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AN ONTOLOGY OF DIRECTIONAL REGULARITY IMPLYING JOINT REGULARITY

Abstract

It is an old idea to consider whether a function on $\mathbb{R}^N$ that is smooth in each variable separately is in fact jointly smooth. It turns out that some uniformity of estimates in each variable is necessary for such a result. More recently, there have been studies of functions that are smooth along integral curves of certain vector fields. Depending on the commutator properties of the vector fields, different types of results may be obtained.

Another recent idea is that if one has smoothness along *all* curves then the uniformity hypothesis may be dropped.

In the present paper we explore all these approaches to the problem in a variety of new norms. We present new, simpler proofs of some classical results. We also explore new theorems in the real analytic category.

0 Preliminaries.

If $0 < \alpha < 1$ and f is a function on $\mathbb{R}^N$ then we say that f belongs to the α-order Lipschitz space on $\mathbb{R}^N$ if it satisfies the condition

$$\sup_{\substack{x\in\mathbb{R}^N \\ 0\neq h\in\mathbb{R}^N}} \frac{|f(x+h)-f(x)|}{|h|^\alpha} \equiv \|f\|_{\Lambda_\alpha(\mathbb{R}^N)} < \infty\,.$$

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For $\alpha = 1$ we modify the condition (following Zygmund) to

$$\sup_{\substack{x\in\mathbb{R}^N \\ 0\neq h\in\mathbb{R}^N}} \frac{|f(x+h)+f(x-h)-2f(x)|}{|h|} \equiv \|f\|_{\Lambda_1(\mathbb{R}^N)} < \infty\,.$$

Inductively, for $\alpha > 1$, we say that $f \in \Lambda_\alpha(\mathbb{R}^N)$ if $f \in C^1$ and $\nabla f \in \Lambda_{\alpha-1}$.

Of course these definitions make sense just as well on any open subset $U \subseteq \mathbb{R}^N$, or more generally on any set $S \subseteq \mathbb{R}^N$. We need only require in the definition that $x, x+h, x-h$ lie in S. If f is defined on an open set $U \subseteq \mathbb{R}^N$, then we say that f is locally Lipschitz α on U, and write $f \in \Lambda_\alpha^{\rm loc}(\mathrm{U})$, if $f\big|_K$ is Lipschitz on K for each compact $K \subseteq U$.

In 1915 S. Bernstein proved the following theorem (see [1, pp. 96–104], [16], and the further discussion in [17, p. 386]):

Theorem 1. *Fix a real number $\alpha > 0$. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function that is Λ_α in each variable separately. That is, for each $j = 1, \dots, N$ and for each $\mathbf{X} = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_{N-1}, x_N)$, the function*

$$f_{\mathbf{X}} : t \longmapsto f(x_1, x_2, \dots, x_{j-1}, t, x_{j+1}, \dots, x_{N-1}, x_N)$$

is Λ_α. Further suppose that there is a constant $C > 0$ such that

$$\|f_{\mathbf{X}}\|_{\Lambda_\alpha} \leq C \tag{$*$}$$

for every $\mathbf{X}$, with C being independent of $\mathbf{X}$. Then $f \in \Lambda_\alpha(\mathbb{R}^N)$.

Certainly it is known that, if the condition $(*)$ is omitted, then the conclusion fails in general. A simple example is

$$f(x_1, x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2}\,.$$

This f is clearly C^∞ in each variable separately, but it fails to be even continuous at the origin. In general, a function that is C^∞ in each variable separately is at best in the first Baire class (see [15]). Variants of the fundamental Theorem 1 are explored in [8].

Let us say a few words about the proof of Theorem 1. Perhaps the most classical proof uses basic Fourier analysis. Recall that the Dirichlet kernel for Fourier series is

$$D_N(t) = \sum_{j=-N}^{N} e^{ijt} = \frac{\sin(N+1/2)t}{\sin(1/2)t}$$

and the Fejér kernel (for Cesaro summability of Fourier series) is

$$K_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) = \frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}t}{\sin\frac{1}{2}t}\right)^2 .$$

Finally, the de la Vallée Poussin kernel for a Fourier series is

$$V_N(t) = 2K_{2N+1}(t) - K_N(t) .$$

The book [6] is a good reference for this material. The chief virtue of the de la Vallée Poussin mean $M_N \equiv f * V_N$ of a function f on the circle $\mathbb{T}$ is that the Fourier coefficients $\widehat{M_N}(j)$ of M_N agree with the Fourier coefficients $\widehat{f}(j)$ of f when $|j| \le N$, yet the Fourier coefficients of M_N trail off to zero in a linear fashion (which is useful for summability). A basic result (which may be found in [6]) is this:

Lemma 2. *Let $\alpha > 0$. Let f be an integrable function on the circle group $\mathbb{T}$. If*

$$\sup_{\mathbb{T}} |f - M_N f| \le C \cdot N^{-\alpha} ,$$

then $f \in \Lambda_\alpha(\mathbb{T})$. The converse is true as well.

This lemma is often formulated in terms of the "trigonometric polynomial of best approximation" to f. It turns out that, for all practical purposes, the de la Vallée Poussin mean gives that best approximation.

To prove Theorem 1, let $f(x_1, \ldots, x_N)$ be a function of N variables that satisfies the hypotheses. Let $M_N^j(f)$ denote the de la Vallée Poussin mean of f in the j^{th} variable. Then one approximates f by $M_N^1(f)$ and then approximates $M_N^1(f)$ by $M_N^2(M_N^1(f))$ and so forth up to the N-variable approximation $M_N^N(M_N^{N-1}(\cdots(M_N^2(M_N^1(f))\cdots))$. The resulting approximation by a trigonometric polynomial of N variables turns out to be sufficient to prove a version of Lemma 2 in the N-variable setting. That is what we need.

It is also possible to prove this result using the calculus of finite differences. To wit, it is easy to see from Lagrange's form of the remainder term in Taylor's formula that, if $1 < \alpha < 2$ and $f \in \Lambda_\alpha(\mathbb{R})$ then, for $x, h \in \mathbb{R}$,

$$f(x+h) = f(x) + h \cdot f'(x) + \mathcal{O}(|h|^\alpha) . \tag{$*$}$$

Coupled with

$$f(x-h) = f(x) - h \cdot f'(x) + \mathcal{O}(|h|^\alpha) ,$$

one sees that

$$f(x+h) + f(x-h) - 2f(x) = \mathcal{O}(|h|^\alpha) .$$

Exploiting the expansion $(*)$ in both variables, and using some linear algebra, one can prove a version of Theorem 1 for functions of two real variables. The result for more variables is similar but more tedious—see [8] for the details.

If one reformulates Bernstein's theorem in the language of Sobolev spaces then a particularly elegant attack on the problem comes to light. Suppose that $H^s(\mathbb{R}^N)$ denotes the usual Sobolev space—see [13] or [5]. Now we have:

Theorem 3. *Let $k > 0$ be a positive integer. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function that satisfies the condition*

$$\|f\mathbf{x}\|_{H^k(\mathbb{R})} \leq C$$

for every choice of $\mathbf{X} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{N-1}, x_N)$. Here $C > 0$ is some fixed constant. Then $f \in H^k(\mathbb{R}^N)$.

PROOF. The proof is simplicity itself, and we include it for its didactic value.

For convenience, multiply f by a C^∞ cutoff function so that the resulting function (still called f) is supported in the unit cube. The hypothesis guarantees that

$$\left(\frac{\partial}{\partial x_j}\right)^\ell f(x) \in L^2\,,$$

with a uniform estimate in j, ℓ, and the variable x, for $j = 1, \ldots, N$ and $0 \leq \ell \leq k$. But then Plancherel's theorem tells us that

$$\xi_j^\ell \widehat{f}(\xi) \in L^2(\mathbb{R}^N)$$

for j and ℓ as above (we use here Fubini's theorem). But now elementary estimates tell us that

$$m(\xi) \cdot \widehat{f} \in L^2$$

for every monomial m of degree not exceeding k. That simply says that $f \in H^k$.

$\square$

It is worth noting that, in the complex analysis of several variables, matters are different. For suppose that $f(z_1, z_2, \ldots, z_n)$ is a function of several complex variables defined on an open set U. It is a theorem of F. Hartogs (see [12]) that, if f is holomorphic in each variable separately, then f is jointly holomorphic (in the sense, for instance, that it has a convergent, n-variable power series expansion about each point). Note that, in Hargogs's result, no uniform estimates are required in each variable. The joint holomorphicity is, in effect, automatic.

On the other hand, Jan Boman [2] has proved the following remarkable result:

Theorem 4. *Let f be a function on $\mathbb{R}^N$. Suppose that for every smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^N$ it holds that $f \circ \gamma \in C^\infty(\mathbb{R})$. Then $f \in C^\infty(\mathbb{R}^N)$.*

Proof. We give here a new proof of this result—considerably simpler than the one originally offered by Boman.

The proof is still by contradiction. So suppose, seeking a contradiction, that f is *not* C^∞. Then there is some k such that f is not C^k. That means that f will *not* satisfy the hypotheses of Theorem 1 for $\alpha = k + 1$. Focus now on a compact cube Q (the closure of an open cube) on which this failure holds. So there will be an index j and a sequence of points $\{p^\ell\} \subseteq Q$ so that $|(\partial^{k+1}/\partial x_j^{k+1})f(p^\ell)| \geq \ell$. Invoking compactness, we may suppose that the p^ℓ converge to a point $p^0 \in Q$. But now it is easy to interpolate a C^∞ curve γ through the p^ℓ in sequence so that $\gamma(t^\ell) = p^\ell$ and $\gamma'(t^\ell)$ is parallel to the unit vector in the j^{th} coordinate direction. We can also arrange that $|\gamma'(t^\ell)| = 1$. Then $f \circ \gamma$ will fail the hypothesis of the theorem. $\square$

Boman's theorem is particularly notable because it makes no hypothesis about uniformity of estimates in the different directions. Yet one still is able to conclude that the function f is genuinely C^∞ as a function of several variables. It is also remarkable that Boman's proof is by contradiction.

One of the purposes of the present paper is to explore variants of Boman's theorem when the function space C^∞ is replaced by some other space, particularly by the space of real analytic functions.

Before we explore variants of Boman's theorem, we take some time to formulate and discuss some invariant versions of Theorem 1 that are useful in analysis on manifolds and nilpotent Lie groups.

It is a pleasure to thank the editor for a number of useful suggestions.

1 Non-Commuting Vector Fields.

A basic theorem in the paper [9] is as follows.

Theorem 5. *Let $U \subseteq \mathbb{R}^N$ be a connected, open set (hereinafter called a domain). Let $X_1, \ldots, X_N$ be smooth vector fields on U. Assume that, at each point of U, the vectors $X_1(x), \ldots, X_N(x)$ form a basis of $\mathbb{R}^N$. Let $\varphi_j^x : t \mapsto \exp_x tX_j$ denote the integral curve of the vector field X_j emanating from the point $x \in U$ (so that $\varphi_j^x(0) = x$). Let $\alpha > 0$. Let $C > 0$ and assume that, for each j and each x, the function*

$$f \circ \varphi_j^x$$

is Λ_α with $\|f \circ \varphi_j^x\|_{\Lambda_\alpha} \leq C$. Then f is locally in Λ_α on U.

This result is like an "invariant" form of the fundamental Theorem 1. For it is not tied to the coordinate axes. It can be formulated in terms of flows that arise from the problem at hand. The proof of this theorem uses the finite differences approach that was outlined in the last section.

The next step in the development of these ideas was the result of [10]. In that paper we proved the following:

Theorem 6. *Let $U \subseteq \mathbb{R}^N$ be a domain. Let $X_1, \ldots, X_k$, $1 \leq k < N$ be smooth vector fields on U. Assume that $X_1, \ldots, X_k$ and the commutators of these vector fields up to order m span $\mathbb{R}^N$ at each point of U. Let $\varphi_j^x : t \mapsto \exp_x tX_j$ denote the integral curve of the vector field X_j emanating from the point $x \in U$ (so that $\varphi_j^x(0) = x$). Let $\alpha > 0$. Let $C > 0$ and assume that, for each j and each x, the function*

$$f \circ \varphi_j^x$$

is Λ_α with $\|f \circ \varphi_j^x\| \leq C$. Then f is locally in $\Lambda_{\alpha/m}$ on U.

What is remarkable about this last result is that one need only assume smoothness in a "small" set of directions—smaller than the number of dimensions. And then the contact structure automatically gives smoothness in the remaining directions.

In fact more can be said in the conclusion of Theorem 5. Let V be a vector field that is a p^{th} order commutator of $X_1, \ldots, X_k$, $2 \leq m$. Then, along integral curves of V, the function f is locally $\Lambda_{\alpha/p}$. The proof of this more refined result is just the same as that of the theorem as enunciated. It is a subtle finite-difference argument (see [11]).

2 Harmonic Functions.

Suppose that f is a given continuous function on the closure of a smoothly bounded domain Ω in $\mathbb{R}^N$. If the restriction of f to $\partial\Omega$ is known to be smooth, then what can be said about the smoothness of f on $\overline{\Omega}$? The answer, of course, is nothing. The simple example on the unit ball given by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \partial B \\ (1 - |x|)^{1/2} \sin(1/[1 - |x|]) & \text{if} \quad x \in B \end{cases}$$

exhibits a function that is real analytic on ∂B but is only $\Lambda_{1/2}$ on $\overline{B}$.

The only hope of relating boundary smoothness to interior smoothness is to have a partial differential equation that mediates between the two. A sample result is this:

Theorem 7. *Let f be a continuous function on the closure of a smoothly bounded domain $\Omega \subseteq \mathbb{R}^N$. Let $X_1, \ldots, X_{N-1}$ be smooth vector fields on $\partial\Omega$ which are linearly independent at each point of the boundary. Let $\alpha > 0$. Assume that f is Λ_α along the integral curves of each of the X_j, with a uniform bound $C > 0$ on the Lipschitz norms. Define u to be the solution of the Dirichlet problem for the Laplacian on Ω with boundary data f. Then $u \in \Lambda_\alpha(\overline{\Omega})$.*

In fact this theorem is true for the solution of the Dirichlet problem for any strongly partial differential elliptic operator of order 2. These ideas are developed in [10].

3 Holomorphic Functions.

The paper that taught us that something special is true for holomorphic functions is [19]. To formulate the fundamental result, we need a bit of terminology. Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded domain. If $P \in \partial\Omega$ then let $\nu = \nu_P$ be the outward unit normal vector at P. The one-dimensional complex linear space $\mathbb{C}\nu$ is called the *complex normal space* $\mathcal{N}_P$ at P. The Hermitian orthogonal complementary space $\mathcal{T}_P$ is the *complex tangent space* at P. Let U be a tubular neighborhood of $\partial\Omega$. If $z \in U$, then let $\pi(z)$ be the well-defined Euclidean orthogonal projection of z to $\partial\Omega$. Then we may define $\mathcal{N}_z \equiv \mathcal{N}_{\pi(z)}$ and $\mathcal{T}_z \equiv \mathcal{T}_{\pi(z)}$.

We call a curve $\gamma : (0,1) \to U \cap \Omega$ *complex tangential* if $\gamma'(t) \in \mathcal{T}_{\gamma(t)}$ for each t. The curve γ is *normalized complex tangential of order k* if $\|g^{(j)}\|_{\sup} \leq 1$ for derivatives of γ of order j, $1 \leq j \leq k$.

Proposition 8. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Suppose that f is holomorphic in Ω and that $f \in \Lambda_\alpha(\Omega)$. Let $\alpha > 0$. If γ is any normalized complex tangential curve of order $[\alpha]+1$ (where square brackets [ ] denote the greatest integer function) then $f \circ \gamma$ is Lipschitz of order 2α.*

This result (see [12, Ch. 8], for the proof) is remarkable for several reasons. First, it gives *free* additional smoothness in certain geometrically distinct directions. Second, it begs the question of "Why an improvement of order 2?" We say just a word about the proof. For simplicity, restrict α to $0 < \alpha < 1/2$. The key fact for this result is the following two estimates. Let $\nu = \nu_z$ represent a complex normal direction at $z \in \Omega$ and $\tau = \tau_z$ represent a complex tangential direction.

$$(\mathbf{1}) \qquad \left|\frac{\partial}{\partial\nu}f(z)\right| \leq C\cdot\delta_\Omega(z)^{\alpha-1}\,.$$

$$(\mathbf{2}) \qquad \left|\frac{\partial}{\partial\tau}f(z)\right| \leq C\cdot\delta_\Omega(z)^{\alpha-1/2}\,.$$

These estimates are established by a clever exploitation of the mean value property on complex analytic discs pointing in the different directions.

The last result was studied and developed in a series of papers. Rudin [18] pioneered the idea of hypothesizing smoothness just in the normal direction. Krantz [9] took that idea to its natural fruition. We state that result in a moment. But first a little notation.

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Let U be a tubular neighborhood of $\partial\Omega$. For each $P \in \partial\Omega$, let e_P be the inward-extending normal segment emanating from P and having length ϵ_0. Here ϵ_0 is chosen so that, for each P, this segment will lie in the tubular neighborhood.

Theorem 9. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Let $\alpha > 0$. Let f be a holomorphic function on Ω. Assume that, for each $P \in \partial\Omega$, the restriction of f to e_P is Λ_α, with the Lipschitz norm being uniformly bounded in P. Then $f \in \Lambda_\alpha(\Omega)$. Further, by Stein's theorem, f is $\Lambda_{2\alpha}$ in complex tangential directions.*

The other key insight, which we mention briefly now, is that one can take into account the Levi geometry of the domain Ω to sharpen the result. We now recall the notion of finite type. Restrict attention to complex dimension 2. Let Ω be a smoothly bounded domain in $\mathbb{C}^2$. We say that $P \in \partial\Omega$ is a *point of finite type m* if there is a nonsingular complex analytic disc that is tangent to $\partial\Omega$ at P to order m, but no such disc which is tangent to order $m+1$. A strongly pseudoconvex point is of finite type 2. The point $(1,0)$ in the boundary of $E_{2p} = \{(z_1,z_2) : |z_1|^2 + |z_2|^{2p} < 1\}$, p a positive integer, is of finite type $2p$. The idea of finite type was first developed in [7] to measure subellipticity of the $\overline{\partial}$-Neumann problem. It has developed into an important geometric tool in several complex variables. Now we have

Theorem 10. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Let $\alpha > 0$. Let f be a holomorphic function on Ω. Assume that, for each $P \in \partial\Omega$, the restriction of f to e_P is Λ_α, with the Lipschitz norm being uniformly bounded in P. Then $f \in \Lambda_\alpha(\Omega)$. Further, let $P \in \partial\Omega$ be a point of finite type m. Then, near P, f is $\Lambda_{m\alpha}$ in complex tangential directions.*

4 New Results in the Vein of Boman.

Our purpose in this section is to present some new results along the lines of Jan Boman's ideas in [2]. We wish to have theorems in the Lipschitz category, and also in the real analytic category.

Theorem 11. *Let f be a function on an open set $U \subseteq \mathbb{R}^N$. Suppose that $0 < \alpha < k \in \mathbb{N}$. Assume that, for every C^k curve $\gamma : (-1,1) \to U$, it holds that $f \circ \gamma \in \Lambda_\alpha^{\mathrm{loc}}$. Then $f \in \Lambda_\alpha^{\mathrm{loc}}(U)$.*

Proof. As in [2], our proof will proceed by contradiction. First suppose for simplicity that $0 < \alpha < 1$. Suppose that f satisfies the hypotheses, yet f is *not* locally Lipschitz α on U. Then there is a compact set $K \subseteq U$ and points $x_j, x_j + h_j \in K$ so that

$$\frac{|f(x_j + h_j) - f(x_j)|}{|h|^\alpha} > j\,. \tag{$**$}$$

Invoking compactness, we may assume that $x_j \to x_0 \in K$ and $x_j + h_j \to x_0 + h_0 \in K$. But now it is a simple matter to interpolate a smooth curve η, in sequence, through the points $x_1, x_1 + h_1, x_2, x_2 + h_2, \ldots$. According to our hypothesis, $f \circ \eta$ is Lipschitz smooth on compact sets. Yet that contradicts $(**)$. $\square$

Our next result is about real analytic functions. The following classical characterization of these objects (see [14]) will prove useful:

Proposition 12. *Let f be a function on a domain $U \subseteq \mathbb{R}^N$. Let V be a relatively compact open subset of U. Suppose that there are constants $C > 0$, $r > 0$ such that, for each $k = 0, 1, \ldots$ and each multi-index β with $|\beta| \le k$, it holds for $x \in V$ that*

$$\left|\frac{\partial^\beta f}{\partial x^\beta}(x)\right| \le C \cdot \frac{k!}{r^k}\,.$$

Then f is real analytic on V.

Now a classical result of F. Browder (see [3], [14]) says this:

Theorem 13. *Let f be a function on an open cube $\mathcal{C} \equiv (-a,a) \times (-a,a) \times \cdots \times (-a,a) \subseteq \mathbb{R}^N$. Assume that there is an $r > 0$ such that, for $j = 1, \ldots, N$ and $k = 0, 1, 2, \ldots$, we have*

$$\left|\frac{d^k f(x_1, x_2, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_N)}{dx^k}\right| \le C \cdot \frac{k!}{r^k}\,.$$

Then f is real analytic on $\mathcal{C}$.

Theorem 14. *Let f be a continuous function on an open set $U \subseteq \mathbb{R}^N$. Let V be a relatively compact, connected open subset of U. Assume that, for every real analytic curve $\gamma : (-1,1) \to V$, it holds that $f \circ \gamma$ is real analytic, and in particular that, on any relatively compact interval $W \subseteq (-1,1)$ and for any $r > 0$ sufficiently small,*

$$\left| \frac{d^k}{dt^k} \left[f \circ \gamma \right](t) \right| \le C \cdot \frac{k!}{r^k} \, .$$

[Here the choice of C and r will, in general, depend on γ.] Then f is real analytic on V.

PROOF. Seeking a contradiction, let us suppose that our f is *not* real analytic on V. Therefore the hypotheses of Browder's theorem will fail for f. Thus, for any $r > 0$, there will be a sequence of points $x^j \in V$ and indices $k_j \to +\infty$ and $m_j \in \{2, \ldots, N-1\}$ such that

$$\left| \frac{d^{k_j} f}{dx^{k_j}} (x_1^j, x_2^j, \ldots, x_{m_j - 1}^j, x, x_{m_j+1}^j, \ldots, x_N^j) \right| > j \cdot \frac{j!}{r^j} \, .$$

Passing to a subsequence, we may suppose that the set $\{x^j\}$ has at most a single limit point as $j \to \infty$. In particular, no x^j is itself an accumulation point of this sequence.

Although the proof may now be completed in the real variable category, it is in fact more expeditious to pass to the complex analytic category (so that we may invoke Mittag-Leffler's theorem—see [4]). What we need to do, in order to obtain the necessary contradiction, is to interpolate a real analytic curve $\gamma : (-1,1) \to V$ through the points $x^1, x^2, \cdots \in V$ in such a way that γ at the point x^j agrees with the curve

$$t \mapsto (x_1^1, x_2^j, \ldots, x_{m_j-1}^j, x_{m_j}^j + t, x_{m_j+1}^j, \ldots, x_{N-1}^j, x_N^j) \qquad (\star)$$

to order $j+1$. [We may, and do, restrict attention to γ on a relatively compact subset $W \subseteq (-1,1)$ so that we may apply Theorem 12. In particular, the γ that we construct may be supposed to satisfy the interpolation condition $(\star)$ at points of W.] Doing so, we will find then that $f \circ \gamma$ fails the condition in Proposition 12. And that is the required contradiction.

Now we turn to the construction of γ. Complexifying γ to a holomorphic function Γ on an open set $X \subseteq \mathbb{C}$ that contains W, we are asking for a holomorphic function on X with specification of a particular Taylor jet at each of the points x^j. Of course Mittag-Leffler's theorem guarantees that this can be done. So the holomorphic function Γ exists. The restriction of Γ to the set $W \subseteq \mathbb{R}$ gives us the real analytic curve γ that we seek. And $f \circ \gamma$ gives the desired contradiction. The proof of the theorem is complete. $\square$

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