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PSEUDOCONTINUITY IS NECESSARY AND SUFFICIENT FOR ORDER-PRESERVING CONTINUOUS REPRESENTATIONS

Abstract

Using some properties of pseudocontinuous functions - a recent generalization of continuous functions - and without invoking Debreu's Open Gap Theorem, we solve the following problem: given a pseudocontinuous function v, find a continuous function u such that $u(x) > u(y)$ if and only if $v(x) > v(y)$. We show that this problem can be solved only for pseudocontinuous functions. Finally, we obtain a new proof on the existence of continuous numerical representations for continuous, transitive and total binary relations.

1 Introduction.

Let X be a topological space and let v be a real function defined on X. We are interested in the following problem $\mathcal{P}$: *find a continuous real function u order-preserving with respect to v, that is: $u(x) > u(x')$ if and only if $v(x) > v(x')$.* This problem plays a role in finding continuous numerical representations of binary relations. Indeed, let $\mathcal{R}$ be a total and transitive binary relation defined on X and let $u : X \longrightarrow \mathbb{R}$. The function u is said to be a numerical representation (utility function in Economics) of $\mathcal{R}$ if $u(x) \geq u(x') \Longleftrightarrow (x, x') \in \mathcal{R}$ - see [5, 3, 1, 6]. If $\mathcal{R}$ is continuous - that is: $\{x' \in X : (x, x') \in \mathcal{R}\}$ and

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$\{x' \in X : (x', x) \in \mathcal{R}\}$ are closed sets for any $x \in X$ - and if X is either connected and separable or second countable, the existence of continuous numerical representations have been obtained in the early papers [5, 3]. More precisely, X is connected and separable in [5] and second countable in [3]. The idea in [3] is the following: I) to obtain a numerical representation v of $\mathcal{R}$; II) to get a continuous function u which is order-preserving with respect to v - that is problem $\mathcal{P}$. Problem I) has been solved in [9] for X second countable and, inspired by [9], in [3] for X connected and separable. Problem II) has required more efforts and it has been solved in [3] with the Open Gap Theorem. Let us emphasize that only the binary relations which are total and transitive can possibly have numerical representations. Such binary relations, also called *rational preferences*, play a significant role in Economics, see, for example, [1, 6]. Concerning the existence of numerical representations, we point out that not all rational preferences have utility functions. For example, the lexicographic ordering is a non-continuous rational preference on $\mathbb{R}^n$ without numerical representations: see [6, 2]. Moreover, we note that the only continuity of a rational preference does not guarantee the existence of numerical representations, as remarked in [10], where it is shown that for every non-separable metric space, there exists a continuous rational preference which cannot be represented by utility functions.

The aim of this paper is to propose, without invoking the Open Gap Theorem, a new solution of problem $\mathcal{P}$ in the case in which X is connected and v is a *pseudocontinuous* function. Pseudocontinuity is a generalization of continuity introduced in [8]. This leads to a new proof on the existence of continuous utility functions for continuous rational preferences. More precisely, in light of [8, Proposition 2.2], the numerical representations of such binary relations are pseudocontinuous functions. Thus, a utility function v solving problem I) is pseudocontinuous and one gets a continuous numerical representation of the rational preference by solving problem $\mathcal{P}$. Finally, we show that $\mathcal{P}$ can be solved only for pseudocontinuous functions.

2 Preliminaries.

In this section, we recall the class of pseudocontinuous functions and we fix new properties. In the following, X denotes a topological space.

For a real function f defined on X, given a net $(x_\alpha)_{\alpha \in A} \subseteq X$, we set (see, for example, [2]):

$$\limsup f(x_\alpha) = \inf_{\alpha_o \in A} \sup_{\alpha \geq \alpha_o} f(x_\alpha) \quad \text{and} \quad \liminf f(x_\alpha) = \sup_{\alpha_o \in A} \inf_{\alpha \geq \alpha_o} f(x_\alpha) .$$

Now, we introduce the pseudocontinuous functions in an equivalent fashion with respect to [8, Definition 2.1], as one can easily verify: the function f is *upper pseudocontinuous* at $x \in X$ if for any $x' \in X$ such that $f(x) < f(x')$ and for any net $(x_\alpha)_{\alpha \in A}$ converging to x, we have $\limsup f(x_\alpha) < f(x')$; f is *lower pseudocontinuous* at x if $-f$ is upper pseudocontinuous at x; f is *pseudocontinuous* at x if it is both upper and lower pseudocontinuous.
The class of pseudocontinuous functions strictly include the class of continuous functions: in [8, Example 4.1] are presented pseudocontinuous functions which are neither upper semicontinuous nor lower semicontinuous.

The following properties will be used in the next section. We omit the proof of the first proposition since it is similar to the proof of [7, Proposition 2.3].

Proposition 2.1. *A function $f : X \longrightarrow \mathbb{R}$ is pseudocontinuous if and only if it satisfies the following property:*

$$f(x) < f(x') \text{ and } x_\alpha \longrightarrow x \text{ and } x'_\alpha \longrightarrow x' \implies \limsup f(x_\alpha) < \liminf f(x'_\alpha).$$

Proposition 2.2. *If X is connected and $f : X \longrightarrow \mathbb{R}$ is pseudocontinuous, then:*

$$f(x) < f(x') \Longrightarrow]f(x), f(x')[\cap f(X) \neq \emptyset \tag{1}$$

PROOF. In light of [8, Proposition 2.1], the sets $\{z \in X : f(z) \geq \lambda\}$ and $\{z \in X : f(z) \leq \lambda\}$ are closed for every $\lambda \in f(X)$. Assume that (1) does not hold: so, there exist two elements x and x' such that $f(x) < f(x')$ and $]f(x), f(x')[\cap f(X) = \emptyset$. Set $A_1 = \{z \in X : f(z) \leq f(x)\}$ and $A_2 = \{z \in X : f(z) \geq f(x')\}$, we have that A_1 and A_2 are non empty, closed and such that $A_1 \cap A_2 = \emptyset$ and $X = A_1 \cup A_2$. Since X is connected, we get a contradiction. $\square$

Proposition 2.3. *Let $f : X \longrightarrow \mathbb{R}$ be pseudocontinuous and let D be the set of points of discontinuity of f. If f is one-to-one over D, then D is at most countable.*

PROOF. Let $x \in D$. Since the function f is not continuous at x, at least one of the followings holds:

$$\text{there exists a net} \quad x_\alpha \longrightarrow x \quad \text{such that} \quad \liminf f(x_\alpha) < f(x) \tag{2}$$

$$\text{there exists a net} \quad x_\alpha \longrightarrow x \quad \text{such that} \quad f(x) < \limsup f(x_\alpha) \tag{3}$$

If (2) occurs, let $q(x)$ be a rational number belonging to $]\liminf_\alpha f(x_\alpha), f(x)[$. If (2) does not occur, let $q(x)$ be a rational number in $]f(x), \limsup_\alpha f(x_\alpha)[$.

So, we obtain a function $q : x \in D \longrightarrow q(x) \in \mathbb{Q}$. Let x' and x'' be two elements of D and suppose that $f(x') < f(x'')$. If $q(x') \in]\liminf_\alpha f(x'_\alpha), f(x')[$, the function f being lower pseudocontinuous at x'', one has:

$$\begin{array}{ll} q(x') < f(x') < \liminf f(x''_\alpha) < q(x'') < f(x'') & \text{if (2) holds for } x''_\alpha \longrightarrow x'' \\ q(x') < f(x') < f(x'') < q(x'') < \limsup f(x''_\alpha) & \text{if (3) holds for } x''_\alpha \longrightarrow x'' \end{array}$$

If $q(x') \in]f(x'), \limsup_\alpha f(x'_\alpha)[$, in light of Proposition 2.1 and the function f being upper pseudocontinuous at x', one has:

$$\begin{array}{ll} q(x') < \limsup f(x'_\alpha) < \liminf f(x''_\alpha) < q(x'') & \text{if (2) holds for } x''_\alpha \longrightarrow x'' \\ q(x') < \limsup f(x'_\alpha) < f(x'') < q(x'') & \text{if (3) holds for } x''_\alpha \longrightarrow x''_2 \end{array}$$

So, q is a one-to-one function with values in the countable set $\mathbb{Q}$. Therefore, the set D is at most countable. $\square$

We remark that, in order to have a set of discontinuities at most countable, pseudocontinuity is not a superfluous condition. In fact, let f be the function defined below:

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ -x & \text{if} \quad x \in \mathbb{Q} \end{cases}$$

The function f is one-to-one but not pseudocontinuous and the set of discontinuities is uncountable.

Finally, we note that some of the above results could be also deduced by using the continuity of a binary relation. For example, Proposition 2.2 could be easily obtained in the following equivalent way: one defines the total binary relation $\mathcal{R}$ such that $[(x, x') \in \mathcal{R}$ and $(x', x) \notin \mathcal{R}]$ if and only if $u(x) > u(x')$ and the result follows being $\mathcal{R}$ continuous from [8, Proposition 2.2] and X being connected. However, following the spirit of the paper, we prove the properties of pseudocontinuous functions through real function arguments.

3 Continuous Order-Preserving Functions for Pseudocontinuous Functions.

Let X be a connected topological space. In this section we solve problem $\mathcal{P}$: *given a pseudocontinuous real function v defined on X, find a continuous real function u which is order-preserving with respect to v, that is $u(x) > u(x')$ if and only if $v(x) > v(x')$.*

First of all, let us remark that pseudocontinuity is a necessary condition for the existence of continuous order-preserving functions. Indeed, assume that a continuous function u is order-preserving with respect to a function v. If we define the total binary relation $\mathcal{R}$ as follows: $[(x, x') \in \mathcal{R}$ and $(x', x) \notin \mathcal{R}]$ if and only if $u(x) > u(x')$, we have that $\mathcal{R}$ is continuous and the function v is a numerical representation of $\mathcal{R}$. So, in light of [8, Proposition 2.2], v is pseudocontinuous.

We give a new solution of problem $\mathcal{P}$ in the following Theorems 3.1 and 3.2. First, we need the following results.

Proposition 3.1. *Let $v : X \longrightarrow \mathbb{R}$ be pseudocontinuous and for each $x \in X$ let $Z_x = \{z \in X : v(z) > v(x)\}$. Now define the function $f_v : X \longrightarrow \mathbb{R}$ by $f_v(x) = \inf_{z \in Z_x} v(z)$ if $Z_x \neq \emptyset$ and $f_v(x) = v(x)$ if $Z_x = \emptyset$. Then, f_v is upper semicontinuous and lower pseudocontinuous.*

PROOF. Let $x \in X$. First, we prove that f_v is upper semicontinuous at x. The case $Z_x = \emptyset$ is obvious. Let $Z_x \neq \emptyset$ and assume that f_v is not upper semicontinuous at x. So, there exists a net $(x_\alpha)_{\alpha \in A}$ converging to x and such that $f_v(x) < f_v(x) + \varepsilon \leq f_v(x_\alpha)$ for any $\alpha \in A$, where ε is a suitable positive real number. Moreover there exists $z' \in Z_x$ such that $v(z') < f_v(x) + \varepsilon \leq f_v(x_\alpha)$. For any $\alpha \in A$, by definition of f_v, two are the possible cases: (i) $v(x_\alpha) < f_v(x_\alpha)$, which implies that $]v(x_\alpha), f_v(x_\alpha)[\cap v(X) = \emptyset$, so $v(z') \leq v(x_\alpha)$; (ii) $v(x_\alpha) = f_v(x_\alpha)$, which leads to $v(z') < v(x_\alpha)$. Hence, in both cases we have $v(z') \leq v(x_\alpha)$ for any $\alpha \in A$. Since the function v is upper pseudocontinuous at x, we get a contradiction. This proves that the function f_v is upper semicontinuous at x[1].
Finally, we prove that f_v is lower pseudocontinuous at x. Let $y \in X$ such that $f_v(y) < f_v(x)$ and let $(x_\alpha)_{\alpha \in A}$ be a net converging to x. So, there exists $y' \in Z_y$ such that $v(y) < v(y') < f_v(x)$. If $v(x) = f_v(x)$, one has $f_v(y) \leq v(y') < \liminf v(x_\alpha) \leq \liminf f_v(x_\alpha)$ and f_v is lower pseudocontinuous at x. On the other hand, if $v(x) < f_v(x)$, we have $]v(x), f_v(x)[\cap v(X) = \emptyset$ and $v(y') \leq v(x)$. Now, if $v(y') < v(x)$, one gets $f_v(y) \leq v(y') < \liminf v(x_\alpha) \leq \liminf f_v(x_\alpha)$, and f_v is lower pseudocontinuous at x. Finally, let $v(y') = v(x)$. If $f_v(y) = v(y') = v(x)$, we have $]v(y), v(x)[\cap v(X) = \emptyset$ and, in light of Proposition 2.2, we get a contradiction. So, we have $f_v(y) < v(y')$ and there exists $y'' \in X$ such that $v(y) < v(y'') < v(y') = v(x)$. Proceeding as above, one obtains that f_v is lower pseudocontinuous at x. ☐

Proposition 3.2. *If $v : X \longrightarrow \mathbb{R}$ is pseudocontinuous, then the function f_v defined in Proposition 3.1 is order-preserving with respect to v.*

[1] Let us remark that the function f_v is upper semicontinuous even in non-connected spaces.

PROOF. First, let $f_v(x) > f_v(y)$. So, there exists y' such that $v(y) < v(y') < f_v(x)$. If $v(x) \leq v(y)$ one has $f_v(x) \leq v(y') < f_v(x)$, that is a contradiction. So, $v(x) > v(y)$. Finally, let $v(x) > v(y)$. In light of Proposition 2.2, there exists y' such that $v(y) < v(y') < v(x)$ and so $f_v(x) > f_v(y)$. $\square$

Consider the function f_v defined in Proposition 3.1 and let $x \in X$. We set $\delta(x) = f_v(x) - \sup\{f_v(z) : f_v(x) > f_v(z)\}$. Let D be the set of points of discontinuity of the function f_v (it is easy to check that D contains the points in which the function v is discontinuous). For any $x \in D$, we consider the following functions $\underline{g}_v(x, \cdot)$ and $\overline{g}_v(x, \cdot)$ defined on X:

$$\underline{g}_v(x,z) = \begin{cases} f_v(z) & \text{if} \quad f_v(z) \geq f_v(x) \\ f_v(z) + \delta(x) & \text{if} \quad f_v(x) > f_v(z) \end{cases}$$

and

$$\overline{g}_v(x,z) = \begin{cases} f_v(z) - \delta(x) & \text{if} \quad f_v(z) \geq f_v(x) \\ f_v(z) & \text{if} \quad f_v(x) > f_v(z) \end{cases}$$

Proposition 3.3. *Let $v : X \longrightarrow \mathbb{R}$ be pseudocontinuous. The functions $\underline{g}_v(x, \cdot)$ and $\overline{g}_v(x, \cdot)$ are order-preserving with respect to v and continuous at x and on $X \backslash D$.*

PROOF. First, let us prove that $\underline{g}_v(x, \cdot)$ is order-preserving with respect to f_v: it is sufficient to observe that $\underline{g}_v(x, x') < f_v(x)$ for any x' such that $f_v(x') < f_v(x)$. In fact, if $\underline{g}_v(x, x') = f_v(x)$ for some x' with $f_v(x') < f_v(x)$, we have $f_v(x') = \sup\{f_v(z) : f_v(x) > f_v(z)\}$. Now, Proposition 2.2 implies that there exists x'' such that $f_v(x') < f_v(x'') < f_v(x)$, and we get a contradiction. On the other hand, there are no x' such that $f_v(x') < f_v(x)$ and $\underline{g}_v(x, x') > f_v(x)$. Similarly for the function $\overline{g}_v(x, \cdot)$.
Finally, we show that $\underline{g}_v(x, \cdot)$ is continuous at x. If $\underline{g}_v(x, \cdot)$ is not upper semicontinuous at x, then there exists a net $(x_\alpha)_{\alpha \in A}$ converging to x such that $f_v(x) = \underline{g}_v(x, x) < \limsup \underline{g}_v(x, x_\alpha)$. Consequently, there exists a subnet $(x_{\alpha'})_{\alpha' \in A'}$ converging to x such that $f_v(x_{\alpha'}) \geq f_v(x)$ for any $\alpha' \in A'$ and $f_v(x) < \limsup \underline{g}_v(x, x_{\alpha'}) = \limsup f_v(x_{\alpha'})$. The function f_v being upper semicontinuous at x, we get a contradiction. If $\underline{g}_v(x, \cdot)$ is not lower semicontinuous at x, we have $\liminf \underline{g}_v(x, x_\alpha) < \underline{g}_v(x, x) = f_v(x)$ for at least a net $(x_\alpha)_{\alpha \in A}$ converging to x and such that $f_v(x_\alpha) < f_v(x)$. Since f_v is lower pseudocontinuous at x, we have $\sup\{f_v(z) : f_v(x) > f_v(z)\} = \liminf f_v(x_\alpha)$. So: $f_v(x) > \liminf \underline{g}_v(x, x_\alpha) = \liminf f_v(x_\alpha) + \delta(x) = f_v(x)$, which is a contradiction.
On the other hand, it is obvious that $\underline{g}_v(x, \cdot)$ is continuous on $X \backslash D$ and by

using the same arguments, one can obtain that $\overline{g}_v(x,\cdot)$ is continuous at x and on $X\backslash D$. □

Finally, the solution of problem $\mathcal{P}$ follows from the next results: the first theorem concerns one-to-one pseudocontinuous functions and the second one deals with the general case.

Theorem 3.1. *Let v be a pseudocontinuous and one-to-one function defined on X. Then, there exists a function u continuous on X and order-preserving with respect to v.*

PROOF. First, we note that, in light of Propositions 3.1 and 3.2, the function f_v is pseudocontinuous and one-to-one over X. Therefore, in light of Proposition 2.3, the set D of discontinuity points of f_v is at most countable. So, the following are the possible cases:

i) D is countable and there is neither a point $x \in D$ such that $f_v(x) \geq f_v(x')$ for all $x' \in D$ nor a point $x \in D$ such that $f_v(x') \geq f_v(x)$ for all $x' \in D$;

ii) D is countable and there exists a point $x \in D$ such that either $f_v(x) \geq f_v(x')$ for all $x' \in D$ or $f_v(x') \geq f_v(x)$ for all $x' \in D$;

iii) D is finite.

Case i): assume that $D = \{\dots, x_{-n}, \dots, x_0, \dots, x_n, \dots\}$ and $... < f_v(x_{-n}) < ... < f_v(x_0) < ... < f_v(x_n) < ...$. For any $k \in \mathbb{Z}$, let $\delta_k = f_v(x_k) - \sup\{f_v(z) : f_v(x_k) > f_v(z)\}$. In this case, problem $\mathcal{P}$ is solved by the function u defined below:

$$u(x) = \begin{cases} f_v(x) + \sum_{i=0}^{n} \delta_{-i} & \text{if} \quad f_v(x_{-n}) > f_v(x) \geq f_v(x_{-n-1}) \\ f_v(x) & \text{if} \quad f_v(x_1) > f_v(x) \geq f_v(x_0) \\ f_v(x) - \sum_{i=1}^{n} \delta_i & \text{if} \quad f_v(x_{n+1}) > f_v(x) \geq f_v(x_n) \end{cases}$$

In fact, let us prove that u is continuous. Assume that $x_{-n} \in D$ with, for example, $n \neq 0$ and let $(x_\alpha)_{\alpha \in A}$ be a net converging to x_{-n}. Since f_v is upper semicontinuous and lower pseudocontinuous at x_{-n}, we have $f_v(x_{-n}) \geq \limsup f_v(x_\alpha) \geq \liminf f_v(x_\alpha) > f_v(x_{-n-1})$. If $f_v(x_{-n}) > \limsup f_v(x_\alpha)$, we get $f_v(x_{-n}) > f_v(x_\alpha) > f_v(x_{-n-1})$ for $\alpha \geq \alpha_o$, where α_o is a suitable index. Hence, for any $\alpha \geq \alpha_o$

$$u(x_\alpha) = f_v(x_\alpha) + \sum_{i=0}^{n} \delta_{-i} = \underline{g}_v(x_{-n}, x_\alpha) + \sum_{i=0}^{n-1} \delta_{-i} \ .$$

In light of Proposition 3.3, the function $\underline{g}_v(x_{-n}, \cdot)$ is continuous at x_{-n} and we have

$$\lim u(x_\alpha) = \lim \underline{g}_v(x_{-n}, x_\alpha) + \sum_{i=0}^{n-1} \delta_{-i} = \underline{g}_v(x_{-n}, x_{-n}) + \sum_{i=0}^{n-1} \delta_{-i} = u(x_{-n}) \ .$$

If the net $(x_\alpha)_{\alpha \in A}$ is such that $f_v(x_{-n}) = \limsup f_v(x_\alpha)$, we can restrict to the case of a subnet $(x_{\alpha'})_{\alpha' \in A'}$ such that $f_v(x_{\alpha'}) \geq f_v(x_{-n})$ for any $\alpha' \in A'$. So, we have $f_v(x_{-n+1}) > f_v(x_{\alpha'}) \geq f_v(x_{-n})$ for any $\alpha' \geq \alpha'_o$ and

$$u(x_{\alpha'}) = f_v(x_{\alpha'}) - \delta_{-n} + \delta_{-n} + \sum_{i=0}^{n-1} \delta_{-i} = \overline{g}_v(x_{-n}, x_{\alpha'}) + \sum_{i=0}^{n} \delta_{-i} \ .$$

The function $\overline{g}_v(x_{-n}, \cdot)$ being continuous at x_{-n} - see Proposition 3.3 -, we obtain:

$$\lim u(x_{\alpha'}) = \lim \overline{g}_v(x_{-n}, x_{\alpha'}) + \sum_{i=0}^{n} \delta_{-i} = \overline{g}_v(x_{-n}, x_{-n}) + \sum_{i=0}^{n} \delta_{-i} = u(x_{-n}) \ .$$

Obviously, the function u is continuous on $X \backslash D$. Finally, Proposition 3.2 guarantees that u is order preserving with respect to f_v, which is order preserving with respect to v.

Case ii): assume that $D = \{\ldots, x_{-n}, \ldots, x_0\}$ and $\ldots < f_v(x_{-n}) < \ldots < f_v(x_0)$. As in the previous case, for any $n \in \mathbb{N}_0$, let $\delta_{-n} = f_v(x_{-n}) - \sup\{f_v(z) : f_v(x_{-n}) > f_v(z)\}$. Using the same arguments of the previous case, it is easy to prove that problem $\mathcal{P}$ is solved by the following function:

$$u(x) = \begin{cases} f_v(x) & \text{if} \quad f_v(x) \geq f_v(x_0) \\ f_v(x) + \sum_{i=0}^{n} \delta_{-i} & \text{if} \quad f_v(x_{-n}) > f_v(x) \geq f_v(x_{-n-1}) \end{cases}$$

If $D = \{x_0, \ldots, x_n, \ldots\}$ and $f_v(x_0) < \ldots < f_v(x_n) < \ldots$, let $\delta_n = f_v(x_n) - \sup\{f_v(z) : f_v(x_n) > f_v(z)\}$ for any $n \in \mathbb{N}_0$. As above, one can prove that problem $\mathcal{P}$ is solved by the following function:

$$u(x) = \begin{cases} f_v(x) & \text{if} \quad f_v(x_0) > f_v(x) \\ f_v(x) - \sum_{i=0}^{n} \delta_i & \text{if} \quad f_v(x_{n+1}) > f_v(x) \geq f_v(x_n) \end{cases}$$

Finally, case iii) becomes trivial after cases i) and ii). $\square$

Theorem 3.2. *Let v be a pseudocontinuous function defined on X. Then, there exists a function u continuous on X and order-preserving with respect to v.*

PROOF. If v is one-to-one, the thesis follows from Theorem 3.1. Otherwise, we consider the quotient topological space $\widetilde{X} = X/\sim$, where $\sim$ is the equivalence relation on X such that $x \sim y \Longleftrightarrow v(x) = v(y)$. We denote by $\widetilde{x}$ the equivalent class to which x belongs. Now, let $\widetilde{v}$ be the one-to-one function defined on $\widetilde{X}$ by $\widetilde{v}(\widetilde{x}) = v(x)$.
We claim that $\widetilde{v}$ is pseudocontinuous in the quotient topology. In fact, in light of [8, Proposition 2.1], we have that the sets:

$$\{z \in X : v(z) < v(x)\} \quad \text{and} \quad \{z \in X : v(z) > v(x)\}$$

are open for any $x \in X$. On the other hand, set

$$\sigma(\widetilde{x}) = \{\widetilde{z} \in \widetilde{X} : \widetilde{v}(\widetilde{z}) < \widetilde{v}(\widetilde{x})\} \quad \text{and} \quad \Sigma(\widetilde{x}) = \{\widetilde{z} \in \widetilde{X} : \widetilde{v}(\widetilde{z}) > \widetilde{v}(\widetilde{x})\} ,$$

we obtain:

$$\bigcup_{\widetilde{z} \in \sigma(\widetilde{x})} \widetilde{z} = \{z \in X : v(z) < v(x)\} \quad \text{and} \quad \bigcup_{\widetilde{z} \in \Sigma(\widetilde{x})} \widetilde{z} = \{z \in X : v(z) > v(x)\} ,$$

which implies that $\sigma(\widetilde{x})$ and $\Sigma(\widetilde{x})$ are open sets in the quotient topology for any $\widetilde{x} \in \widetilde{X}$. Hence, in light of [8, Proposition 2.1], $\widetilde{v}$ is a pseudocontinuous function.
Now, since the topological space $\widetilde{X}$ is connected (see, for example, [4, Corollary 5.9]), Theorem 3.1 ensures that there exists a function $\widetilde{u}$ continuous on $\widetilde{X}$ and order-preserving with respect to $\widetilde{v}$. Set $u(x) = \widetilde{u}(\widetilde{x})$ for any $x \in X$, we obtain that u is continuous and order-preserving with respect to v. □

We conclude observing that Theorem 3.2 allows to obtain a new proof of Eilenberg's Theorem [5]: *Any continuous rational preference $\mathcal{R}$ defined on a connected and separable topological space admits continuous utility functions.* In fact, following [3], the function v:

$$v(x) = \sum_{n \in \mathcal{N}(x)} \frac{1}{2^n} \quad \text{if } \mathcal{N}(x) \neq \emptyset \qquad \text{and} \qquad v(x) = 0 \quad \text{if } \mathcal{N}(x) = \emptyset ,$$

where $\mathcal{N}(x) = \{n \in \mathbb{N} : (x, z_n) \in \mathcal{R} \text{ and } (z_n, x) \notin \mathcal{R}\}$ and $\overline{\{z_n : n \in \mathbb{N}\}} = X$, is a numerical representation of $\mathcal{R}$, which is pseudocontinuous in light of [8, Proposition 2.2]. So, Eilenberg's Theorem follows from Theorem 3.2.

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