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Enrico Zoli, Istituto Tecnico Industriale Statale "Berenini", via Alfieri 4 – 43036 Fidenza (PR), Italy. email: zoli@math.unifi.it

ON A ZERO–INFINITY LAW OF OLSEN

Abstract

Let μ be a translation-invariant metric measure on $\mathbb{R}$ with the following scaling property: for every $\lambda \in (0,1)$ there exists $b(\lambda) > \lambda$ with $\mu(\lambda X) \geq b(\lambda)\mu(X)$ for all $X \subseteq \mathbb{R}$. If X is a $\mathbb{Z}$-invariant subset of $\mathbb{R}$ with $X/q \subseteq X$ for some $q \in \mathbb{N} \setminus \{1\}$, then $\mu(X) = 0$ or $\mu(X \cap O) = \infty$ for every non-empty open set O. This refines an earlier result by Olsen.

The reader is supposed to be familiar with the rudiments of geometric measure theory. We shall closely follow Mattila's notation [4]. For us, a *dimension function* is a continuous map $h : [0,\infty) \to [0,\infty)$ with $h(r) = 0$ if and only if $r = 0$. The Hausdorff h-measure $\mathcal{H}^h$ (the packing h-measure $\mathcal{P}^h$) on $\mathbb{R}$ is then a metric (outer) measure. A dimension function h is said to be *strongly concave at 0* if $\liminf_{r\to 0^+} h(\lambda r)/h(r) > \lambda$ for all $\lambda \in (0,1)$. For instance, for any $s \in (0,1)$ the map $r \mapsto r^s$ is strongly concave at 0. A subset X of $\mathbb{R}$ is *$\mathbb{Z}$-invariant* if $X + z = X$ for all $z \in \mathbb{Z}$.

Given a subset X of $\mathbb{R}$, it is important to find an open set O and a translation-invariant metric measure μ on $\mathbb{R}$ such that $\mu(X \cap O) \in (0,\infty)$. Addressing a question raised by Mauldin, several authors have recently shown that for many subsets of $\mathbb{R}$ of interest in number theory this search is vain [1],[3],[7].[1] Similarly, a few years before, Olsen [6] proved the following zero-infinity law: *Let h be a dimension function that is strongly concave at 0. If X is a $\mathbb{Z}$-invariant subset of $\mathbb{R}$ with $X/q \subseteq X$ for some $q \in \mathbb{N} \setminus \{1\}$, then either $\mathcal{H}^h(X) = 0$ or $\mathcal{H}^h(X \cap [0,1]) = \infty$.*

Aim of this note is to point out the following refinement of Olsen's result:

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[1] Examples include: dense additive subgroups of $\mathbb{R}$ with Lebesgue measure zero; the set of Liouville numbers; the sets of non-normal numbers in the sense of Besicovitch–Eggleston.

Theorem 1. *Let μ be a translation-invariant metric measure on $\mathbb{R}$ with the following scaling property:*[2]

$$\text{for every } \lambda \in (0,1) \text{ there exists } b(\lambda) > \lambda \text{ with } \mu(\lambda X) \geq b(\lambda)\mu(X) \text{ for all } X \subseteq \mathbb{R}. \quad \text{(SP)}$$

If X is a $\mathbb{Z}$-invariant subset of $\mathbb{R}$ with $X/q \subseteq X$ for some $q \in \mathbb{N} \setminus \{1\}$, then $\mu(X) = 0$ or $\mu(X \cap O) = \infty$ for every non-empty open set O.

A frequently occurring set to which Theorem 1 (but not the afore-cited results in [1],[3],[7]) can be applied is that of badly ψ-approximable numbers: given an arbitrary function $\psi : \mathbb{N} \to [0, \infty)$, the set

$$\left\{ x \in \mathbb{R} : \text{ there exists } c > 0 \text{ with } \left| x - \frac{p}{q} \right| > c\psi(q) \text{ for all } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}$$

satisfies the needed hypotheses for X.

Note that sums of measures with (SP) have (SP) in their turn; moreover, many Hausdorff and packing h-measures satisfy (SP):

Proposition 2. *If the dimension function h is strongly concave at 0, then both Hausdorff and packing h-measures on $\mathbb{R}$ satisfy* (SP).

The proofs, now.

PROOF OF THEOREM 1. Let us first prove the weaker dichotomy $\mu(X) = 0$ or $\mu(X \cap [0,1]) = \infty$. Suppose $\mu(X \cap [0,1]) < \infty$ (observe that this implies $\mu(X \cap [0,n]) < \infty$ for all $n \in \mathbb{N}$, by the invariance assumptions on X and μ). Define $\alpha := qb(1/q) - 1$ and choose $n \in \mathbb{N}$ such that

$$\alpha n > 1. \tag{1}$$

By our invariance assumptions on X and μ, we have on the one hand

$$\mu(X \cap [0, 2n]) - \mu(X \cap [0, 2n-1]) \leq \mu(X \cap [0,1]); \tag{2}$$

on the other, using also (SP) and that μ is metric,

$$\begin{aligned} \mu(X \cap [0,2n]) &\geq \mu\left(\frac{X}{q} \cap [0,2n]\right) = \mu\left(\frac{X \cap [0,2nq]}{q}\right) \\ &\geq b(1/q)\mu(X \cap [0,2nq]) \geq b(1/q)\mu\left(X \cap \bigcup_{i=1}^{q} [2n(i-1), 2ni-1]\right) \\ &= qb(1/q)\mu(X \cap [0,2n-1]). \end{aligned} \tag{3}$$

[2]Compare with the similar scaling law analyzed in detail in [2],[5].

By (3) and (2) we then have

$$\begin{aligned}
\alpha n\mu(X\cap[0,1]) &= \alpha\sum_{i=1}^{n}\mu(X\cap[2(i-1),2i-1]) \\
&= \alpha\mu\left(X\cap\bigcup_{i=1}^{n}[2(i-1),2i-1]\right)\leq\alpha\mu(X\cap[0,2n-1]) \\
&= qb(1/q)\mu(X\cap[0,2n-1])-\mu(X\cap[0,2n-1]) \\
&\leq\mu(X\cap[0,2n])-\mu(X\cap[0,2n-1])\leq\mu(X\cap[0,1]);
\end{aligned}$$

from this, in view of (1) we obtain $\mu(X\cap[0,1])=0$ (since the alternative $\mu(X\cap[0,1])=\infty$ is excluded by hypothesis) and therefore $\mu(X)=0$.

It remains to prove that, if $\mu(X\cap[0,1])=\infty$, then $\mu(X\cap O)=\infty$ for any fixed non-empty open set O. For such an O there exist $z\in\mathbb{Z}$ and $n\in\mathbb{N}$ with $[z/q^n,(z+1)/q^n]\subseteq O$; the conclusion now follows from the chain:

$$\mu(X\cap O)\geq\mu\left(\frac{X}{q^n}\cap\left[\frac{z}{q^n},\frac{z+1}{q^n}\right]\right)\geq\frac{\mu(X\cap[z,z+1])}{q^n}=\frac{\mu(X\cap[0,1])}{q^n},$$

a further application of (SP) and the invariance of X and μ. □

Proof of Proposition 2. We prove the packing h-measure case only, since the other, very similar, is essentially contained in the first part of [6, Proposition 4]. Fix $\lambda\in(0,1)$. By assumption, there exist $\epsilon>0$ and $r\in(0,1]$ such that $h(\lambda\delta)/h(\delta)\geq\lambda+\epsilon$ for all $\delta\in(0,r)$. Fix now $A\subseteq\mathbb{R}$. Since for any countable collection $(A_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ we have $A\subseteq\bigcup_{n=1}^{\infty}A_n$ if and only if $\lambda A\subseteq\bigcup_{n=1}^{\infty}\lambda A_n$, by definition of packing h-measure $\mathcal{P}^h$ it is enough to show that $P^h(\lambda B)\geq(\lambda+\epsilon)P^h(B)$ for all $B\subseteq\mathbb{R}$. If $(I_n)_{n=1}^{\infty}$ is an arbitrary δ-fine packing of B, then $(\lambda I_n)_{n=1}^{\infty}$ is a $\lambda\delta$-fine packing of λB. Hence

$$P^h_{\lambda\delta}(\lambda B)\geq\sum_{n=1}^{\infty}h(d(\lambda I_n))=\sum_{n=1}^{\infty}\frac{h(\lambda d(I_n))}{h(d(I_n))}h(d(I_n))\geq(\lambda+\epsilon)\sum_{n=1}^{\infty}h(d(I_n)).$$

By the arbitrariness of the δ-fine packing $(I_n)_{n=1}^{\infty}$ of B we first have $P^h_{\lambda\delta}(\lambda B)\geq(\lambda+\epsilon)P^h_{\delta}(B)$; then, by letting $\delta\to 0^+$, we obtain $P^h(\lambda B)\geq(\lambda+\epsilon)P^h(B)$. □

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