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FOURIER COEFFICIENTS AND GENERALIZED LIPSCHITZ CLASSES IN UNIFORM METRIC

Abstract

In this paper we give some equivalence relations between behavior of Fourier coefficients of a special kind and smoothness of functions. A necessary and sufficient condition for existence of Schwartz derivative is also obtained.

1 Introduction.

Let $\{c_k\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers such that

$$\sum_{k\in\mathbb{Z}} |c_k| < \infty. \tag{1.1}$$

Then

$$f(x) := \sum_{k\in\mathbb{Z}} c_k e^{ikx} \tag{1.2}$$

is a continuous 2π-periodic function ($f \in C_{2\pi}$) and series (1.2) is the Fourier series of f. In the case $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}_+$ (that is for cosine series with nonnegative coefficients) R.P.Boas [2, p.45] showed that $f \in Lip(\alpha)$ $(0 < \alpha < 1)$ if and only if $\sum_{k=n}^{\infty} c_k = O(n^{-\alpha})$ or, equivalently, $\sum_{k=1}^{n} kc_k = O(n^{1-\alpha})$. The case $\alpha = 1$ is more complicated. Similar results are obtained in [2] for sine series. S.Yu.Tikhonov [6] proved Boas-type results for moduli of continuity of an arbitrary order $\beta > 0$ in the case of sine and cosine series with positive

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coefficients. In his paper there are many references to other generalizations of Boas' result. F. Móricz [3] considered series (1.2) where $kc_k \geq 0$ for all $k \in \mathbb{Z}$ and established Boas-type results for $Lip(\alpha)$ and $lip(\alpha)$, $0 < \alpha \leq 1$, and Zygmund classes $\Lambda_*(1)$ and $\lambda_*(1)$ (see below). He also characterized the differentiability of f at point under additional condition $\sum_{|k|>n} |c_k| = o(n^{-1})$, $n \to \infty$. In another article [4] F. Móricz extended some results from [3] to the case of the second modulus of continuity.

The aim of our paper is to generalize F. Móricz's results for moduli of continuity of natural orders and some special classes of complex coefficients $\{c_k\}_{k\in\mathbb{Z}}$. After this article had been completed the author found that F. Móricz in [5] established Theorems 1 and 2 of the present article for $\omega(\delta) = \delta^\alpha$, $0 < \alpha < m$, and $j^m c_j \geq 0$ in parts (ii), (iii). However, the methods of proof in [5] and this article are different and our results are more general.

2 Definitions and Lemmas.

Let $g \in C_{2\pi}$, $\hat{g}(k) = (2\pi)^{-1} \int_0^{2\pi} g(t)e^{-ikt}\,dt$ for $k \in \mathbb{Z}$ and $S_n(g)(x) = \sum_{|k|\leq n} \hat{g}(k)e^{ikx}$ for $n \in \mathbb{Z}_+$. For $k \in \mathbb{N}$ let introduce

$$\Delta_h^k g(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(x + (k-j)h)$$

and $\omega_k(g,t) = \sup_{|h|\leq t} \|\Delta_h^k g(\cdot)\|$, where $\|g\| = max_{x\in[0,2\pi]}|g(x)|$. Denote by Φ the set of continuous increasing on $[0, 2\pi]$ functions ω such that $\omega(0) = 0$, $\omega(t) > 0$ when $t \in (0, 2\pi]$ and $\omega(2t) \leq C\omega(t)$, $t \in [0, \pi]$. If $\omega \in \Phi$ and $\sum_{k=n}^{\infty} k^{-1}\omega(k^{-1}) = O(\omega(n^{-1}))$ for all $n \in \mathbb{N}$, then $\omega \in B$. If $\omega \in \Phi$, $m \in (0, \infty)$ and $\sum_{k=1}^{n} k^{m-1}\omega(k^{-1}) = O(n^m\omega(n^{-1}))$, then $\omega \in B_m$. The classes B and B_m were introduced by N.K.Bari and S.B.Stechkin [1]. By definition $H^{\omega,k} = \{f \in C_{2\pi} : \omega_k(f,t) = O(\omega(t))\}$ and $h^{\omega,k} = \{f \in C_{2\pi} : \omega_k(f,t) = o(\omega(t)), t \to 0\}$. Spaces $H^{\omega,1}$ ($h^{\omega,1}$) for $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, are called $Lip(\alpha)$ ($lip(\alpha)$) and $H^{\omega,2}$ ($h^{\omega,2}$) for $\omega(t) = t$ are called $\Lambda_*(1)$ ($\lambda_*(1)$). A function $\gamma(t)$ will be called almost increasing (almost decreasing) if there exists a constant $K := K(\gamma) \geq 1$ such that $K\gamma(t) \geq \gamma(u)$ ($K\gamma(u) \geq \gamma(t)$) for $0 \leq u \leq t \leq \pi$. We need some lemmas.

Lemma 1. (See [6]) *If $f \in C_{2\pi}$, $k \in \mathbb{N}$ and $V_n(f) = \sum_{k=n}^{2n-1} S_k(f)/n$, then*

$$n^{-k}\|V_n^{(k)}(f)\| \leq C(k)\omega_k(f, 1/n), \quad n \in \mathbb{N},$$

where $g^{(k)}$ is the k-th derivative of g.

Lemma 2. (See [1]). *(i) Let $\omega \in \Phi$. Then $\omega \in B$ if and only if there exists $\alpha \in (0,1)$ such that $t^{-\alpha}\omega(t)$ is almost increasing.*

(ii) Let $\omega \in \Phi$. Then $\omega \in B_k$, $k \in \mathbb{N}$, if and only if there exists $\alpha \in (0,k)$ such that $t^{\alpha-k}\omega(t)$ is almost decreasing.

Lemma 3. *Let $\omega \in \Phi$, $\{\mu_j\}_{j=1}^{\infty}$ be a nonnegative sequence and $m \in \mathbb{N}$.*

(i) If $\omega \in B_m$, then condition

$$\sum_{j=n}^{\infty} \mu_j = O(\omega(1/n)) \tag{2.1}$$

implies

$$\sum_{j=1}^{n} j^m \mu_j = O(n^m \omega(1/n)). \tag{2.2}$$

(ii) If $\omega \in B$, then (2.2) implies (2.1).

Lemma 3 is the corollary of Lemma 2 and Lemma 4.4 from [6].

Lemma 4. *If the conditions of Lemma 3 are fulfilled then both statements of Lemma 3 remain valid if O in (2.1) and (2.2) is replaced by o.*

PROOF. (i) Put $s_k := \sum_{j=1}^{k} j^m \mu_j$ and $r_k = \sum_{j=k}^{\infty} \mu_j$. For every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that $r_k < \varepsilon\omega(1/k)$ whenever $k > k_0$. This inequality, condition $\omega \in B_m$ and summation by parts give

$$s_n = \sum_{j=1}^{n} (j^m - (j-1)^m) r_j - n^m r_{n+1} \le C_1 \sum_{j=1}^{n} j^{m-1} r_j \le C_1 r_1 \sum_{j=1}^{k_0} j^{m-1} +$$

$$+ C_1 \varepsilon \sum_{j=1}^{n} j^{m-1} \omega(j^{-1}) \le C_2 + \varepsilon C_3 n^m \omega(n^{-1})$$

for $n > k_0$. But from Lemma 2 it follows that $n^{m-\alpha}\omega(n^{-1})$ is almost increasing sequence for some $\alpha \in (0,m)$ and $\lim_{n\to\infty} n^m \omega(n^{-1}) = +\infty$. Therefore $C_2 = o(n^m \omega(n^{-1}))$ and $s_n = o(n^m \omega(n^{-1}))$.

(ii) For every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that $s_k \le \varepsilon k^m \omega(k^{-1})$ whenever $k > k_0$. Then for $n > k_0$ we have by condition $\omega \in B$

$$r_n = \sum_{k=n}^{\infty} k^{-m}(s_k - s_{k-1}) = \sum_{k=n}^{\infty} (k^{-m} - (k+1)^{-m}) s_k - n^{-m} s_{n-1} \le$$

$$C_4 \sum_{k=n}^{\infty} k^{-m-1} \varepsilon k^m \omega(k^{-1}) \le C_5 \varepsilon \omega(n^{-1}).$$

Lemma is proved. □

3 Main Results.

Theorem 1. *(i) If* $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$, $m \in \mathbb{N}$, $\omega \in B$ *and*

$$\sum_{|k|\le n} |k^m c_k| = O(n^m \omega(n^{-1})), \tag{3.1}$$

then $f \in H^{\omega,m}$.

(ii) Let $m \in \mathbb{N}$ *be even and* $\{c_j\}_{j\in\mathbb{Z}} = \{a_j + b_j i\}_{j\in\mathbb{Z}}$ *be a sequence of complex numbers such that all* a_j, $j \in \mathbb{Z}$, *are non-negative or non-positive and the same is true for* b_j. *Then* $f \in H^{\omega,m}$ *implies (3.1).*

(iii) Let $m \in \mathbb{N}$ *be odd and* $\{c_j\}_{j\in\mathbb{Z}} = \{a_j + b_j i\}_{j\in\mathbb{Z}}$ *be a sequence of complex numbers such that all* ja_j, $j \in \mathbb{Z}$, *are non-negative or non-positive and the same is true for* jb_j. *Then* $f \in H^{\omega,m}$ *implies (3.1).*

Proof. Further we shall consider the symmetric m-th difference

$$\dot{\Delta}_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + (m-2j)h/2).$$

If f has Fourier series (1.2), then in view of (1.1)

$$\dot{\Delta}_h^m f(x) = \sum_{k\in\mathbb{Z}} (2i)^m (\sin kh/2)^m c_k e^{ikx}. \tag{3.2}$$

We set $n = [1/|h|]$ for $h \neq 0$. Using the inequality $|\sin x| \le |x|$ and Lemma 3(ii) we have

$$|\dot{\Delta}_h^m f(x)| \le \sum_{|k|\le n} |c_k||2kh/2|^m + \sum_{|k|>n}^{\infty} |c_k| = O(|h|^m \sum_{|k|=1}^{n} |k^m c_k| + \omega(n^{-1})) =$$

$$= O(|h|^m n^m \omega(n^{-1}) + \omega(n^{-1})) = O(\omega(n^{-1})). \tag{3.3}$$

Applying the standard procedure we obtain $f \in H^{\omega,m}$.

(ii) By definition we find for $h \in (0, \pi)$ and $n = [1/h]$ that

$$C_1\omega(h) \ge |\dot{\Delta}_h^m f(0)| = \left| \sum_{k\in\mathbb{Z}} c_k (2\sin kh/2)^m \right| \ge$$

$$\left| \sum_{k\in\mathbb{Z}} a_k (2\sin kh/2)^m \right| \ge C_2 \sum_{|k|\le n} |a_k||kh|^m$$

and

$$\sum_{|k|\le n} |k^m a_k| = O(h^{-m}\omega(h)) = O(n^m\omega(n^{-1})). \tag{3.4}$$

Similarly to (3.4), the relation

$$\sum_{|k|\le n} |k^m b_k| = O(n^m\omega(n^{-1})). \tag{3.4$'$}$$

is valid. Combining (3.4) and (3.4′) gives $\sum_{|k|\le n} |k^m c_k| = O(n^m\omega(n^{-1}))$. This proves (ii).

(iii) Using the definition of $V_n(f)$ we have

$$V_n^{(m)}(f)(0) = i^m \left(\sum_{|k|\le n} k^m(a_k + b_k i) + \sum_{|k|=n+1}^{2n-1} (2 - |k|/n) k^m (a_k + b_k i) \right).$$

Similarly to (3.4) and (3.4′) by Lemma 1 the following estimates hold:

$$C_3\omega(n^{-1}) \ge n^{-m}\|V_n^{(m)}(f)\| \ge n^{-m}|V_n^{(m)}(f)(0)| \ge n^{-m} \sum_{|k|\le n} |k^m a_k|, \tag{3.5}$$

$$C_3\omega(n^{-1}) \ge n^{-m}|V_n^{(m)}(f)(0)| \ge n^{-m} \sum_{|k|\le n} |k^m b_k|. \tag{3.5$'$}$$

Combining (3.5) and (3.5′) gives $\sum_{|k|\le n} |k^m c_k| = O(n^m\omega(n^{-1}))$. This proves (iii). □

Theorem 2. *(i) If* $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$, $\omega \in B$, $m \in \mathbb{N}$ *and*

$$\sum_{|k|\le n} |k^m c_k| = o(n^m\omega(n^{-1})), \tag{3.6}$$

then $f \in h^{\omega,m}$.

(ii) If m *and* $\{c_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions of either Theorem 1 (ii) or Theorem 1 (iii), then* $f \in h^{\omega,m}$ *implies (3.6).*

The proof of part (i) is similar to that of theorem 1 (i) with using Lemma 4 (ii) instead of Lemma 3 (ii).

Proof. of (ii). Let m be even and $\omega_k(f, 1/n) < \varepsilon\omega(n^{-1})$ for $n > n_0(\varepsilon)$. Then we have for $n > n_0$ (see (3.4))

$$\varepsilon\omega(1/n) > |\dot{\Delta}_{1/n}^m| \ge \left| \sum_{k\in\mathbb{Z}} a_k (2\sin k/2n)^m \right| \ge (2/\pi)^m \sum_{|k|\le n} |a_k (k/n)^m|$$

and $\sum_{|k|\le n}|k^m a_k| = o(n^m\omega(n^{-1}))$. Similarly, we find that $\sum_{|k|\le n}|k^m b_k| = o(n^m\omega(n^{-1}))$ and $\sum_{|k|\le n}|k^m c_k| = o(n^m\omega(n^{-1}))$. If m is odd, then we have analogously to (3.5)

$$C_1\varepsilon\omega(n^{-1}) \ge n^{-m}|V_n^{(m)}(f)(0)| \ge n^{-m}\sum_{|k|\le n}|k^m a_k|$$

and $\varepsilon\omega(n^{-1}) \ge n^{-m}\sum_{|k|\le n}|k^m b_k|$. These two inequalities give $\sum_{|k|\le n}|k^m c_k| = o(n^m\omega(n^{-1})$. Theorem 2 is proved. □

Corollary 1. *Let $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ and $m \in \mathbb{N}$ satisfy the conditions of Theorem 1 (ii). If $\omega \in B \cap B_m$, then statements $f \in H^{\omega,m}$ and*

$$\sum_{|k|>n}|c_k| = O(\omega(n^{-1})) \tag{3.7}$$

are equivalent.

Proof. The implication (3.7) $\Rightarrow f \in H^{\omega,m}$ follows from (3.3) and Lemma 3 (i). Conversely, if $f \in H^{\omega,m}$, then by inequality $\|f - V_n(f)\| \le C_1\omega(n^{-1})$ (see [7, chapter III, Theorem (13.5)] and Jackson-Stechkin Theorem (D) in [1]) we have

$$\begin{aligned} C_1\omega(n^{-1}) &\ge |f(0) - V_n(f)(0)| \\ &= \left|\sum_{|k|=n+1}^{2n-1}(|k|/n-1)(a_k+b_ki) + \sum_{|k|=2n}^{\infty}(a_k+b_ki)\right| \\ &\ge 2^{-1/2}\left(\sum_{|k|=2n}^{\infty}|a_k| + \sum_{|k|=2n}^{\infty}|b_k|\right). \end{aligned}$$

By the property $\omega(2t) \le C\omega(t)$ we have $\sum_{|k|=n}^{\infty}|a_k| = O(\omega(n^{-1}))$ and $\sum_{|k|=n}^{\infty}|b_k| = O(\omega(n^{-1}))$. These relations imply $\sum_{|k|=n}^{\infty}|c_k| = O(\omega(n^{-1}))$. □

Corollary 2. *Let $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ and $m \in \mathbb{N}$ satisfy conditions of Theorem 1 (ii). If $\omega \in B \cap B_m$, then statements $f \in h^{\omega,m}$ and $\sum_{|k|>n}|c_k| = o(\omega(n^{-1}))$ are equivalent.*

The proof is similar to that of Corollary 1, but we use Lemma 4 (i) instead of Lemma 3 (i).

For the last theorem we need the so called Schwartz derivative of order m instead of the ordinary derivative. By definition, function f has the Schwartz derivative of order $m \in \mathbb{N}$ in point x and this derivative equals to A if there exists $\lim_{h\to 0} h^{-m}\dot{\Delta}_h^m f(x) = A$.

Theorem 3. *If $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$, $m \in \mathbb{N}$, and*

$$\sum_{|k|>n} |c_k| = o(n^{-m}), \tag{3.8}$$

then Schwartz derivative of order m exists in point x and equals to A if and only if the formally differentiated series

$$\sum_{k\in\mathbb{Z}} (ik)^m c_k e^{ikx}$$

converges to A.

Proof. By (3.2) we have for $n = [1/|h|]$, $h \neq 0$,

$$\dot{\Delta}_h^m f(x) = \sum_{|k|\le n} (2i\sin kh/2)^m c_k e^{ikx} + \sum_{|k|>n} (2i\sin kh/2)^m c_k e^{ikx} =: A_h + B_h.$$

In virtue of (3.8), we see that $|B_h| \le 2^m \sum_{|k|>n} |c_k| = o(h^m)$, $h \to 0$. Further we use the estimate $\sin t = t + O(t^3)$, $t \in [0,1]$, and find that

$$A_h = \sum_{|k|\le n} (ik)^m c_k e^{ikx} h^m + \sum_{j=1}^{m} \sum_{|k|\le n} i^m c_k e^{ikx} O((kh)^{m-j}(kh)^{3j}) =: A_h^{(0)} + \sum_{j=1}^{m} A_h^{(j)}$$

Since $\omega(t) = t^m \in B_{m+2j}$, $j \in \mathbb{N}$ (see [1]), then by lemma 4 (i) the relation $\sum_{|k|>n} |c_k| = o(n^{-m})$ implies $\sum_{|k|\le n} |k^{m+2j} c_k| = o(n^{m+2j} n^{-m})$. Therefore $|A_h^{(j)}| = o(n^{2j}|h|^{2j+m}) = o(|h|^m)$ and $\lim_{h\to 0} h^{-m}\dot{\Delta}_h^m f(x)$ exists if and only if there exists

$$\lim_{h\to 0} h^{-m} A_h^{(0)} = \lim_{n\to\infty} \sum_{|k|\le n} (ik)^m c_k e^{ikx} = \sum_{k\in\mathbb{Z}} (ik)^m e^{ikx}$$

Theorem is proved. □

Remark 1. *For $m = 2$ part "if" of Theorem 3 follows from regularity of Riemann's method of summation (see [7, chapter IX, Theorem (2.4)]).*

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