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ON THE CONNECTIVITY OF ATTRACTORS OF ITERATED MULTIFUNCTION SYSTEMS

Abstract

The aim of the paper is to give sufficient and necessary conditions when the attractor of an iterated multifunction system, composed by multifunctions with compact and connected values, is a connected set.

1 Introduction.

We start by a short presentation of iterated multifunction systems (IMSs) which are a generalization of iterated function systems (IFSs). We will also fix the notations.

Iterated function systems were conceived in the present form by John Hutchinson [7] and popularized by Michael Barnsley [2] and are one of the most common and most general ways to generate fractals. Many of the important examples of functions and sets with special and unusual properties in analysis turns out to be fractal sets and a great part of them are attractors of IFSs. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems (IIFSs) or more generally to multifunction systems and to study them. A very good survey article for multifunction systems where one can found also an extended and recent bibliography is [1]. A recent such extension of the IFS theory can be found in [11], where the Lipscomb's space–which was an important example

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in dimension theory–can be obtained as an attractor of an IIFS defined in a very general setting. In this setting the attractor can be a closed and bounded set in contrast with the classical theory where only compact sets are considered. Although the fractal sets are defined with measure theory-being sets with noninteger Hausdorff dimension [4],[5]-it turns out that they have interesting topological properties as we can see from the above example [11]. One of the most important result in these direction is Theorem 1.2 below (see [13] for a proof) which states when the attractor of an IFS is a connected set. We want to extend this result to IMSs and point out the differences between the two cases. Theorem 1.2 is not valid for IMSs (see Example 3.1). Even if we take an IMS which contains only one multifunction, the attractor could not be connected. So we have to take multifunctions with special properties. We require for a multifunction to be with compact and connected values. This choice is suggested by Theorem 1.3, which states that the fixed set of at multicontraction with compact and connected values is connected and compact.

The paper is divided in three parts. The first part is the introduction. The second part contains the main result (Theorem 2.1). The last part contains some examples and remarks.

For a set X, $P^*(X)$ denotes the subsets of X with the empty set thrown down. For a metric space (X, d) $K^*(X)$ denotes the set of compact subsets of X with the empty set thrown down.

Definition 1.1. *The generalized Hausdorff-Pompeiu semidistance is an application $h : P^*(X) \times P^*(X) \to [0, +\infty]$ defined by*

$$h(A, B) = \max(d(A, B), d(B, A))$$

where $d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} (\inf_{y \in B} d(x, y))$.

Definition 1.2. *Let (X, d_X) and (Y, d_Y) be two metric spaces. For a function $f : X \to Y$ let us denote by $\mathbf{Lip}(f) \in [0, +\infty]$ the Lipschitz constant associated to f, that is*

$$\mathbf{Lip}(f) = \sup_{x,y \in X; x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

f is a Lipschitz function if $\mathbf{Lip}(f) < +\infty$ and a contraction if $\mathbf{Lip}(f) < 1$.

Concerning the Hausdorff-Pompeiu semidistance we have the following important properties:

Proposition 1.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Then:*

1. *If H and K are two nonempty subsets of X then $h_X(H,K) = h_X(\overline{H}, \overline{K})$, where h_X is the Hausdorff-Pompeiu semidistance associated with the distance d_X.*

2. *If $(H_i)_{i\in I}$ and $(K_i)_{i\in I}$ are two families of nonempty subsets of X then*
$$h_X(\overline{\underset{i\in I}{\cup} H_i}, \overline{\underset{i\in I}{\cup} K_i}) = h_X(\underset{i\in I}{\cup} H_i, \underset{i\in I}{\cup} K_i) \leq \sup_{i\in I} h_X(H_i, K_i).$$

3. *If H and K are two nonempty subsets of X and $f : X \to Y$ is a function then*
$$h_Y(f(K), f(H)) \leq \mathbf{Lip}(f) \cdot h_X(K,H).$$

4. *If $(H_n)_{n\geq 1} \subset P^*(X)$ is a sequence of compact and connected sets and $H \in P^*(X)$ is a closed set such that $h_X(H, H_n) \to 0$ when $n \to \infty$ then H is a compact and connected set.*

Proof. See [1], [6] or [12]. □

It is well-known that $(K^*(X), h)$ is a metric space and it is complete if X is such (see [1], [5], or [6]).

Definition 1.3. *An iterated function system on a metric space (X,d) consists in a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X and is denoted by $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$. For an IFS $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, $F_{\mathcal{S}} : K^*(X) \to K^*(X)$ is the function defined by $F_{\mathcal{S}}(B) = \overset{n}{\underset{k=1}{\cup}} f_k(B)$.*

The function $F_{\mathcal{S}}$ is a contraction with $\mathbf{Lip}(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} \mathbf{Lip}(f_k)$. Using Banach contraction theorem there exists for an IFS a unique set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. More generally we have the case of multifunctions (see [1] for more details). We present the basic definitions and results, which we need later.

Definition 1.4. *Let X and Y be two sets. A multifunction F from the set X into the set Y is a function $F : X \to P^*(Y)$. We will denote the multifunction by $F : X \rightarrowtail Y$. For a nonvoid set $A \subset X$ $F(A) = \underset{x\in A}{\cup} F(x)$.*

Definition 1.5. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A multifunction $F : X \rightarrowtail Y$ is said to be with compact, bounded or connected values if for every $x \in X$ the set $F(x)$ is compact, bounded or connected. If $F : X \rightarrowtail Y$ is a multifunction with compact values we will denote by $F : X \overset{c}{\rightarrowtail} Y$.*

Definition 1.6. *Let (X, d_X) and (Y, d_Y) be two metric spaces and $F : X \rightarrowtail Y$ be a multifunction. Then* $\mathbf{Lip}(F) \in [0, +\infty]$ *is the Lipschitz constant associated to the function* $F : (X, d_X) \to (P^*(Y), h_Y)$, *that is* $\mathbf{Lip}(F) = \sup\limits_{x,y \in X; x \neq y} \dfrac{h_Y(F(x), F(y))}{d_X(x,y)}$. *$F$ is a Lipschitz multifunction if* $\mathbf{Lip}(F) < +\infty$ *and a multicontraction if* $\mathbf{Lip}(F) < 1$.

Let (X, d_X) and (Y, d_Y) be two metric spaces. For a multifunction $F : X \rightarrowtail Y$, $\Phi_F : P^*(X) \to P^*(X)$ is the function defined by $\Phi_F(A) = \overline{F(A)} = \overline{\bigcup\limits_{x \in A} F(x)}$.

If $F : X \overset{c}{\rightarrowtail} X$ is a multicontraction and (X, d) is a complete metric space, from the Banach's contraction theorem, there exists a unique compact nonvoid set $A(F)$ such that $\Phi_F(A(F)) = F(A(F)) = A(F)$. The set $A(F)$ is named the attractor of the multifunction F. More precisely we have the following theorem.

Theorem 1.1. *[1] Let (X, d) be a complete metric space and $F : X \overset{c}{\rightarrowtail} X$ be a multifunction with $c = \mathbf{Lip}(F) < 1$. Then there exists a unique $A(F) \in K^*(X)$ such that $\Phi_F(A(F)) = F(A(F)) = A(F)$. Moreover, for any $H_0 \in K^*(X)$ the sequence $(H_n)_{n \geq 0}$ defined by $H_{n+1} = \Phi_F(H_n)$ is convergent to $A(F)$. For the speed of the convergence we have the following estimation*

$$h(H_n, A(F)) \leq \frac{c^n}{1-c} h(H_0, H_1).$$

Let us note that the theorem is also true for multicontraction with bounded values with the only differences that $\Phi_F(A(F)) = \overline{F(A(F))} = A(F)$ and the sets $A(F)$ and H_n are bounded closed nonvoid, but we need only the compact case.

Definition 1.7. *Let (X, d) be a metric space. An iterated multifunction system, an IMS for short, consists in a family of multicontractions with compact values $(F_k)_{k=\overline{1,n}}$, where $F_k : X \rightarrowtail X$, and it is denote by $\mathcal{S} = (X, (F_k)_{k=\overline{1,n}})$. For an IMS $F_{\mathcal{S}} : X \rightarrowtail X$ is the multifunction defined by $F_{\mathcal{S}}(x) = \bigcup\limits_{k=1}^{n} F_k(x)$.*

We remark that $F_{\mathcal{S}}$ is a multifunction with compact values.

Let us note that the IMS $\mathcal{S} = (X, (F_k)_{k=\overline{1,n}})$ can always be reduced to one multifunction, namely $F_{\mathcal{S}} = \bigcup\limits_{k=1}^{n} F_k$. Moreover $\mathbf{Lip}(F_{\mathcal{S}}) \leq \max\limits_{k=\overline{1,n}} \mathbf{Lip}(F_k) < 1$. By replacing the single multicontraction from Theorem 1.1 by a family of (multivalued) contractions one can obtain the results from [1],[2],[4],[7],[9],[12].

Notation. For two nonvoid sets A and B B^A denotes the set of functions from A to B.

By $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$, where $\mathbb{N}_n^* = \{1, 2, ..., n\}$. The elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as words $\omega = \omega_1\omega_2...\omega_n$. If $\omega = \omega_1\omega_2...\omega_n$ and $n \geq m$ then $[\omega]_m = \omega_1\omega_2...\omega_m$. Let (X, d) be a complete metric space and $\mathcal{S} = (X, (F_k)_{k=\overline{1,n}})$ be an IMS. Then $F_\omega = F_{\omega_1} \circ F_{\omega_2} \circ ... \circ F_{\omega_m}$ for $\omega = \omega_1\omega_2...\omega_m \in \Lambda_m(\mathbb{N}_n^*)$. For a set $H \subset X$ $H_\omega = \Phi_{F_\omega}(H)$.

Remark 1.1. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (F_k)_{k=\overline{1,n}})$ be an IMS and $A = A(\mathcal{S})$ the attractor of $\mathcal{S}$. Then $A = \underset{\omega\in\Lambda_m(\mathbb{N}_n^*)}{\cup} A_\omega$.*

Definition 1.8. *Let (X, d) be a metric space and $(A_i)_{i\in I}$ a family of nonvoid subset of X. The family $(A_i)_{i\in I}$ is said to be connected if for every $i, j \in I$ there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_m = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1, 2, .., n-1\}$.*

Definition 1.9. *A metric space (X, d) is arcwise connected if for every $x, y \in X$ there exists a continuous function $\varphi : [0, 1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.*

In the case of IFSs, concerning the attractor's connectivity, we have the following theorem (see [13]).

Theorem 1.2. *Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an IFS with $c = \max_{k=\overline{1,n}} \mathbf{Lip}(f_k) < 1$ and $A(\mathcal{S})$ the attractor of $\mathcal{S}$. The following are equivalent:*

1. *The family $(A_i)_{i=\overline{1,n}}$ is connected where $A_i = f_i(A(\mathcal{S}))$.*
2. *$A(\mathcal{S})$ is arcwise connected.*
3. *$A(\mathcal{S})$ is connected.*

We want to find a similar result for IMSs. For an IMS point 2) and 3) are not equivalent as we can see from Example 3.1. The next results are well known.

Lemma 1.1. *([3], 6.3.1 p. 457) Let (X, d) be a metric space and K a compact subset of X. Then K is connected if and only if for every $x, y \in K$ and every $\varepsilon > 0$ there exists $(x_k)_{k=\overline{1,n}} \subset K$ such that $x_1 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \varepsilon$ for every $i \in \{1, 2, ..., n-1\}$.*

Lemma 1.2. *([1] or [6]) Let (X, d) be a complete metric space and $F : X \overset{c}{\rightarrowtail} X$ be a Lipschitz multifunction with bounded values. If the multifunction F has connected values and the set H is connected then the sets $F(H)$ and $\Phi_F(H)$ are connected.*

Corollary 1.1. *Let (X,d) be a complete metric space and $F_i : X \overset{c}{\rightarrowtail} X$ for $i \in \{1,2,...,n\}$ be Lipschitz multifunctions with connected values. If the set H is connected then the sets $F_1 \circ F_2 \circ ... \circ F_n(H)$ and are $\Phi_{F_1 \circ F_2 \circ ... \circ F_n}(H)$ also connected.*

The next result is a particular case of the main Theorem 2.1.

Theorem 1.3. *Let (X,d) be a complete metric space and $F : X \overset{c}{\rightarrowtail} X$ be a multicontraction with connected values. Let $A(F)$ be the attractor of F. Then $A(F)$ is compact and connected.*

PROOF. (see also [10] Theorem 1 for a generalization) Let $x_0 \in X$, $H_0 = \{x_0\}$ and $(H_n)_{n \geq 0}$ be the sequence defined by $H_{n+1} = F(H_n)$. H_0 is compact and connected. From Lemma 2.1 it follows by induction that H_n is compact and connected. From the Theorem 1.1 we have $H_n \to A(F)$. From Proposition 1.1 point 4 it follows that $A(F)$ is connected. □

2 The Main Result.

The main result of the paper is the following theorem.

Theorem 2.1. *Let (X,d) be a complete metric space, $\mathcal{S} = (X,(F_k)_{k=\overline{1,n}})$ be an IMS where the multifunctions F_k has compact and connected values, $c = \max\limits_{k=\overline{1,n}} \mathbf{Lip}(F_k) < 1$ and $A = A(\mathcal{S})$ be the attractor of $\mathcal{S}$. The following are equivalent:*

1. *The family $(A_i)_{i=\overline{1,n}}$ is connected, where $A_i = F_i(A(\mathcal{S}))$.*
2. *$A(\mathcal{S})$ is connected.*

PROOF. We remark first that, from Theorem 1.1, $A(\mathcal{S}) \in K^*(X)$. Then $A_i = F_i(A(\mathcal{S})) \in K^*(X)$ for every $i \in \{1,2,...,n\}$ and, more generally, $A_\omega = F_\omega(A(\mathcal{S})) \in K^*(X)$ for every $\omega \in \Lambda_p = \Lambda_p(\mathbb{N}_n^*)$ and $p \in \mathbb{N}^*$.

2) $\Rightarrow$ 1). Let $M = \{j \in \{1,2,...,n\}|$ there exist $(i_k)_{k=\overline{1,m}}$ such that $i_1 = 1$, $i_m = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,2,...,m-1\}\}$.

Set $V_1 = \underset{j \in M}{\cup} A_j$ and $V_2 = \underset{j \notin M}{\cup} A_j$. Then $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = A(\mathcal{S})$ and V_1 and V_2 are compact sets. Because $A(\mathcal{S})$ is connected and $V_1 \neq \emptyset$ (because $A_1 \subset V_1$) it follows that $A(\mathcal{S}) = V_1$ and so $M = \{1,2,...,n\}$. This means that the family $(A_i)_{i=\overline{1,n}}$ is connected.

1) $\Rightarrow$ 2). We will prove first that the families of sets $(A_i)_{i \in \Lambda_p}$ are connected. This will be done by induction. First step is the hypothesis.

The induction step. Let us suppose that the family $(A_\omega)_{\omega\in\Lambda_p}$ is connected. We want to prove that the family $(A_\omega)_{\omega\in\Lambda_{p+1}}$ is connected. Let $\omega_1,\omega_2\in\Lambda_{p+1}$. Then $\omega_1=\omega_1^1\omega_2^1\omega_3^1...\omega_p^1\omega_{p+1}^1$, $\omega_2=\omega_1^2\omega_2^2\omega_3^2...\omega_p^2\omega_{p+1}^2$, $[\omega_1]_p=\omega_1^1\omega_2^1\omega_3^1...\omega_p^1$, $[\omega_2]_p=\omega_1^2\omega_2^2\omega_3^2...\omega_p^2$ and $[\omega_1]_p,[\omega_2]_p\in\Lambda_p$. Because the family $(A_i)_{i\in\Lambda_p}$ is connected there exists $\alpha_1,\alpha_2,...,\alpha_l\in\Lambda_p$ such that $\alpha_1=[\omega_1]_p$, $\alpha_l=[\omega_2]_p$ and $A_{\alpha_j}\cap A_{\alpha_{j+1}}\neq\emptyset$ for $j\in\{1,2,...,l-1\}$. Let $x_j\in A_{\alpha_j}\cap A_{\alpha_{j+1}}$. We have $x_{j-1},x_j\in A_{\alpha_j}$ and $A_{\alpha_j}=\underset{k=\overline{1,n}}{\cup}A_{\alpha_j k}=\underset{k=\overline{1,n}}{\cup}F_{\alpha_j}(A_k)$. Let $A_{\alpha_j k_1^j}$ and $A_{\alpha_j k_2^j}$ be such that $x_{j-1}\in A_{\alpha_j k_1^j}$ and $x_j\in A_{\alpha_j k_2^j}$. We can choose $k_1^1=\omega_{p+1}^1$ and $k_2^l=\omega_{p+1}^2$. Because the family $(A_i)_{i=\overline{1,n}}$ is connected it follows that the family $(F_{\alpha_j}(A_i))_{i=\overline{1,n}}$ is also connected. Then there exists $(i_k^j)_{k=\overline{1,m(j)}}\subset\{1,2,...,n\}$ such that $i_1^j=k_1^j$, $i_{m(j)}^j=k_2^j$ and $A_{\alpha_j i_k^j}\cap A_{\alpha_j i_{k+1}^j}=F_{\alpha_j}(A_{i_k^j})\cap F_{\alpha_j}(A_{i_{k+1}^j})\neq\emptyset$ for every $k\in\{1,2,...,m(j)-1\}$. The sequence $(i_k^j)_{k=\overline{1,m(j)}}$ can be taken without repetition. In this case $m(j)\le n$. We can suppose that $m(j)=n$ by taken $i_{m(j)}^j=i_{m(j)+1}^j=...=i_n^j=k_2^j$.

We consider the functions $f,g:\mathbb{N}\to\mathbb{N}$ defined by $f(m)=[\frac{m-1}{n}]+1$ and $g(m)=m-[\frac{m-1}{n}]n$, where $\mathbb{N}$ denotes the set of natural numbers and $[x]$ denotes the greater integer less or equal to x.

Let $t:\{1,2,...,nl\}\to\Lambda_{p+1}$ be defined by $t(m)=\alpha_{f(m)}i_{g(m)}^{f(m)}$. We have $A_{t(m)}\cap A_{t(m+1)}\neq\emptyset$, $A_{t(1)}=A_{\alpha_1 i_1^1}=A_{[\omega_1]_p k_1^1}=A_{[\omega_1]_p\omega_{p+1}^1}=A_{\omega_1}$ and $A_{t(nl)}=A_{\alpha_l i_n^l}=A_{[\omega_2]_p k_2^l}=A_{[\omega_2]_p\omega_{p+1}^2}=A_{\omega_2}$. This proves that the family of sets $(A_\omega)_{\omega\in\Lambda_{p+1}}$ is connected.

For the proof of the fact that $A(\mathcal{S})$ is a connected set we will used Lemma 1.1. Let $\varepsilon>0$ and $x_0,z_0\in A$ be fixed. Let m be such that $c^m d(A)<\varepsilon/2$ and $B=\{x_0\}$. Then there exists $\alpha_1,\alpha_2\in\Lambda_m$ such that $x_0\in A_{\alpha_1}$ and $z_0\in A_{\alpha_2}$. Because the family of sets $(A_\omega)_{\omega\in\Lambda_m}$ is connected it follows that there exists $(\omega_j)_{j=\overline{0,l}}\subset\Lambda_m$ such that $\omega_0=\alpha_1$, $\omega_l=\alpha_2$ and $A_{\omega_j}\cap A_{\omega_{j+1}}\neq\emptyset$ for every $j\in\{0,1,...,l-1\}$. Let us fixed $x_j\in A_{\omega_{j-1}}\cap A_{\omega_j}$ for $j\in\{1,2,...,l\}$. Set $x_{l+1}=z_0$.

Let $\omega\in\Lambda_p$ with $p\ge m$ and let $B_\omega=F_\omega(B)$. Since the set $B=\{x_0\}$ is connected and compact it follows, from Corollary 1.1 that the set B_ω is connected and compact. We have $h(B_\omega,A_\omega)\le c^{|\omega|}h(B,A)\le c^{|\omega|}d(A)\le c^m d(A)<\varepsilon$ and $B_\omega\subset A_\omega$ (because $B\subset A$).

Since $h(B_{\omega_j},A_{\omega_j})<\varepsilon$, there exists for every $j\in\{0,1,...,l\}$ $y_j,y_j'\in B_{\omega_j}$ such that $d(y_j,x_j)<\varepsilon$ and $d(y_j',x_{j+1})<\varepsilon$. Since B_{ω_j} is a connected set there exists $(t_i^j)_{i=\overline{2,m(j)}}\subset B_{\omega_j}$ such that $t_2^j=y_j$, $t_{m(j)}^j=y_j'$ and $d(t_i^j,t_{i+1}^j)<\varepsilon$ for $i\in\{2,3,...,m(j)-1\}$ and $j\in\{0,1,...,l\}$. Set $t_1^j=x_j$ for $j\in\{0,1,...,l\}$ and $m=\max\{m(0),m(1),...,m(l)\}$. By repeating the last therms of the sequences

$(t_i^j)_{i=\overline{1,m(j)}}$ we can suppose that $m(j) = m$.

We consider the functions $t, s : \mathbb{N} \to \mathbb{N}$ defined by $t(i) = [\frac{i-1}{m}]$ and $s(i) = i - [\frac{i-1}{m}]m$.

Let $(z_i)_{i=\overline{1,(l+1)m+1}}$ be the sequence defined by $z_i = t_{s(i)}^{t(i)}$ for $i \in \{1, 2, ..., (l+1)m\}$ and $z_{(l+1)m+1} = z_0$. We have $z_1 = x_0$, $z_{(l+1)m+1} = z_0$ and $d(z_i, z_{i+1}) < \varepsilon$. This ends the proof. □

3 Examples.

Example 3.1. *Let (X, d) be a complete metric space, K a nonvoid compact set in X and $F : X \rightarrowtail X$ be a multifunction defined by $F(x) = K$ for every $x \in X$. Then $\Phi_F(H) = K$ for every $H \in P^*(X)$, F is a compact multifunction compact values, $\mathbf{Lip}(F) = 0$ and $A(F) = K$. If K is also connected, F is a multifunction with connected values. Because K could be a connected set but not arcwise connected it follows that the sentences 2) and 3) from Theorem 1.2 could not be equivalent for multifunctions. From Theorem 2.1 it follows that sentences 1) and 3) from Theorem 1.2 could not be equivalent for multifunctions. Let us consider the case when K is a disconnected set. The IMS $\mathcal{S} = (X, (F))$ is connected but $A(\mathcal{S}) = A(F) = K$ is not a connected set. This does not contradict Theorem 1.3 because F is not a connected multifunction.*

Example 3.2. *Let (X, d) be a metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an iterated function system. $\mathcal{S}$ can be seen as an iterated multifunction system $(X, (F_k)_{k=\overline{1,n}})$ where $F_k(x) = \{f_k(x)\}$ for every $k \in \{1, 2, ..., n\}$. In this case the equivalence 1) ⇔ 3) from Theorem 1.2 is a particular case of Theorem 3.1.*

Example 3.3. *Let (X, d_X) and (Y, d_Y) be two metric spaces such that Y is compact and $(Z = X \times Y, d)$ be the product metric space with the maxim distance, that is $d((x, y), (x_1, y_1)) = \max\{d(x, x_1), d(y, y_1)\}$. Let $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be an iterated function system on the metric space X. Let $\mathcal{S}' = (Z, (F_k)_{k=\overline{1,n}})$ be an iterated multifunction system defined by $F_k(x, y) = \{f_k(x)\} \times Y$ for every $k = \overline{1, n}$. Then*

$$\begin{aligned} h_Z(F_k(x, y), F_k(x', y')) &= h_Z(\{f_k(x)\} \times Y, \{f_k(x')\} \times Y) \\ &= h_X(\{f_k(x)\}, \{f_k(x')\}) = d_X(f_k(x), f_k(x')) \end{aligned}$$

and so $\mathbf{Lip}(F_k) = \mathbf{Lip}(f_k)$.

The attractor of the IMS $\mathcal{S}'$ is $A(\mathcal{S}') = A(\mathcal{S}) \times Y$. F_k are connected multifunctions if and only if Y is a connected space. If Y is not a connected space then $A(\mathcal{S}') = A(\mathcal{S}) \times Y$ is not a connected set. If Y is a connected *space then $A(\mathcal{S}')$ is a connected set if and only if $A(\mathcal{S})$ is a connected set. This is equivalent, from Theorem 1.2, with the fact that the family of sets $(f_k(A(\mathcal{S})))_{k=\overline{1,n}}$ is connected. This is also equivalent with the fact that the family of sets $(f_k(A(\mathcal{S}))\times Y)_{k=\overline{1,n}}$ is connected. In this way we obtain a proof of Theorem 2.1 in this particular case.*

We also remark that in this case $A(\mathcal{S}')$ is arcwise connected if and only if it is connected and Y is arcwise connected.

Example 3.4. *We consider the multifunctions $F, G : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \{\frac{x}{3}, \frac{x}{2}\}$ and by $G(x) = \{\frac{x+2}{3}, \frac{x+1}{2}\}$. The attractor of the IMS $\mathcal{S} = (\mathbb{R}, (F, G))$ is $[0, 1]$. Indeed*

$F_{\mathcal{S}}([0, 1]) = F([0, 1]) \cup G([0, 1]) = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1] = [0, 1]$.

We remark that the attractor is connected but the multifunctions F, G are not connected.

Example 3.5. *We consider the unit interval, $[0, 1]$, with the usual distance, $a \in [0, 1)$ and the multifunctions $F, G : [0, 1] \rightarrowtail [0, 1]$ defined by $F(x) = [0, ax]$ and by $G(x) = [1 - ax, 1]$. Then $F([0, 1]) = [0, a]$ and $G([0, 1]) = [1 - a, 1]$. It follows that the attractor of the IMS $\mathcal{S} = ([0, 1](F, G))$ is $[0, 1]$ if $a \geq 1/2$ and $[0, a] \cup [1 - a, 1]$ if $a < 1/2$. It is obvious that the attractor is connected if and only if $F([0, 1]) \cap G([0, 1]) = \emptyset$.*

Example 3.6. *(an extension of a m-dimensional generalization of the Sierpinsky triangle). Let $\mathbb{R}^m$ be endowed with the euclidead metric (with $m \geq 2$), n be a natural number between 3 and $m+1$, $a_1, a_2, ..., a_n$ be linear independent points from $\mathbb{R}^m$ and $G_i : \mathbb{R}^m \rightarrowtail \mathbb{R}^m$ be multifunctions with compact and connected values such that $\mathbf{Lip}(G_i) \leq 1/6$ and $0_{\mathbb{R}^m} \in G_i(x)$ for $i \in \{1, 2, ..., n\}$. Let also, for $i \in \{1, 2, ..., n\}$, $F_i : \mathbb{R}^m \rightarrowtail \mathbb{R}^m$ be multifunctions defined by $F_i(x) = \dfrac{x + a_i}{2} + G_i(x)$. Then $\mathcal{S} = (\mathbb{R}^m, (F_k)_{k=\overline{1,n}})$ is an IMS such that $A(\mathcal{S})$ is a connected set.*

Indeed, it is easy to see that F_i are multicontractions with compact and connected values such that $\mathbf{Lip}(F_i) \leq 2/3$, $a_i \in F_i(a_i)$ for every $i \in \{1, 2, ..., n\}$ and that $\dfrac{a_i + a_j}{2} \in F_i(a_j) \cap F_j(a_i) \subset F_i(A(\mathcal{S})) \cap F_j(A(\mathcal{S}))$ for every $i, j \in \{1, 2, ..., n\}$. From Theorem 2.1 it results that $A(\mathcal{S})$ is a connected set. In this case we can establish that $A(\mathcal{S})$ is a connected set without finding $A(\mathcal{S})$.

Example 3.7. *Let us consider the space $X = [0, 1]^2$ endowed with the maximum metric, that is $d((x, y), (x_1, y_1)) = \max\{|x - x_1|, |y - y_1|\}$, the function $g : (0, 1] \to [0, 1]$ defined by $g(x) = \dfrac{1}{2}(1 + \sin(\dfrac{1}{x}))$, the Warsaw sine curve $C = \{0\} \times [0, 1] \cup \{(x, y) \in [0, 1]^2 | x > 0$ and $y = \dfrac{1}{2}g(x)\}$ and π :*

$[0,1]^2 \to [0,1]$ *be the function defined by* $\pi(x,y) = x$. *Let us denote by* $C_{a,b}$ *the set* $C \cap \pi^{-1}([a,b])$, *where* $0 \le a \le b \le 1$. *We consider the multifunctions* $F_1, F_2, F_3, : C \rightarrowtail C$ *defined by* $F_1(x,y) = [0,1] \times \{0\}$, $F_2(x,y) = C_{1/2,1}$, $F_3(0,y) = [0,1] \times \{0\}$ *and by* $F_3(x,y) = C_{x/10,3x/4}$ *if* $x > 0$ *and the "IMS"* $\mathcal{S} = (C, (F_1, F_2, F_3))$. *It can be seen that* C *is a connected but not arcwise connected set,* $\mathbf{Lip}(F_1) = \mathbf{Lip}(F_2) = 0$, F_1, F_2, F_3 *are continuous multifunctions with compact and arcwise connected values such that* $F_{\mathcal{S}}(C) = C$ *and for every* $(x,y) \in C$ *the sequence* $(F_{\mathcal{S}}^{[n]}(x,y))_n$ *is convergent to* C, *where* $F_{\mathcal{S}}^{[n]}$ *denotes* $F_{\mathcal{S}} \circ F_{\mathcal{S}} \circ ... \circ F_{\mathcal{S}}$. *We also remark that* F_3 *is not a multicontraction. To see this let us consider* $x = \dfrac{1}{2n\pi}$, *where* $n \ge 2$ *is a natural number, and a* $\varepsilon > 0$. *Then if* ε *is small enough we have*

$$\begin{aligned} h(F_3(x,g(x)), F_3(x-\varepsilon, g(x-\varepsilon))) &= h(C_{x/10,3x/4}, C_{(x-\varepsilon)/10,3(x-\varepsilon)/4}) \\ &\ge d((\frac{x-\varepsilon}{10}, g(\frac{x-\varepsilon}{10})), C_{x/10,3x/4}) \\ = d((\frac{x}{10}, g(\frac{x}{10})), (\frac{x-\varepsilon}{10}, g(\frac{x-\varepsilon}{10}))) &\ge |g(\frac{x}{10}) - g(\frac{x-\varepsilon}{10})|. \end{aligned}$$

If F_3 *is a contraction, taking account that* $|g'(x)| = \dfrac{1}{x^2} > 1$, *we have*

$$\begin{aligned} |g(x) - g(x-\varepsilon)| &= d((x,g(x)), (x-\varepsilon, g(x-\varepsilon)) \\ \ge h(F_3(x,g(x)), F_3(x-\varepsilon, g(x-\varepsilon))) &\ge |g(\frac{x}{10}) - g(\frac{x-\varepsilon}{10})|. \end{aligned}$$

This is in contradiction with the fact that $\lim_{\varepsilon \to 0} \dfrac{g(x) - g(x-\varepsilon)}{g(\frac{x}{10}) - g(\frac{x-\varepsilon}{10})} = 10$.

Example 3.8. *In [8], an example of a locally connected continuum which is not an IFS attractor is given.*

Open question. It is an open question that if in Theorem 2.1 we suppose that the multifunctions have arcwise connected (and compact) values then $A(\mathcal{S})$ is arcwise connected. The same question can be put for Theorem 1.3. I think that the answer is not true but I have not found a counterexample. Example 3.7. is an attempt to find an example. The only problem is that F_3 is not a contraction. At least in the case of Theorem 2.1 it seems to me that to find a counterexample is not very easy. In this case the attractor $A = A(F)$ of a multicontraction F with compact and arcwise connected values must contain at least two arcwise connected components which are dense in

A. It results from the fact that every arcwise connected component of A is dense in A. To see this let us denote by A_x the arcwise connected component of an $x \in A$ and remark that for every $x \in A$ and natural number n there exists an $x_n \in A$ such that $x \in F^{[n]}(x_n)$. But the set $F^{[n]}(x_n)$ is arcwise connected and so $F^{[n]}(x_n) \subset A_x$. Also $h(F^{[n]}(x_n), A) \leq \dfrac{\mathbf{Lip}^n(F)}{1-\mathbf{Lip}(F)}\delta(A)$, where $\delta(A) = \sup\limits_{x,y \in A} d(x,y)$. This gives the desired conclusion. Although such sets exists one should also define the multifunctions and can encounter similar problems with those from Example 3.7. The difference between the two cases, connected and arcwise connected, consists of the fact that the limit (in the Hausdorff-Pompeiu distance) of a sequence of compact and arcwise connected sets could not be arcwise connected.

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