\mathbb{R}^n]$, by Theorem 27 it extends to a bounded linear operator on $\mathbf{KS}^2[\mathbb{R}^n]$. Now use the fact that convolution is commutative to get that $\mathfrak{C}_f$ is a bounded linear operator on $\mathbf{L}^1[\mathbb{R}^n]$ for all $f \in \mathbf{KS}^2[\mathbb{R}^n]$. Another application of Theorem 27 completes the proof. □

We now return to $\mathfrak{M}[\mathbb{R}^n]$.

Definition 29. *A uniformly bounded sequence $\{\mu_k\} \subset \mathfrak{M}[\mathbb{R}^n]$ is said to converge weakly to μ ($\mu_n \xrightarrow{w} \mu$) if, for every bounded uniformly continuous function $h(\mathbf{x})$,*

$$\int_{\mathbb{R}^n} h(\mathbf{x})d\mu_n \to \int_{\mathbb{R}^n} h(\mathbf{x})d\mu.$$

Theorem 30. *If $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbf{R}^n]$, then $\mu_n \xrightarrow{s} \mu$ (strongly) in $\mathbf{KS}^p[\mathbb{R}^n]$.*

Proof. Since the characteristic function of a closed cube is a bounded uniformly continuous function (a.e.), $\mu_n \xrightarrow{w} \mu$ in $\mathfrak{M}[\mathbb{R}^n]$ implies that

$$\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) d\mu_n \to \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) d\mu$$

for each k, so that $\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathbf{KS}^p} = 0$. □

A little reflection gives:

Theorem 31. *The space $\mathbf{KS}^2[\mathbb{R}^n]$ is a commutative Banach algebra with unit.*

In closing, it is clear that all bounded linear operators on $\mathbf{L}^p[\mathbb{R}^n]$ have extensions to $\mathbf{KS}^2[\mathbb{R}^n]$. It is easy to see that they also have densely defined closed extensions to $\mathbf{KS}^p[\mathbb{R}^n]$ for $p \neq 2$. We have not been able to show that these extensions are bounded.

5.1 Markov Processes.

In the study of Markov processes, two of the natural spaces on which to formulate the theory; $\mathbf{C}_b[\mathbb{R}^n]$, the space of bounded continuous functions, or $\mathbf{UBC}[\mathbb{R}^n]$, the bounded uniformly continuous functions, do not have the expected properties. It is well-known that the semigroups associated with Markov processes, whose generators have unbounded coefficients, are not necessarily strongly continuous when defined on $\mathbf{C}_b[\mathbb{R}^n]$. This means that the generator of such a semigroup does not exist in the standard sense. As a consequence, a number of weaker (equivalent) definitions have been developed in the literature. For a good discussion of this and related problems see Lorenzi and Bertoldi [43].

Definition 32. *A sequence of functions $\{f_n\}$ in $\mathbf{C}_b[\mathbb{R}^n]$ is said to converge to f in the mixed topology, written τ^M-$\lim f_n = f$, if $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \leqslant M$ and $\|f_n - f\|_\infty \to 0$ uniformly on every compact subset of $\mathbb{R}^n$.*

Theorem 33. *If $\{f_n\}$ converges to f in the mixed topology on $\mathbf{C}_b[\mathbb{R}^n]$, then $\{f_n\}$ converges to f in the norm topology of $\mathbf{KS}^p[\mathbb{R}^n]$ for each $1 \le p \le \infty$.*

Proof. It is easy to see that both $\mathbf{C}_b[\mathbb{R}^n]$ and $\mathbf{UBC}[\mathbb{R}^n]$ are subsets of $\mathbf{KS}^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Thus, it suffices to prove that τ^M-$\lim f_n = f$ implies that $\lim_{n\to\infty} \|f_n - f\|_{\mathbf{KS}}^p = 0$. This now follows from the fact that each box used in our definition of the $\mathbf{KS}^p[\mathbb{R}^n]$ norm is a compact subset of $\mathbb{R}^n$. □

Theorem 34. *Suppose that $\hat{T}(t)$ is a transition semigroup defined on $\mathbf{C}_b[\mathbb{R}^n]$ with weak generator $\hat{A}$. Let $T(t)$ be the extension of $\hat{T}(t)$ to $\mathbf{KS}^p[\mathbb{R}^n]$. Then $T(t)$ is strongly continuous, and the extension A of $\hat{A}$ to $\mathbf{KS}^p[\mathbb{R}^n]$ is the strong generator of $T(t)$.*

PROOF. First observe that the dual of $\mathbf{C}_b[\mathbb{R}^n]$ is $\mathfrak{M}[\mathbb{R}^n]$, which is contained in $\mathbf{KS}^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$. Thus, we can apply Theorem 27 to show that $\hat{T}(t)$ has a bounded extension to $\mathbf{KS}^2[\mathbb{R}^n]$. It is easy to show that the extended operator $T(t)$ is a semigroup. Since the τ^M topology on $\mathbf{C}_b[\mathbb{R}^n]$ is stronger than the norm topology on $\mathbf{KS}^2[\mathbb{R}^n]$, we see that the generator A of $T(t)$ is strong. □

5.2 Feynman Path Integral.

Historically, the mathematics community has had two responses to the introduction of a new mathematical idea or method into physics. The first response has been to fit the idea or method into an existing framework. The second and more exciting response is when such an idea or method leads to the development of a new branch of mathematics.

The most prevalent and successful response has been in finding an existing mathematical structure that will reasonably accommodate the physical theory and provide (at least) the framework for mathematical rigor. An excellent example of this is the introduction of matrix algebra into the Heisenberg formulation of quantum theory (i.e., matrix mechanics) by Born and Jordan [6]. This made it possible for Schrödinger to show that, in the nonrelativistic case, his wave mechanics was equivalent to Heisenberg's theory. This was later shown to be rigorously true mathematically via the unitary equivalence between $\mathbf{l}_2$ and $\mathbf{L}^2$ as separable Hilbert spaces (c.f., von Neumann [57]). However, even in this case, we should not conclude that this is the complete story. There have always been physical advantages in looking at and working with some problems using the Heisenberg formulation. In fact, in 1964, Dirac strongly suggested on physical grounds that, at the quantum field level, Heisenberg's formulation is much more fundamental (see [7], page 130). Furthermore, recent studies strongly indicate that the mathematical concept of isometric isomorphism need not be sufficient for physical equivalence. (For example, it is known, [27], that the Dirac operator is nonlocal in time, while the square-root operator is nonlocal in space, but they are unitarily equivalent.)

In some rare but important instances, there is no obvious mathematical structure which can completely accommodate the theory in the manner presented by physicists. In this case, mathematicians have extended and/or adapted an existing mathematical theory, developed new mathematical structures or suggested (in frustration) that any conclusions derived from the use of these ideas or methods are at least suspect. Over the last sixty years, all of the above positions have appeared in response to Feynman's introduction of his path integral into quantum theory.

Since his path integral is the object of this section, let us consider the simple case of a free particle in nonrelativistic quantum theory in $\mathbb{R}^3$:

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t}-\frac{\hbar^2}{2m}\Delta\psi(\mathbf{x},t)=0,\ \psi(\mathbf{x},s)=\delta(\mathbf{x}-\mathbf{y}). \tag{5.2}$$

The solution can be computed directly:

$$\psi(\mathbf{x},t)=K\left[\mathbf{x},\ \ t;\ \mathbf{y},\ \ s\right]=\left[\frac{2\pi i\hbar(t-s)}{m}\right]^{-3/2}\exp\left[\frac{im}{2\hbar}\frac{\left|\mathbf{x}-\mathbf{y}\right|^2}{(t-s)}\right].$$

Feynman wrote the above solution to equation (5.2) as

$$K\left[\mathbf{x},\,t;\ \mathbf{y},\,s\right]=\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x}\mathcal{D}\mathbf{x}(\tau)\exp\left\{\tfrac{im}{2\hbar}\int_s^t\ \left|\tfrac{d\mathbf{x}}{dt}\right|^2d\tau\right\}, \tag{5.3}$$

where

$$\begin{gathered}\int_{\mathbf{x}(s)=y}^{\mathbf{x}(t)=x}\mathcal{D}\mathbf{x}(\tau)\exp\left\{\tfrac{im}{2\hbar}\int_s^t\left|\tfrac{d\mathbf{x}}{dt}\right|^2d\tau\right\}=: \\ \lim_{N\to\infty}\left[\tfrac{m}{2\pi i\hbar\varepsilon(N)}\right]^{3N/2}\int_{\mathbb{R}^3}\prod_{j=1}^{N}d\mathbf{x}_j\exp\left\{\tfrac{i}{\hbar}\sum_{j=1}^{N}\left[\tfrac{m}{2\varepsilon(N)}\left(\mathbf{x}_j-\mathbf{x}_{j-1}\right)^2\right]\right\},\end{gathered} \tag{5.4}$$

with $\varepsilon(N)=(t-s)/N$.

Problems.

Feynman's objective was to develop an approach to quantum theory which would avoid the use of a Hamiltonian. Equations (5.2)-(5.4) can be viewed as an attempt to "apparently" define an integral over the space of all continuous paths of the exponential of an integral of the classical Lagrangian on configuration space. Thus, his objective was (partly) accomplished.

However, this approach (using the Lagrangian directly) has led to a new method for quantizing physical systems, called the path integral method. It is now used almost exclusively by large groups (in all branches of physics) and has also been used (formally) by researchers in both mathematics and mathematical physics. Thus, we must conclude that Feynman's formulation (as he proposed it) is both physically and mathematically distinct from those of Heisenberg and Schrödinger. (Feynman showed that existence of the other two representations implies his path integral representation. However, it has not been shown that a path integral representation implies the other two.)

From a mathematical point of view, this leads to a number of problems:

- The kernel $\mathbf{K}[\mathbf{x},\ t;\ \mathbf{y},\ s]$ and $\delta(\mathbf{x})$ are not in $L^2[\mathbf{R}^3]$, the standard space for quantum theory.
- The kernel $\mathbf{K}[\mathbf{x},\ t;\ \mathbf{y},\ s]$ cannot be used to define a measure.

Thus, a natural question is: Does there exist a separable Hilbert space containing $\mathbf{K}[\mathbf{x}, t;\ \mathbf{y}, s]$ and $\delta(\mathbf{x})$ which also allows the convolution and Fourier transform as bounded operators? A positive answer to this question is necessary if we are to make sense of equation (5.4) and have a representation space for the Feynman formulation of quantum theory (as presented).

The properties of $\mathbf{KS}^2[\mathbb{R}^n]$ derived earlier suggest that it may be a more appropriate Hilbert space, compared to $\mathbf{L}^2[\mathbb{R}^n]$, for the Feynman formulation. It is easy to prove that both the position and momentum operators have closed densely defined extensions to $\mathbf{KS}^2[\mathbb{R}^n]$. Furthermore, the extensions of $\mathfrak{F}$ and $\mathfrak{C}$ insure that both the Schrödinger and Heisenberg theory have faithful representations on $\mathbf{KS}^2[\mathbb{R}^n]$.

Since $\mathbf{KS}^2[\mathbb{R}^n]$ contains the space of measures, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $\mathbf{KS}^2[\mathbb{R}^n]$ topology. (For example, $[\sin(\lambda\cdot\mathbf{x})/(\lambda\cdot\mathbf{x})]\in\mathbf{KS}^2[\mathbb{R}^n]$ and converges strongly to $\delta(\mathbf{x})$.) Thus, the Feynman kernel [17] generates a finitely additive set function defined on the algebra of sets B, such that $\mathcal{E}_B(|\mathbf{y}|)\in\mathcal{BV}$ by: (with $m=1$ and $\hbar=1$)

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B]=\int_B(2\pi i(t-s))^{-n/2}\exp\{i|\mathbf{x}-\mathbf{y}|^2\big/2(t-s)\}d\mathbf{y}$$

is in $\mathbf{KS}^2[\mathbb{R}^n]$ and $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B]\|_{\mathrm{KS}}\leqslant 1$, while $\|\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B]\|_{\mathfrak{M}}=\infty$ (the total variation norm) and

$$\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,B]=\int_{\mathbb{R}^n}\mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,\tau,d\mathbf{z}]\mathbb{K}_{\mathbf{f}}[\tau,\mathbf{z}\,;\,s,B],\ \text{(HK-integral)}.$$

Definition 35. *Let $\mathbf{P}_k=\{t_0,\tau_1,t_1,\tau_2,\cdots,\tau_k,t_k\}$ be an HK-partition for a function $\delta_k(s),\ s\in[0,t]$ for each k, with $\lim_{k\to\infty}\Delta\mu_k=0$ (mesh). Set $\Delta t_j=t_j-t_{j-1},\tau_0=0$ and, for $\psi\in\mathbf{KS}^2[\mathbb{R}^n]$, define*

$$\int_{\mathbb{R}^{n[0,t]}}\mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda\mathbf{x}(\tau)\ ;\mathbf{x}(0)]=e^{-\lambda t}\sum_{k=0}^{[\![\lambda t]\!]}\frac{(\lambda t)^k}{k!}\left\{\prod_{j=1}^{k}\int_{\mathbb{R}^n}\mathbb{K}_{\mathbf{f}}[t_j,\mathbf{x}(\tau_j)\,;\,t_{j-1},d\mathbf{x}(\tau_{j-1})]\right\},$$

and

$$\int_{\mathbb{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\mathbf{x}(0)]\psi[\mathbf{x}(0)] = \lim_{\lambda\to\infty}\int_{\mathbb{R}^{n[0,t]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}_\lambda\mathbf{x}(\tau)\ ;\mathbf{x}(0)]\psi[\mathbf{x}(0)] \tag{5.5}$$

whenever the limit exists.

Remark 36. *In the above definition we have used the Poisson process. This is not accidental but appears naturally from a physical analysis of the information that is knowable in the micro-world (see [23]). In fact, it has been suggested by Kolokoltsov [38] that such jump processes often provide another effective way to give meaning to the Feynman path integral, and also offers a nice approach to Feynman diagrams.*

However, the term $k = 0$ in our definition still seems strange. This term represents the probability that no information appears. Thus, in case $k = 0$, the product does not contribute and the right-hand side reduces to $e^{-\lambda t}$. In the limit, this term becomes zero in equation (5.5) (see also Section 5 in [23]).

The next result is now elementary, since $\mathbf{KS}^2[\mathbb{R}^n]$ is closed under convolution.

Theorem 37. *The function $\psi(\mathbf{x}) \equiv 1 \in \mathbf{KS}^2[\mathbb{R}^n]$ and*

$$\int_{\mathbb{R}^{n[t,s]}} \mathbb{K}_{\mathbf{f}}[\mathcal{D}\mathbf{x}(\tau)\ ;\mathbf{x}(s)] = \mathbb{K}_{\mathbf{f}}[t,\mathbf{x}\,;\,s,\mathbf{y}] = \tfrac{1}{\sqrt{[2\pi i(t-s)]^n}}\exp\{i|\mathbf{x}-\mathbf{y}|^2\big/2(t-s)\}.$$

The above result is what Feynman was trying to obtain without the appropriate space. A more general (sum over paths) result, that covers almost all application areas, will appear later, where these spaces have been used to provide a generalization of the constructive representation theory for the Feynman operator calculus (see [24] and also [23] for other applications).

Discussion.

A natural reaction to any suggestion that we replace the Lebesgue integral by one based on a finitely additive measure would be negative. After all, we would lose all of the advantages of the powerful theorems (dominated convergence theorem, monotone convergence theorem, etc) that depend on the countable additivity of the measure. Those strongly vested in using $\mathbf{L}^2$ for the C^*-algebra approach to quantum theory via the GNS construction may also feel obliged to object to such a proposed change. These are all reasonable concerns. However, we do not lose any of the powerful theorems found via countable additivity. First of all, the HK-measure is an extension of the

Lebesgue measure so that all of its power is still available to us. In fact, Henstock has extended each of the standard theorems to the HK-integral (see [33]). Those concerned with the C^*-algebra approach to quantum theory need not be concerned since $\mathbf{KS}^2$ is a separable Hilbert space and is also amenable to the GNS construction. If we treat $\mathbf{K}[\mathbf{x},\ t;\ \mathbf{y},\ s]$ as the kernel for an operator acting on good initial data, then a partial solution has been obtained by a number of workers. (See [24] for references to all the important contributions in this direction.)

A related approach to the Feynman path integral can be found in the work of Fujiwara and Kumano-go (see [18], [19] and references therein). For a survey of this approach, see [20]. They have systematically developed a time-slicing approximation method that covers a large portion of classical quantum theory. They restrict themselves to scalar potentials with polynomial growth. However, their method seems general enough to eventually include the additional cases. (They show the power of their approach by providing an analytic formula for the second term of the semi-classical asymptotic expansion of the Feynman path integral.)

5.3 Examples.

A standard method is to compute the Wiener path integral for the problem under consideration and then use analytic continuation in the mass to provide a rigourous meaning for the Feynman path integral. The following example provides a path integral representation for a problem that cannot be solved using analytic continuation via a Gaussian kernel (see Gill and Zachary [26]). It is shown that, if the vector potential $\mathbf{A}$ is constant, $\mu = mc/\hbar$, and $\boldsymbol{\beta}$ is the standard beta matrix, then the solution to the square-root equation for a spin 1/2 particle:

$$i\hbar\partial\psi(\mathbf{x},t)/\partial t = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 + m^2c^4}\right\}\psi(\mathbf{x},t),\ \ \psi(\mathbf{x},0) = \psi_0(\mathbf{x}),$$

is given by:

$$\psi(\mathbf{x},t) = \mathbf{U}[t,0]\psi_0(\mathbf{x}) = \int\limits_{\mathbb{R}^3} \exp\left\{\frac{ie}{2\hbar c}\left(\mathbf{x}-\mathbf{y}\right)\cdot\mathbf{A}\right\}\mathbf{K}\left[\mathbf{x},t\,;\,\mathbf{y},0\right]\psi_0(\mathbf{y})d\mathbf{y},$$

where

$$\mathbf{K}\left[\mathbf{x},t\,;\,\mathbf{y},0\right]=\frac{ict\mu^2\beta}{4\pi}\begin{cases}\frac{-H_2^{(1)}\left[\mu\left(c^2t^2-||\mathbf{x}-\mathbf{y}||^2\right)^{1/2}\right]}{[c^2t^2-||\mathbf{x}-\mathbf{y}||^2]}, & ct<-||\mathbf{x}||,\\ \frac{-2iK_2\left[\mu\left(||\mathbf{x}-\mathbf{y}||^2-c^2t^2\right)^{1/2}\right]}{\pi[||\mathbf{x}-\mathbf{y}||^2-c^2t^2]}, & c\,|t|<\,||\mathbf{x}||,\\ \frac{H_2^{(2)}\left[\mu\left(c^2t^2-||\mathbf{x}-\mathbf{y}||^2\right)^{1/2}\right]}{[c^2t^2-||\mathbf{x}-\mathbf{y}||^2]}, & ct>\,||\mathbf{x}||.\end{cases}$$

The function $K_2(\,\cdot\,)$ is a modified Bessel function of the third kind of second order, while $H_2^{(1)},\ H_2^{(2)}$ are Hankel functions (see Gradshteyn and Ryzhik [30]). Thus, we have a kernel that is far from the standard form. This example was first introduced in [25], where we only considered the kernel for the Bessel function term. In that case, it was shown that, under appropriate conditions, that term will reduce to the free-particle Feynman kernel and, if we set $\mu=0$, we get the kernel for a (spin 1/2) massless particle. In closing this section, we remark that the square-root operator is unitarily equivalent to the Dirac operator (in the case discussed).

5.4 The Kernel Problem.

Since any semigroup that has a kernel representation will automatically generate a path integral via the reproducing property, a fundamental question is: Under what general conditions can we expect a given (time-independent) generator of a semigroup to have an associated kernel? In this section we discuss a class of general conditions for unitary groups. It will be clear that the results of this section carry over to semigroups with minor changes.

Let $A(\mathbf{x},\ \mathbf{p})$ denote a $k\times k$ matrix operator $[A_{ij}(\mathbf{x},\ \mathbf{p})]$, $i,\ j=1,2,\cdots,k$, whose components are pseudodifferential operators with symbols $a_{ij}(\mathbf{x},\ \boldsymbol{\eta})\in\mathbb{C}^\infty(\mathbb{R}^{\mathbf{n}}\times\mathbb{R}^{\mathbf{n}})$ and we have, for any multi-indices α and β,

$$\left|a_{ij(\beta)}^{(\alpha)}(\mathbf{x},\boldsymbol{\eta})\right|\leqslant C_{\alpha\beta}(1+|\boldsymbol{\eta}|)^{m-\xi|\alpha|+\delta|\beta|}, \tag{5.6}$$

where

$$a_{ij(\beta)}^{(\alpha)}(\mathbf{x},\boldsymbol{\eta})=\partial^\alpha\mathbf{p}^\beta a_{ij}(\mathbf{x},\boldsymbol{\eta}),$$

with $\partial_l=\partial/\partial\eta_l$ and $p_l=(1/i)(\partial/\partial x_l)$. The multi-indices are defined in the usual manner by $\alpha=(\alpha_1,\cdots,\alpha_n)$ for integers $\alpha_j\geq 0$, and $|\alpha|=\sum_{j=1}^n\alpha_j$, with similar definitions for β. The notation for derivatives is $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ and $\mathbf{p}^\beta=p_1^{\beta_1}\cdots p_n^{\beta_n}$. Here, $m,\ \beta$, and δ are real numbers satisfying $0\leq\delta<\xi$. Equation (5.6) states that each $a_{ij}(\mathbf{x},\ \boldsymbol{\eta})$ belongs to the symbol class $S_{\xi,\delta}^m$ (see [49]).

Let $a(\mathbf{x},\boldsymbol{\eta}) = [a_{ij}(\mathbf{x},\boldsymbol{\eta})]$ be the matrix-valued symbol for $A(\mathbf{x},\boldsymbol{\eta})$, and let $\lambda_1(\mathbf{x},\boldsymbol{\eta}) \cdots \lambda_k(\mathbf{x},\boldsymbol{\eta})$ be its eigenvalues. If $|\cdot|$ is the norm in the space of $k \times k$ matrices, we assume that the following conditions are satisfied by $a(\mathbf{x},\boldsymbol{\eta})$. For $0 < c_0 < |\boldsymbol{\eta}|$ and $\mathbf{x} \in \mathbb{R}^n$ we have

1. $\left| a^{(\alpha)}_{(\beta)}(\mathbf{x},\boldsymbol{\eta}) \right| \le C_{\alpha\beta} \, |a(\mathbf{x},\boldsymbol{\eta})| \, (1 + |\boldsymbol{\eta}|) |)^{-\xi|\alpha|+\delta|\beta|}$ (hypoellipticity),
2. $\lambda_0(\mathbf{x},\boldsymbol{\eta}) = \max\limits_{1\leqslant j\leqslant k} \operatorname{Re} \lambda_j(\mathbf{x},\boldsymbol{\eta}) < 0$,
3. $\frac{|a(\mathbf{x},\boldsymbol{\eta})|}{|\lambda_0(\mathbf{x},\boldsymbol{\eta})|} = O\left[(1 + |\boldsymbol{\eta}|)^{(\xi-\delta)/(2k-\varepsilon)} \right], \ \varepsilon > 0.$

We assume that $A(\mathbf{x},\mathbf{p})$ is a selfadjoint generator of a unitary group $U(t,0)$, so that

$$U(t,0)\psi_0(\mathbf{x}) = exp[(i/\hbar)tA(\mathbf{x},\mathbf{p})]\psi_0(\mathbf{x}) = \psi(\mathbf{x},t)$$

solves the Cauchy problem

$$(i\hbar)\partial\psi(\mathbf{x},t)/\partial t = A(\mathbf{x},\mathbf{p})\psi(\mathbf{x},t), \quad \psi(\mathbf{x},t) = \psi_0(\mathbf{x}). \tag{5.7}$$

Definition 38. *We say that $Q(\mathbf{x},t,\boldsymbol{\eta},0)$ is a symbol for the Cauchy problem (5.7) if $\psi(\mathbf{x},t)$ has a representation of the form*

$$\psi(\mathbf{x},t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\mathbf{x},\boldsymbol{\eta})} Q(\mathbf{x},t,\boldsymbol{\eta},0)\hat{\psi}_0(\boldsymbol{\eta})d\boldsymbol{\eta}. \tag{5.8}$$

It is sufficient that ψ_0 belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, which is contained in the domain of $A(\mathbf{x},\mathbf{p})$, in order that (5.8) makes sense.

Following Shishmarev [49], and using the theory of Fourier integral operators, we can define an operator-valued kernel for $U(t,0)$ by

$$K(\mathbf{x},t\,;\,\mathbf{y},0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\mathbf{x}-\mathbf{y},\boldsymbol{\eta})} Q(\mathbf{x},t,\boldsymbol{\eta},0)d\boldsymbol{\eta},$$

so that

$$\psi(\mathbf{x},t) = U(t,0)\psi_0(\mathbf{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} K(\mathbf{x},t\,;\,\mathbf{y},0)\psi_0(\mathbf{y})d\mathbf{y}. \tag{5.9}$$

The following results are due to Shishmarev [49].

Theorem 39. *If $A(\mathbf{x},\mathbf{p})$ is a selfadjoint generator of a strongly continuous unitary group with domain D, $\mathcal{S}(\mathbb{R}^n) \subset D$ in $L^2(\mathbb{R}^n)$, such that conditions (1)-(3) are satisfied, then there exists precisely one symbol $Q(\mathbf{x},t,\boldsymbol{\eta},0)$ for the Cauchy problem (5.7).*

Theorem 40. *If we replace our condition (3) in Theorem 39 by the stronger condition*

(3′) $\frac{|a(\mathbf{x},\boldsymbol{\eta})|}{|\lambda_0(\mathbf{x},\boldsymbol{\eta})|} = O\left[(1+|\boldsymbol{\eta}|)^{(\xi-\delta)/(3k-1-\varepsilon)}\right],\ \varepsilon > 0, |\boldsymbol{\eta}| > c_0,$

then the symbol $Q(\mathbf{x}, t, \boldsymbol{\eta}, 0)$ *of the Cauchy problem (5.7) has the asymptotic behavior near* $t = 0$*:*

$$Q(\mathbf{x}, t, \boldsymbol{\eta}, 0) = exp[-(i/\hbar)ta(\mathbf{x}, \boldsymbol{\eta})] + o(1),$$

uniformly for $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^{\mathbf{n}}$*.*

Now, using Theorem 40 we see that, under the stronger condition (3'), the kernel $K(\mathbf{x}, t\,; \mathbf{y}, 0)$ satisfies

$$K(\mathbf{x}, t; \mathbf{y}, 0) = \int_{\mathbb{R}^n} \exp[i(\mathbf{x}-\mathbf{y}, \boldsymbol{\eta}) - (i/\hbar)ta(\mathbf{x}, \boldsymbol{\eta})] \frac{d\boldsymbol{\eta}}{(2\pi)^{n/2}} + \int_{\mathbb{R}^n} \exp[i(\mathbf{x}-\mathbf{y}, \boldsymbol{\eta})] \frac{d\boldsymbol{\eta}}{(2\pi)^{n/2}} o(1).$$

In order to see the power of $\mathbf{KS}^2(\mathbb{R}^n)$, first note that $A(\mathbf{x}, \mathbf{p})$ has a selfadjoint extension to $\mathbf{KS}^2(\mathbb{R}^n)$ which also generates a unitary group. This means that we can construct a path integral in the same (identical) way as was done for the free-particle propagator (i.e., for all Hamiltonians with symbols in $\mathcal{S}^m_{\alpha,\delta}$). Furthermore, it follows that the same comment applies to any Hamiltonian that has a kernel representation, independent of its symbol class. The important point of this discussion is that no initial data or Gaussian form for the kernel is required!

6 The Lebesgue Measure Problem on $\mathbb{R}^\infty$.

6.1 Background.

It is well-known that physics has been and continues to be a powerful source of research inspiration for both pure and applied mathematics. In some cases, physics insight has also given new approaches to problems in mathematics that appeared hopeless. This section has two objectives. The first objective is to show how elementary physics explains why the standard way of thinking about Lebesgue measure on finite–dimensional space does not apply in the infinite-dimensional case. The second and more important objective is to show how a slight change in thinking about the cause for problems with unbounded measures on $\mathbb{R}^\infty$ makes the construction of Lebesgue measure (on a reasonable version of $\mathbb{R}^\infty$) possible. This allows us to provide a version of Lebesgue measure for every separable Banach space that has a Schauder basis. As an application, we construct a version of Gaussian measure on the space of continuous paths.

6.2 Motivation.

In order to understand the problem and part of our motivation, let I be a countable set and, for each $i \in I$, let $(X_i,\ \mathfrak{B}_i,\ m_i)$ be a measure space, where X_i is a complete separable metric space, $\mathfrak{B}_i$ is the Borel σ-algebra generated by the open sets of X_i and m_i is a probability measure on X_i. Let $\mathbf{L}^2[X_i,\ \mathfrak{B}_i,\ m_i] = \mathbf{L}^2[X_i]$ be the set of complex-valued functions $f(x)$ in X_i such that $|f(x)|^2$ is integrable with respect to m_i. If Δ_2 is the natural tensor product norm for Hilbert spaces; then, for any pair X_i and X_j, $\mathbf{L}^2[X_i] \hat{\otimes}^{\Delta_2} \mathbf{L}^2[X_j] = \mathbf{L}^2[X_i \times X_j]$. Let $\phi_i \in \mathbf{L}^2[X_i]$ with $\|\phi_i\|_{\mathbf{L}^2[X_i]} = 1$ and, with $\phi = \otimes_{i \in I}\phi_i$, construct the incomplete tensor product space of von Neumann [58], $\mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s$. Let $X = \times_{i\in I} X_i$ and $\mathfrak{B} = \hat{\otimes}_{i\in I}\mathfrak{B}_i$ (the smallest σ-algebra containing $\times_{i\in I}\mathfrak{B}_i$). Recall that a tame function in $\mathbf{L}^2[X]$ is any function $f \in \mathbf{L}^2[X]$ which only depends on a finite number of variables.

Theorem 41. (Guichardet [32]) $\mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s \cong \mathbf{L}^2[X]$.

PROOF. Let $J_N = \{i_1, \cdots i_N\} \subset I$ (where N is finite but arbitrary), let $f(x_{i_1}, \cdots x_{i_N})$ be a tame function in $\mathbf{L}^2[X]$, and define $\tilde{f}(x_{i_1}, \cdots x_{i_N}) = f(x_{i_1}, \cdots x_{i_N}) \otimes (\otimes_{i\in I\setminus J_N}\phi_\nu)$ so that $\tilde{f}(x_{i_1}, \cdots x_{i_N}) \in \mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s$. Define a function $\Lambda :\ \mathbf{L}^2[X] \to \mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s$ by $\Lambda(f) = \tilde{f}$. It is easy to check that Λ is well-defined and it is easy to see that: (1) $\Lambda(af_1 + bf_2) = a\Lambda(f_1) + b\Lambda(f_2)$ (Λ is a linear mapping); (2) $\|\Lambda(f)\|_{\Delta_2} = \|f\|_{\mathbf{L}^2[X]}$ (Λ is an isometric mapping); and (3) $\Lambda(f_1) = \Lambda(f_2) \Rightarrow f_1 = f_2$ (Λ is a one-to-one mapping). Since the set of tame functions is dense in $\mathbf{L}^2[X]$ and the set of all $\tilde{f}$ is dense in $\mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s$, it follows that, for any f in $\mathbf{L}^2[X]$, we can define $\Lambda(f) = \lim_{n\to\infty}\Lambda(f_n)$, where $\{f_n\}$ is any sequence of tame functions converging to f. Since the extension to $\mathbf{L}^2[X]$ is one-to-one, Λ defines an isometric isomorphism of $\mathbf{L}^2[X]$ onto $\mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s$. $\square$

Now observe that this theorem should be true if each $X_i = \mathbb{R}$ and each $m_i = \lambda$ (Lebesgue measure). In this case, $\mathbf{L}^{\Delta_2}_{\otimes}[\phi]^s \cong \mathbf{L}^2[\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty), \hat{\lambda}_\infty]$, where $\hat{\lambda}_\infty$ is some version of Lebesgue measure on $\mathbb{R}^\infty$. However, there is a "folk theorem" to the effect that there is no reasonable version of Lebesgue measure on $\mathbb{R}^\infty$.

The above is interesting on a number of levels. However, our interest in the general question is mainly motivated by modeling considerations for infinite dimensional systems. In many cases the Hilbert space structure appears as a natural state space. In other cases, both the Hilbert space structure and probability measures are imposed for mathematical convenience and appear somewhat artificial. On the other hand, all reasonable models of infinite dimensional physical systems require some constraint on the effects on all but a finite number of variables. Thus, what is needed, in general, is the imposition of constraints that preserve the freedom associated with functions of independent variables (in some well-defined sense). This necessarily implies a theory of Lebsegue measure on (a version of) $\mathbb{R}^\infty$.

6.3 Rotational and Translational Invariance.

On finite-dimensional space it is useful to think of Lebesgue measure in terms of geometric objects (e.g., volume, surface area, etc.). Thus, it is natural to expect that this measure will leave these objects invariant under translations and rotations, so that rotational and translational invariance is an intrinsic property of Lebesgue measure. However, we then find ourselves disappointed when we try to use these properties to help define Lebesgue measure on $\mathbb{R}^\infty$. On the other hand, if we replace Lebesgue measure by the infinite product Gaussian measure, μ_∞, on $\mathbb{R}^\infty$, we get countable additivity but lose rotational invariance. Furthermore, the μ_∞ measure of l_2 is zero. On the other hand, another approach is to use the standard projection method onto finite dimensional subspaces to construct a probability measure directly on l_2. In this case, we recover rotational invariance but not translation invariance (but we lose countable additivity). A nice discussion of this and related issues can be found in Dunford and Schwartz [12] (see pg. 402).

The above problems, along with the natural need for infinite products of probability measures served to concentrate research in another direction. However, the lack of any definitive understanding of the cause for this lack of invariance on $\mathbb{R}^\infty$ has led to the general sense that it is very difficult, if not impossible, to construct a version of Lebesgue measure on $\mathbb{R}^\infty$.

From a physical point of view, the angular momentum operator is the generator for the rotational group, while the momentum operator is the generator for the translation group. Thus, physically we interpret the failure of full rotational and translation invariance for an infinite dimensional system as a constraint induced by the requirement of finite total angular and linear momenta (i.e., physical systems with infinite amounts of angular or linear momentum are ill-defined). Thus, such systems are necessarily ill-defined as geometric objects. To see why we recover rotational but not translational invariance for l_2 for Gaussian measures, see below. One could ask similar questions about $\mathbb{C}_0[0,1]$, the continuous functions $x(t)$ on $[0,1]$ with $x(0) = 0$ (i.e., Wiener space). (This is unrelated to the mathematical desirability of countable additivity.)

To begin, we first consider the following classical setup. Suppose we have an infinite system of particles with positions $\{\mathbf{x}_i(t)\}$ and momenta $\{\mathbf{p}_i(t)\},\ 1 \leq i < \infty$ $(t \in [0,1])$. For each $i,\ \mathbf{x}_i(t),\ \mathbf{p}_i(t)$ are mappings from $\mathbb{R}^3 \to L^2[0,1]$. In this case, the total angular momentum is defined by:

$$\mathbf{J} = \sum\nolimits_{i=1}^{\infty} \mathbf{x}_i \times \mathbf{p}_i.$$

(Under standard physical conditions, we assume that $\mathbf{J}$ is independent of t.) It follows that, if our phase space is l_2, then

$$\|\mathbf{J}\|_2 \leqslant \sum\nolimits_{i=1}^{\infty} \|\mathbf{x}_i \times \mathbf{p}_i\|_2 \leqslant \tfrac{1}{2} \sum\nolimits_{i=1}^{\infty} \left[\|\mathbf{x}_i\|_2^2 + \|\mathbf{p}_i\|_2^2\right],$$

so that the total angular momentum $\mathbf{J}$ is finite. On the other hand, the total linear momentum, $\mathbf{L} = \sum_{i=1}^{\infty} \mathbf{p}_i$, need not be bounded, so we do not have translation invariance. In the case of $\mathbb{C}_0[0,1]$ both bounds can fail. (The correct setup is the quantum level, where both $\mathbf{x}$, the position, and $\mathbf{p}$, the momentum, of a particle become operators. However, this requires a substantial detour, which will not materially change the above picture.)

Remark 42. *We note that the ray representation (ambiguity) in conventional quantum theory may be viewed as the precursor to the ultraviolet divergence problem in quantum field theory where the ray, along with the wave function, becomes an operator. (A more precise discussion and proof of the physical cause for the ultraviolet divergence in quantum electrodynamics, as conjectured by Dyson [15], can be found in [23].)*

6.4 Lebesgue Measure on $\mathbb{R}_I^\infty$.

It is instructive to review the historical approach to the construction of infinite product measures, $\{\mu_k,\ k \in \mathbb{N}\}$, on $\mathbb{R}^\infty$. In the standard approach, the chosen topology defines open sets to be the (Cartesian) product of an arbitrary finite number of open sets in $\mathbb{R}$, while the remaining (infinite number) are copies of $\mathbb{R}$. The first success was Kolmogorov's work on the foundations of probability theory [37]. This naturally led to the condition that $\mu_k(\mathbb{R})$ be finite for all but a finite number of k. Thus, any attempt to construct Lebesgue measure on this space starts out a failure (since the measure of every open set is infinite). It appears that any attempt to define Lebesgue theory must face this restriction. However, Kolmogorov's approach is not the only way to induce a total measure of one for the spaces under consideration. An alternate approach is to use Lebesgue measure and replace the (tail end of the) infinite product of copies of $\mathbb{R}$ by infinite products of copies of a unit interval (with, as will be seen, a few additions). In this case, we can still construct probability measures, but now the door is open to reconsider Lebesgue measure.

The above discussion suggests that, contrary to the long held belief, it may well be possible to construct Lebesgue measure on (a reasonable version of) $\mathbb{R}^\infty$. The purpose of this section is to provide such a construction. Let λ be Lebesgue measure on $\mathbb{R}$.

Set $I = [-\frac{1}{2}, \frac{1}{2}]$ and, for each n, define $\mathbb{R}_I^n = \mathbb{R}^n \times I_n$, where $I_n = \mathop{\times}\limits_{i=n+1}^{\infty} I$. In order to avoid confusion, we set $\bar{I}^c = \emptyset$, where $\bar{I} = \times_{J=1}^{\infty} I$.

Definition 43. *If $A_n = A \times I_n,\ B_n = B \times I_n$ are any sets in $\mathbb{R}_I^n$, then we define:*

1. $A_n \cup B_n = A \cup B \times I_n$,
2. $A_n \cap B_n = A \cap B \times I_n$, *and*
3. $B_n^c = B^c \times I_n$.

We can now define the topology for $\mathbb{R}_I^n$ via the following class of open sets:

$$\mathfrak{O}_n = \left\{ U_n : U_n = U \times I_n,\ U \text{ open in } \mathbb{R}^n \right\}.$$

Note, that (1) and (2) is the same as the standard approach. However, (3) is different, so that both the set operations and the above class of sets, $\mathfrak{O}_n$, are not quite the same as those used for the topology of infinite product spaces.

If $\mathfrak{B}(\mathbb{R}_I^n)$ is the Borel σ-algebra for $\mathbb{R}_I^n$ (i.e., the smallest σ-algebra generated by $\mathfrak{O}_n$), then it is easy to see that $\mathbb{R}_I^n \subset \mathbb{R}_I^{n+1}$ and $\mathfrak{B}(\mathbb{R}_I^n) \subset \mathfrak{B}(\mathbb{R}_I^{n+1})$. Since they are both increasing sequences, we can define $\bar{\mathbb{R}}_I^\infty$ and $\bar{\mathfrak{B}}(\mathbb{R}_I^\infty)$ by:

$$\bar{\mathbb{R}}_I^\infty = \lim_{\mathrm{n}\to\infty} \mathbb{R}_{\mathrm{I}}^{\mathrm{n}} = \bigcup_{\mathrm{k}=1}^{\infty} \mathbb{R}_{\mathrm{I}}^{\mathrm{k}}$$

and

$$\bar{\mathfrak{B}}(\mathbb{R}_I^\infty) = \lim_{\mathrm{n}\to\infty} \mathfrak{B}(\mathbb{R}_{\mathrm{I}}^{\mathrm{n}}) = \bigcup_{\mathrm{k}=1}^{\infty} \mathfrak{B}(\mathbb{R}_{\mathrm{I}}^{\mathrm{k}}).$$

Let $\mathbb{R}_I^\infty$ be the closure of $\bar{\mathbb{R}}_I^\infty$ in the induced topology from $\mathbb{R}^\infty$ (see below). Let $\mathfrak{B}(\mathbb{R}_I^\infty)$ be the smallest σ-algebra containing $\bar{\mathfrak{B}}(\mathbb{R}_I^\infty)$. From our construction, it is clear that a set of the form $A = A_n \times \left(\times_{k=n+1}^{\infty} \mathbb{R}\right)$ is not in $\bar{\mathbb{R}}_I^\infty$ for any n, so that $\bar{\mathbb{R}}_I^\infty \neq \mathbb{R}^\infty$. It may well be that $\mathbb{R}^\infty \in \mathfrak{B}(\mathbb{R}_I^\infty)$ and, $\mathbb{R}_I^\infty = \mathbb{R}^\infty$. From our construction, it is clear that $\mathbb{R}_I^\infty \subset \mathbb{R}^\infty$ (We will see momentarily that they are actually equal.)

On the other hand, for any set A of the form $A = \mathop{\times}\limits_{i=1}^{\infty} A_i \in \mathfrak{B}(\mathbb{R}_I^{\infty})$ with $\sum_{i=1}^{\infty} |\lambda(A_i) - 1| < \infty$, we can define a finite-valued infinite dimensional Lebesgue measure of A by:

$$0 < \lambda_{\infty}(A) = \prod\nolimits_{i=1}^{\infty} \lambda(A_i). \tag{6.1}$$

The natural topology for $\mathbb{R}_I^{\infty}$ is that induced as a closed subspace of $\mathbb{R}^{\infty}$. Thus, if $X = (x_n)$, $Y = (y_n)$ are sequences in $\mathbb{R}_I^{\infty}$,

$$d(X, Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Theorem 44. *$\mathbb{R}_I^{\infty} = \mathbb{R}^{\infty}$.*

Proof. We know that $\mathbb{R}_I^{\infty} \subset \mathbb{R}^{\infty}$. Let $X = (x_1, x_2, x_3, \cdots)$ be any point in $\mathbb{R}^{\infty}$ and let $\{Y_k\}$, $Y_k = (y_{1,k}, y_{2,k}, y_{3,k}, \cdots)$, be a sequence converging to X. Thus, given any $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for $n \geq N(\varepsilon)$, $d(X, Y_n) < \varepsilon/2$. Now choose $K(\varepsilon)$ such that for $n_k > K(\varepsilon)$,

$$\sum\nolimits_{l=n_k+1}^{\infty} \frac{1}{2^l} \frac{|y_{l,n}|}{1 + ||y_{l,n}||} < \varepsilon/2.$$

Define $\bar{Y}_n = (y_{1,n}, y_{2,n}, y_{3,n}, \cdots, y_{n_k,n}, 0, 0, \cdots)$, so that $\bar{Y}_n \in \mathbb{R}_I^{\infty}$. Furthermore, for $n \geq max\{N(\varepsilon),\ K(\varepsilon)\}$, we have

$$d(X, \bar{Y}_n) \leqslant d(X, Y_n) + d(Y_n, \bar{Y}_n) < \varepsilon.$$

Thus, X is a limit point of $\mathbb{R}_I^{\infty}$ and, since this space is closed, we see that $\mathbb{R}^{\infty} = \mathbb{R}_I^{\infty}$. □

Definition 45. *We call $\mathbb{R}_I^{\infty}$ the essentially bounded version of $\mathbb{R}^{\infty}$.*

It may be expected that we lose some of the pathology of $\mathbb{R}^{\infty}$ by replacing it with $\mathbb{R}_I^{\infty}$. However, this is not true. Infinite product measure for unbounded measures induces problems because it can fail to make sense for two additional reasons (other than being unbounded). This is best explained by considering the following simple example. Let A_i have measure $1 + \varepsilon$ for all i. It is easy to see that $\lambda_{\infty}(A) = \prod_{i=1}^{\infty} \lambda(A_i) = \infty$. On the other hand, if each A_i has measure $1 - \varepsilon$, then $\lambda_{\infty}(A) = \prod_{i=1}^{\infty} \lambda(A_i) = 0$. Thus, the class of sets $A \in \mathfrak{B}(\mathbb{R}_I^{\infty})$ for which $0 < \lambda_{\infty}(A) < \infty$ is relatively small. It follows that sets of measure zero need not be small (in the normal sense), nor need sets of infinite measure be large.

Theorem 46. *$\lambda_\infty(\cdot)$ is a measure on $\mathfrak{B}(\mathbb{R}_I^n)$, equivalent to n-dimensional Lebesgue measure on $\mathbb{R}^n$.*

Proof. If $A = \mathop{\times}\limits_{i=1}^{\infty} A_i \in \mathfrak{B}(\mathbb{R}_I^n)$, then $A_i = I$ for $i > n$ in (6.1) so that the above series always converges. Furthermore, $\lambda_\infty(A) = \prod_{i=1}^n \lambda(A_i) = \lambda_n(\mathop{\times}\limits_{i=1}^{n} A_i)$. Since sets of the type $A_i = \mathop{\times}\limits_{i=1}^{n} A_i$ generate $\mathfrak{B}(\mathbb{R}^n)$, we see that $\lambda_\infty(\cdot)$, restricted to $\mathbb{R}_I^n$, is equivalent to $\lambda_n(\cdot)$. □

It is instructive to see how naturally the measurable functions on $\mathbb{R}_I^n$ are related to those on $\mathbb{R}^n$. Let $\mathcal{E}_{I_n}$ be the characteristic function of I_n. If we let $\mathfrak{L}(X)$ represent the class of measurable functions on the set X, then for each measurable function $f_n \in \mathfrak{L}(\mathbb{R}^n)$ we identify $f_{n,I} \in \mathfrak{L}(\mathbb{R}_I^n)$ by $f_{n,I} = f_n \otimes \mathcal{E}_{I_n}$.

It is not obvious that $\lambda_\infty(\cdot)$ extends to a countably additive measure on $\mathfrak{B}(\mathbb{R}_I^\infty)$. The proof of this requires additional effort.

Definition 47. *Let*

$$\Delta_0 = \{K_n = K^n \times I_n \in \mathfrak{B}(\mathbb{R}_I^n) : K_n \text{ is compact and } 0 < \lambda_\infty(K_n) < \infty\},$$

$$\Delta = \{P_N = \bigcup\nolimits_{i=1}^N K_{n_i},\ N \in \mathbb{N};\ K_{n_i} \in \Delta_0 \text{ and } \lambda_\infty(K_{n_l} \cap K_{n_m}) = 0,\ l \neq m\}.$$

Definition 48. *If $P_N \in \Delta$, we define*

$$\lambda_\infty(P_N) = \sum\nolimits_{i=1}^N \lambda_\infty(K_{n_i}).$$

Since $P_N \in \mathfrak{B}(\mathbb{R}_I^n)$ for some n, and $\lambda_\infty(\cdot)$ is a measure on $\mathfrak{B}(\mathbb{R}_I^n)$, the next result is easy:

Lemma 49. *If $P_{N_1},\ P_{N_2} \in \Delta$ then:*

1. *If $P_{N_1} \subset P_{N_2}$, then $\lambda_\infty(P_{N_1}) \le \lambda_\infty(P_{N_2})$.*
2. *If $\lambda_\infty(P_{N_1} \cap P_{N_2}) = 0$, then $\lambda_\infty(P_{N_1} \cup P_{N_2}) = \lambda_\infty(P_{N_2}) + \lambda_\infty(P_{N_2})$.*

Definition 50. *If $G \subset \mathbb{R}_I^\infty$ is any open set, we define:*

$$\lambda_\infty(G) = \lim_{N \to \infty} \sup\left\{\lambda_\infty(P_N) :\ P_N \in \Delta,\ P_N \subset G, \right\}.$$

Theorem 51. *If $\mathfrak{O}$ is the class of open sets in $\mathfrak{B}(\mathbb{R}_I^\infty)$, we have:*

1. *$\lambda_\infty(\mathbb{R}_I^\infty) = \infty$.*
2. *If $G_1,\ G_2 \in \mathfrak{O}$, $G_1 \subset G_2$, then $\lambda_\infty(G_1) \le \lambda_\infty(G_2)$.*

3. If $\{G_k\} \subset \mathfrak{O}$, then

$$\lambda_\infty(\bigcup\nolimits_{k=1}^{\infty} G_k) \leq \sum\nolimits_{k=1}^{\infty} \lambda_\infty(G_k).$$

4. If the G_k are disjoint, then

$$\lambda_\infty(\bigcup\nolimits_{k=1}^{\infty} G_k) = \sum\nolimits_{k=1}^{\infty} \lambda_\infty(G_k).$$

PROOF. The proof of (1) is standard. To prove (2), observe that

$$\{P_N : \ P_N \subset G_1\} \subset \left\{\bar{P}_N : \bar{P}_N \subset G_2\right\},$$

so that $\lambda_\infty(G_1) \leq \lambda_\infty(G_2)$. To prove (3), let $P_N \subset \bigcup_{k=1}^{\infty} G_k$. Since P_N is compact, there is a finite number of the G_k which cover P_N, so that $P_N \subset \bigcup_{k=1}^{L} G_k$. Now, for each G_k, there is a $P_{N_k} \subset G_k$. Furthermore, as P_N is arbitrary, we can assume that $P_N = \bar{P}_N = \bigcup_{k=1}^{L} P_{N_k}$. Since there is a n such that all $P_{N_k} \in \mathfrak{B}(\mathbb{R}_I^n)$, we may also assume that $\lambda_\infty(P_{N_l} \cap P_{N_m}) = 0,\ l \neq m$. We now have that

$$\lambda_\infty(P_N) = \sum_{k=1}^{L} \lambda_\infty(P_{N_k}) \leqslant \sum_{k=1}^{L} \lambda_\infty(G_k) \leqslant \sum_{k=1}^{\infty} \lambda_\infty(G_k).$$

It follows that

$$\lambda_\infty(\bigcup\nolimits_{k=1}^{\infty} G_k) \leq \sum\nolimits_{k=1}^{\infty} \lambda_\infty(G_k).$$

If the G_k are disjoint, observe that if $P_N \subset \bar{P}_M$,

$$\lambda_\infty(\bar{P}_M) \geq \lambda_\infty(P_N) = \sum_{k=1}^{L} \lambda_\infty(P_{N_k}).$$

It follows that

$$\lambda_\infty(\bigcup\nolimits_{k=1}^{\infty} G_k) \geq \sum\nolimits_{k=1}^{L} \lambda_\infty(G_k).$$

This is true for all L so that this, combined with (3), gives our result. □

If F is an arbitrary compact set in $\mathfrak{B}(\mathbb{R}_I^\infty)$, which is not a subset of $\mathbb{R}_I^n$ for some n, we define

$$\lambda_\infty(F) = \inf\left\{\lambda_\infty(G) : \ F \subset G,\ G \text{ open}\right\}. \tag{6.2}$$

Theorem 52. *Equation (6.1) is consistent with Definition 47 and the results of Lemma 49.*

Definition 53. *Let A be an arbitrary set in $\mathbb{R}_I^\infty$.*

1. *The outer measure (on $\mathbb{R}_I^\infty$) is defined by:*
$$\lambda_\infty^*(A) = \inf\left\{\lambda_\infty(G) : \, A \subset G, \; G \text{ open}\right\}.$$
We let $\mathfrak{L}_0$ be the class of all A with $\lambda_\infty^*(A) < \infty$.
2. *If $A \in \mathfrak{L}_0$, we define the inner measure of A by*
$$\lambda_{\infty,(*)}(A) = \sup\left\{\lambda_\infty(F) : \, F \subset A, \; F \text{ compact}\right\}.$$
3. *We say that A is a bounded measurable set if $\lambda_\infty^*(A) = \lambda_{\infty,(*)}(A)$, and define the measure of A, $\lambda_\infty(A)$, by $\lambda_\infty(A) = \lambda_\infty^* A)$.*

Theorem 54. *Let A, B and $\{A_k\}$ be arbitrary sets in $\mathbb{R}_I^\infty$ with finite outer measure.*

1. $\lambda_{\infty,(*)}(A) \leq \lambda_\infty^*(A)$.
2. *If $A \subset B$ then $\lambda_\infty^*(A) \leq \lambda_\infty^*(B)$ and $\lambda_{\infty,(*)}(A) \leq \lambda_{\infty,(*)}(B)$.*
3. $\lambda_\infty^*(\bigcup_{k=1}^\infty A_k) \leq \sum_{k=1}^\infty \lambda_\infty^*(A_k)$.
4. *If they are disjoint, $\lambda_{\infty,(*)}(\bigcup_{k=1}^\infty A_k) \geq \sum_{k=1}^\infty \lambda_{\infty,(*)}(A_k)$.*

Proof. The proofs follow closely along the same lines as similar results in Jones [36] (see pages 42-52). The proofs of (1) and (2) are straightforward. To prove (3), let ε be given. Then, for each k, there exists an open set G_k such that $A_k \subset G_k$ and $\lambda_\infty(G_k) < \lambda_\infty^*(A_k) + \varepsilon 2^{-k}$. Since $(\bigcup_{k=1}^\infty A_k) \subset (\bigcup_{k=1}^\infty G_k)$, we have
$$\begin{aligned}\lambda_\infty^*\left(\bigcup\nolimits_{k=1}^\infty A_k\right) &\leqslant \lambda_\infty\left(\bigcup\nolimits_{k=1}^\infty G_k\right) \leqslant \sum\nolimits_{k=1}^\infty \lambda_\infty(G_k) \\ &< \sum\nolimits_{k=1}^\infty \left[\lambda_\infty^*(A_k) + \varepsilon 2^{-k}\right] = \sum\nolimits_{k=1}^\infty \lambda_\infty^*(A_k) + \varepsilon.\end{aligned}$$
Since ε is arbitrary, we are done.

To prove (4), let $F_1, F_2, \ldots, F_N$ be compact subsets of $A_1, A_2, \ldots, A_N$, respectively. Since the A_k are disjoint,
$$\lambda_{\infty,(*)}\left(\bigcup\nolimits_{k=1}^\infty A_k\right) \geqslant \lambda_\infty\left(\bigcup\nolimits_{k=1}^N F_k\right) = \sum\nolimits_{k=1}^N \lambda_\infty(F_k).$$
Thus,
$$\lambda_{\infty,(*)}\left(\bigcup\nolimits_{k=1}^\infty A_k\right) \geq \sum\nolimits_{k=1}^N \lambda_{\infty,(*)}(A_k).$$
Since N is arbitrary, we are done. □

The next two theorems are implicit in the last one.

Theorem 55. *(Regularity) If A is a bounded measurable set, then for every $\varepsilon > 0$ there exist a compact set F and an open set G such that $F \subset A \subset G$, with $\lambda_\infty(G \setminus F) < \varepsilon$.*

Theorem 56. *(Countable Additivity) If the family $\{A_k\}$ are disjoint bounded measurable sets and $A = \bigcup_{k=1}^{\infty} A_k$, then $\lambda_\infty(A) = \sum_{k=1}^{\infty} \lambda_\infty(A_k)$.*

Definition 57. *Let A be an arbitrary set in $\mathbb{R}_I^\infty$. We say that A is measurable if $A \cap M \in \mathfrak{L}_0$ for all $M \in \mathfrak{L}_0$. In this case, we define $\lambda_\infty(A)$ by:*

$$\lambda_\infty(A) = \sup\left\{\lambda_\infty(A \cap M) : \, M \subset \mathfrak{L}_0\right\}.$$

We let $\mathfrak{L}$ be the class of all measurable sets A.

Proofs for the following results are direct adaptations of those in Jones [36] (see pages 48-52).

Theorem 58. *A and $\{A_k\}$ be arbitrary sets in $\mathfrak{L}$.*

1. *If $\lambda_\infty^*(A) < \infty$, then $A \in \mathfrak{L}_0$ if and only if $A \in \mathfrak{L}$. In this case, $\lambda_\infty(A)$ is the same number.*
2. *$\mathfrak{L}$ is closed under countable unions, countable intersections, differences and complements.*
3. $$\lambda_\infty\Big(\bigcup\nolimits_{k=1}^{\infty} A_k\Big) \le \sum\nolimits_{k=1}^{\infty} \lambda_\infty(A_k).$$
4. *If they are disjoint,*
$$\lambda_\infty\Big(\bigcup\nolimits_{k=1}^{\infty} A_k\Big) = \sum\nolimits_{k=1}^{\infty} \lambda_\infty(A_k).$$
5. *If $A_k \subset A_{k+1}$ for all k, then*
$$\lambda_\infty\Big(\bigcup\nolimits_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} \lambda_\infty(A_k).$$
6. *If $A_{k+1} \subset A_k$ for all k, then*
$$\lambda_\infty\Big(\bigcap\nolimits_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} \lambda_\infty(A_k).$$

Thus, we see that $\lambda_\infty(\cdot)$ is a regular countably additive measure on the measurable sets of $\mathbb{R}_I^\infty$. More important is the fact that the development is no more difficult than the corresponding theory for Lebesgue measure on $\mathbb{R}^n$.

6.5 Separable Banach Spaces.

One may get the impression that $\mathbb{R}_I^\infty$ offers no other (essentially important) advantages than those obtained for Lebesgue measure. This is incorrect (in the extreme). Let $\mathcal{B}$ be any separable Banach space. Recall that (see Diestel [10] page 32):

Definition 59. *A sequence (e_n) is called a Schauder basis for $\mathcal{B}$ if, for $f \in \mathcal{B}$, there is a unique sequence (a_n) of scalars such that*

$$f = \lim_{n\to\infty} \sum\nolimits_{k=1}^{n} a_k e_k.$$

Definition 60. *The sequence (e_n) is called norm-decreasing for $f = \sum a_k e_k$ if $\left\| \sum\limits_{k=1}^{m} a_k e_k \right\|_{\mathcal{B}} \le \left\| \sum\limits_{k=1}^{m+n} a_k e_k \right\|_{\mathcal{B}}$ for $m, n \ge 1$.*

Let $\mathbf{S}$ be the set of all sequences (a_n) for which $\lim_{n\to\infty} \sum_{k=1}^{n} a_k e_k$ exists in $\mathcal{B}$. Define

$$|||(a_n)||| = \sup_n \left\| \sum\nolimits_{k=1}^{n} a_k e_k \right\|_{\mathcal{B}}.$$

Theorem 61. *The operator*

$$T : (\mathbf{S},\ |||\cdot|||) \to (\mathcal{B},\ \|\cdot\|_{\mathcal{B}})$$

is a norm-decreasing isomorphsim from $\mathbf{S}$ onto $\mathcal{B}$.

It is known that most of the natural separable Banach spaces, and all that have any use for applications in analysis, have a Schauder basis. In particular, it is easy to see from Theorem 60 and the definition of a Schauder basis that, for any sequence $(a_n) \in \mathbf{S}$ representing a function $f \in \mathcal{B}$, we have $\lim_{n\to\infty} a_n = 0$. It follows that every separable Banach space (with a Schauder basis) is isomorphic to a subspace of $\mathbb{R}_I^\infty$. Thus, if we define $\mathcal{B}_I = \mathcal{B} \cap \mathbb{R}_I^\infty$ and $\mathfrak{B}(\mathcal{B}_I) = \mathcal{B} \cap \mathfrak{B}(\mathbb{R}_I^\infty)$, we have:

Theorem 62. *The restriction of λ_∞ is a version of Lebesgue measure on $\mathcal{B}_I$.*

6.6 Gaussian measure.

As an equally important application, we can now return to take another look at infinite product Gaussian measure. The canonical Gaussian measure on $\mathbb{R}$ is defined by:

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{|x|^2}{2} \right\} d\lambda(x).$$

Recall that $\mu_\infty = \otimes_{k=1}^{\infty}\mu$ is countably additive on $\mathbb{R}^\infty$, but its measure of l_2 is zero. If we introduce a scaled version of Gaussian measure on $\mathbb{R}_I^\infty$, we can resolve this difficulty. Let $\left\{\sigma_k^2\right\}$ be a family of variances chosen so that $\prod_{k=2}^{\infty}\mu_k(I) > 0$ (it suffices for $\mu_k(I) = [1-\frac{1}{k^2}]$), where

$$\mu_k(I) = \frac{1}{\sqrt{2\pi\sigma_k^2}}\int_{-1/2}^{1/2}\exp\left\{-\frac{|x_k|^2}{2\sigma_k^2}\right\}d\lambda(x_k).$$

(This is not possible via the conventional approach.) Now, set $\mu_1 = \mu$ and define $\mu_{I,\infty} = \otimes_{k=1}^{\infty}\mu_k$. We call $\mu_{I,\infty}$ the scaled version of Gaussian measure. Since $\mathbb{R}^n \subset \mathcal{B}_I$ and $\mu_{I,\infty}(\mathbb{R}_I^n) = \prod_{k=n+1}^{\infty}\mu_k(I)$ is positive (for every n), we see that $\mu_{I,\infty}(\mathcal{B}_I) = b > 0$.

Definition 63. *We call $\frac{1}{b}\mu_{I,\infty} = \mu_{b,\infty}$ the scaled version of Gaussian measure for $\mathcal{B}_I$.*

Theorem 64. *The measure $\mu_{b,\infty}$ is a countably additive version of Gaussian measure on $\mathcal{B}_I$.*

In particular, observe that we obtain a countably additive (scaled) version of Gaussian measure for both l_2 and $\mathbb{C}_0[0,1]$ (the continuous functions $x(t)$ on $[0,1]$ with $x(0) = 0$). Since all finite dimensional distributions are Gaussian, we obtain an equivalent version of Wiener measure on $\mathbb{C}_0[0,1]$.

7 $L^2[\mathbb{R}_I^\infty]$.

Although we can formally define the space of Lebesgue integrable functions in the obvious way, a complete study requires the development of the infinite tensor product Banach space theory. This would take us too far from our original limited objective. (Such a project is in preparation and will be presented at a later date.) However, because of von Neumann's development of infinite tensor product Hilbert spaces [58], much of the L^2 theory is within reach with the tools at hand and provides some closure to our original motivating question.

For each fixed n, let $(\mathbb{R}^n,\ \mathfrak{B}(\mathbb{R}^n),\ \lambda_n)$ be the Lebesgue measure space and let $I_n = \mathop{\times}\limits_{i=n+1}^{\infty} I$, where $I = [-\frac{1}{2},\frac{1}{2}]$. Let $\mathbf{L}^2[\mathbb{R}^n,\ \mathfrak{B}(\mathbb{R}^n),\ \lambda_n] = \mathbf{L}^2[\mathbb{R}^n]$ be the set of complex-valued functions $f(x)$ on $\mathbb{R}^n$ such that $|f(x)|^2$ is integrable with respect to λ_n. Let $\mathbf{L}^2[\mathbb{R}_I^n]$ be the special tensor product space defined by

$$L^2[\mathbb{R}_I^n] = \otimes_{i=1}^{n}L^2[\mathbb{R}] \otimes \left(\otimes_{i=n+1}^{\infty}\mathcal{E}_I\right)$$

and let

$$\bar{\mathbf{L}}^2[\mathbb{R}_I^\infty] = \lim_{\mathrm{n}\to\infty}\mathbf{L}^2[\mathbb{R}_{\mathrm{I}}^{\mathrm{n}}] = \bigcup_{\mathrm{n}=1}^{\infty}\mathbf{L}^2[\mathbb{R}_{\mathrm{I}}^{\mathrm{n}}].$$

As in Section 6.4, we let $\mathbf{L}^2[\mathbb{R}_I^\infty]$ be the completion of $\bar{\mathbf{L}}^2[\mathbb{R}_I^\infty]$ in the natural Hilbert space norm. We can now relate this space to our question in the first section.

However, we first need a result from von Neumann [58]. Let $\mathbb{N}$ be the natural numbers, let $\{e_n,\ n \in \mathbf{N} = \mathbb{N} \cup \{0\}\}$ be a complete orthonormal basis for $\mathbf{L}^2(\mathbb{R})$, and let $E_0 = \otimes_{i \in I} \mathcal{E}_I$. We define $\mathbf{F}$ to be the set of all functions $f:\ I \to \mathbf{N}$ such that $f(i) = 0$ for all but a finite number of i. Let $F(f)$ be the image of $f \in \mathbf{F}$ (i.e., $F(f) = \{f(i),\ i \in I\}$), and set $E_{F(f)} = \otimes_{i \in I} e_{i,f(i)}$, where $f(i) = 0$ implies that $e_{i,0} = \mathcal{E}_I$ and $f(i) = n$ implies $e_{i,n} = e_n$.

Theorem 65. *The set $\{E_{F(f)},\ f \in \mathbf{F}\}$ is a complete orthonormal basis for $\mathbf{L}_\otimes^{\Delta_2}[E_0]^s$.*

From Theorem 41, we see that $\mathbf{L}_\otimes^{\Delta_2}[E_0]^s$ is isometrically isomorphic to $\mathbf{L}^2[\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty), \hat{\lambda}_\infty]$. On the other hand, every vector in the basis set for $\mathbf{L}_\otimes^{\Delta_2}[E_0]^s$ is in $\mathbf{L}^2[\mathbb{R}_I^\infty, \mathfrak{B}(\mathbb{R}_I^\infty), \lambda_\infty]$. Since both spaces are closed and complete, we have:

Theorem 66. *The spaces $\mathbf{L}^2[\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty), \hat{\lambda}_\infty]$ and $\mathbf{L}^2[\mathbb{R}_I^\infty, \mathfrak{B}(\mathbb{R}_I^\infty), \lambda_\infty]$ are isomorphic.*

We cannot conclude that $\mathfrak{B}(\mathbb{R}_I^\infty)$ and $\mathfrak{B}(\mathbb{R}^\infty)$ are equal, since they are generated from different classes of cylinder sets.

In closing, we briefly discuss the Fourier transform in relation to $\mathbf{L}^2[\mathbb{R}_I^\infty, \mathfrak{B}(\mathbb{R}_I^\infty), \lambda_\infty]$. Our main objective is to point out the advantage in treating it as a bounded linear operator. If

$$\mathfrak{F}_j(g)(x_j) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{-ix_j y_j\} g(y_j) dy_j,$$

we can define $\mathfrak{F}(f_n)(X)$ on $\mathbf{L}^2[\mathbb{R}_I^n]$ by

$$\mathfrak{F}(f_n)(X) = \otimes_{i=1}^n \mathfrak{F}_i \otimes \left(\otimes_{i=n+1}^\infty \mathfrak{F}_i\right) \left(f \otimes \mathcal{E}_{I_n}\right)(X).$$

Viewed as an operator, $\mathfrak{F}$ is a isometric isomorphism on $\mathbf{L}^2[\mathbb{R}_I^n]$.

Theorem 67. *The operator $\mathfrak{F}$ is an isometric isomorphism on $\mathbf{L}^2[\mathbb{R}_I^\infty]$.*

Proof. Let $f \in \mathbf{L}^2[\mathbb{R}_I^\infty]$. Since it is a Hilbert space, there exists a sequence of functions $\{f_k \in \mathbf{L}^2[\mathbb{R}_I^{n_k}],\ n_k \in \mathbb{N}\}$ such that $\lim_{k\to\infty} \|f - f_k\|_2 = 0$. Furthermore, since the sequence converges, it is a Cauchy sequence. Hence, given $\varepsilon > 0$, there exists a $N(\varepsilon)$ such that $m, n \geq N(\varepsilon)$ implies that $\|f_m - f_n\|_2 < \varepsilon$. Since $\mathfrak{F}$ is an isometry, $\|\mathfrak{F}(f_m) - \mathfrak{F}(f_n)\|_2 < \varepsilon$, so that the sequence $\mathfrak{F}(f_k)$

is also a Cauchy sequence in $\mathbf{L}^2[\mathbb{R}_I^\infty]$. Thus, there is a $\hat{f} \in \mathbf{L}^2[\mathbb{R}_I^\infty]$ with $\lim_{k\to\infty} \left\| \hat{f} - \mathfrak{F}(f_k) \right\|_2 = 0$, and we can define $\mathfrak{F}(f) = \hat{f}$. It is easy to see that $\hat{f}$ is unique. $\square$

This is quite interesting since the Fourier transform can only be defined on $\mathbb{R}^\infty$ for sequences in $\mathbb{R}_0^\infty$ (the set of sequences that are zero except for a finite number of terms). Furthermore, we induce a definition of the Fourier transform on $\mathbf{L}^2[\mathcal{B}_I]$ for every separable Banach space (with a basis).

Conclusion.

In this paper we have shown how to construct a natural class of separable Banach spaces $\mathbf{KS}^p$ which parallels the standard $\mathbf{L}^p$ spaces but contains them as dense compact embeddings. These spaces are of particular interest because they contain the Henstock-Kurzweil integrable functions and the HK-measure, which generalizes the Lebesgue measure. We have also constructed the corresponding spaces $\mathbf{KS}^{m,p}$ of Sobolev type.

We have used $\mathbf{KS}^2$ to construct the free-particle path integral in the manner originally intended by Feynman. We have suggested that $\mathbf{KS}^2$ has a claim as the natural representation space for the Feynman formulation of quantum theory in that it allows representations for both the Heisenberg and Schrödinger representations, a property not shared by $\mathbf{L}^2$.

Since any semigroup that has a kernel representation will generate a path integral on $\mathbf{KS}^2$, via our theory, we have also identified a general class of (time-independent) generators of a semigroups that have an associated kernel.

In the analytical theory of Markov processes, it is well-known that, in general, the semigroup $T(t)$ associated with the process is not strongly continuous on $\mathbf{C}_b[\mathbb{R}^n]$, the space of bounded continuous functions or $\mathbf{UBC}[\mathbb{R}^n]$, the bounded uniformly continuous functions. We have shown that the weak generator defined by the mixed locally convex topology on $\mathbf{C}_b[\mathbb{R}^n]$ is a strong generator on $\mathbf{KS}^p[\mathbb{R}^n]$ (i.e., $T(t)$ is strongly continuous on $\mathbf{KS}^p[\mathbb{R}^n]$ for $1 \leq p \leq \infty$).

We have also offered a physical explanation for the failure of both rotational and translational invariance for Lebesgue measure on $\mathbb{R}^\infty$. We have shown that this problem is not related to the mathematical desirability of countable additivity.

We have shown that what appears to be a minor change in the way we represent $\mathbb{R}^\infty$ makes it possible to define a version of both Lebesgue and Gaussian measure (countably additive) on every (classical) separable Banach space. In particular, we obtain a version equivalent to Wiener measure on $\mathbb{C}_0[0, 1]$.

Remark 68. *After preparing the final draft of this work, we discovered that, in a series of papers, A. M. Vershik (see [53], [54], [55] and references contained therein) has discussed an "infinite-dimensional analogue of Lebesgue measure" that is constructed in a different manner than we have done in the present paper. Roughly stated, he considers the weak limit as $n \to \infty$ of invariant measures on certain homogeneous spaces (hypersurfaces of high dimension) of the Cartan subgroup of the Lie groups* $\mathbf{SL}(n, \mathbb{R})$ *(i.e., the subgroups of diagonal matrices with unit determinant).*

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