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A FEW RESULTS ON ARCHIMEDEAN SETS

Abstract

In a 1990 paper by R. Mabry, it is shown that for any constant $a \in (0,1)$ there exist sets A on the real line with the property that for any bounded interval I, $\frac{\mu(A \bigcap I)}{\mu(I)} = a$, where μ is any Banach measure. Many of the constructed sets are Archimedean sets, which are sets that satisfy $A + t = A$ for densely many $t \in \mathbb{R}$. In that paper it is shown that if A is an arbitrary Archimedean set, then for a *fixed* μ, $\frac{\mu(A \bigcap I)}{\mu(I)}$ is constant. (This constant is called the μ-shade of A and is denoted $\mathrm{sh}_\mu A$.) A problem is then proposed: For any Archimedean set A, any fixed Banach measure μ, and any number b between 0 and $\mathrm{sh}_\mu A$, does there exist a subset B of A such that $\frac{\mu(B \bigcap I)}{\mu(I)} = b$ for any bounded interval I? In this paper, we partially answer this question. We also derive a lower bound formula for the μ-shade of the difference set of an arbitrary Archimedean set. Finally, we generalize an intersection result from Mabry's original paper.

1 Introduction.

In this paper we assume the standard definitions for the sum of sets and the scalar multiple of a set. That is, $C + t = \{c + t | c \in C\}$ and $sC = \{sc | c \in C\}$.

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We also define $A-A=\{a_1-a_2|a_1,a_2\in A\}$ to be the difference set of a given set $A\subseteq\mathbb{R}$.

Let μ be a finitely additive, isometry-invariant extension of the Lebesgue measure on $2^{\mathbb{R}}$. Then μ is a measure with the property that $\mu(E+t)=\mu(E)$ and $\mu(-E)=\mu(E)$ for every $t\in\mathbb{R}$ and every set $E\subset\mathbb{R}$. Also, $\mu(E)=\lambda(E)$ if E is Lebesgue measurable and λ is the Lebesgue measure. (Such a measure is called a *Banach measure*; such measures exist as a consequence of the axiom of choice, which we freely assume.) Mabry [4] has shown that for each $\alpha\in[0,1]$ there exist sets K called *shadings*, with the following property: Given any bounded Lebesgue measurable set $E\subset\mathbb{R}$ with positive measure and any Banach measure μ, $\mu(K\cap E)/\mu(E)=\alpha$. It is clear that this "shade density" or *shade* is an extension of the usual Lebesgue density.

We will now briefly review some of the fundamental ideas used in [4]. To show a shading exists in the case where α is of the form $1/a$, where $a\in\mathbb{N},\mathbb{N}=\{1,2,\cdots\}$, define an equivalence relation $\sim$ on $\mathbb{R}$ as follows: $x\sim y\Leftrightarrow x-y\in h\mathbb{Z}+\mathbb{Z}$, where h is a fixed irrational number. Let Γ be a set of numbers consisting of exactly one element from each equivalence class so formed. (That is, let Γ be a selector for $\sim$.) Finally, by letting $K_{a,b}=\Gamma+h(a\mathbb{Z}+b)+\mathbb{Z}$, where $b\in\mathbb{Z}$, it can be shown $K_{a,b}$ has shade $\frac{1}{a}$. To see this, first note that $\mathbb{R}=\Gamma+h\mathbb{Z}+\mathbb{Z}$. Also, $K_{a,b+c}=K_{a,b}+r_c$, where r_c is any element of the set $h\left(a\mathbb{Z}+c\right)+\mathbb{Z}$. Since h is irrational, this set is dense and so we may choose $r_c<\varepsilon$, where ε is any arbitrary positive number. If J is an arbitrary interval and $J^+=J\bigcup(J+\varepsilon)$, then

$$\begin{aligned} a\mu(K_{a,b}\bigcap J) &= \sum_{c=0}^{a-1}\mu(K_{a,b+c}\bigcap(J+r_c))\leq\sum_{c=0}^{a-1}\mu(K_{a,b+c}\bigcap J^+) \\ &= \mu((\bigcup_{c=0}^{a-1}K_{a,b+c})\cap J^+)=\mu(\mathbb{R}\cap J^+)=\mu(J)+\varepsilon. \end{aligned}$$

Similarly, $a\mu(K_{a,b}\bigcap J)\geq\mu(J)-\varepsilon$. Since ε was arbitrary, the result follows. It is also shown in Mabry's paper that shadings of irrational shade can be constructed by taking countable unions of the $K_{a,b}$'s. The extension from intervals J to arbitrary Lebesgue measurable sets E is demonstrated in Theorem 3.11 in Mabry's paper.

2 Subsets of Archimedean Sets and other μ-Shadings.

In [6], Simoson defines an *Archimedean set* A to be a set with the property that $A+t=A$ for densely many $t\in\mathbb{R}$. (We call such t's the *translators* of

A, and denote this set of such t by $\tau(A)$). It is easy to see that the shadings $K_{a,b}$ are Archimedean sets. One of the results proved in [4] is that if A is an Archimedean set, then for each *fixed* Banach measure μ, the quantity $\frac{\mu(A\cap I)}{\mu(I)}$ is constant for any bounded interval I of positive Lebesgue measure (Theorem 6.1 in [4]). This quantity is called the μ-shade of A, denoted $\mathrm{sh}_\mu A$, and the set itself is referred to as a μ-shading. Problem 4 is then posed: Given an Archimedean set A and a number $b \in (0, \mathrm{sh}_\mu A)$, does there exist an Archimedean subset B of A such that $\mathrm{sh}_\mu B = b$? The next theorem is a partial answer to this question.

Theorem 2.1. *Let μ be a fixed Banach measure, let A be an Archimedean set of positive μ-shade a, and let $0 < b < a$. If $\tau(A)$ has two numbers t_1, t_2 such that $\dfrac{t_1}{t_2}$ is irrational, then there exists a subset B of A that has μ-shade b.*

Proof. Let $b = \dfrac{a}{n}$ for some integer $n \geq 2$. Define an equivalence relation on A as follows: for $x, y \in A, x \sim y \Leftrightarrow x - y \in t_1\mathbb{Z} + t_2\mathbb{Z}$. Let Γ_A be a selector for $\sim$ and consider the set $\Gamma_A + t_1\mathbb{Z} + t_2\mathbb{Z}$. Since $\Gamma_A \subseteq A$ and $\tau(A)$ is an additive group, this set is contained in A. Also, since any element $t \in A$ is equivalent to some $\gamma_t \in \Gamma_A$, $t - \gamma_t \in t_1\mathbb{Z} + t_2\mathbb{Z}$ which implies $t \in \Gamma_A + t_1\mathbb{Z} + t_2\mathbb{Z}$ and so $A \subseteq \Gamma_A + t_1\mathbb{Z} + t_2\mathbb{Z}$. We conclude that $A = \Gamma_A + t_1\mathbb{Z} + t_2\mathbb{Z}$. We now claim that $B = \Gamma_A + t_1(n\mathbb{Z}) + t_2\mathbb{Z}$ is a subset of A having μ-shade b. The rest of the proof is similar to Theorem 3.6 in [4]. Let I be a bounded, nontrivial interval, let $\varepsilon > 0$, and let $r_i \in (t_1(n\mathbb{Z} + i) + t_2\mathbb{Z}) \bigcap \left(0, \dfrac{\varepsilon}{n}\right)$ for $i = 1, 2, \cdots, n-1$, and $r_0 = 0$. (Note: we can do this because $t_1(n\mathbb{Z} + i) + t_2\mathbb{Z} = t_2\left(\dfrac{t_1}{t_2}(n\mathbb{Z} + i) + \mathbb{Z}\right)$ is a dense set.) Now let $B_i = \Gamma_A + t_1(n\mathbb{Z} + i) + t_2\mathbb{Z}$ and $I^+ = I \cup (I + \epsilon)$. Note that A is the disjoint union of the $B_i, i = 0, 1, \cdots, n-1$. Then

$$\begin{aligned} n\mu(B \cap I) &= \sum_{i=0}^{n-1} \mu((B \cap I) + r_i) = \sum_{i=0}^{n-1} \mu(B_i \cap (I + r_i)) \\ &\leq \sum_{i=0}^{n-1} \mu\left(B_i \cap I^+\right) \leq \sum_{i=0}^{n-1} \mu(B_i \cap I) + \varepsilon \\ &= \mu(A \cap I) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $n\mu(B \cap I) \leq \mu(A \cap I)$. Similarly we can show $n\mu(B \cap I) \geq \mu(A \cap I)$. The result follows.

Now consider the general case. Let $\sum_{i=1}^{\infty} \frac{d_i}{2^i}$ be a binary expansion of $\frac{b}{a}$; here $d_i = 0$ or $d_i = 1$ for all i. Define $K_{v,w}^{(A)} = \Gamma_A + t_1 (v\mathbb{Z} + w) + t_2\mathbb{Z}$, where Γ_A is the same as in the first case and $v, w \in \mathbb{N}, w < v$. Then as in Case 1, $K_{v,w}^{(A)}$ is an Archimedean subset of A with μ-shade $\frac{a}{v}$. Let $B = \bigcup_{i \in M} K_{2^i,2^{i-1}}^{(A)}$, where $M = \{i : d_i = 1\}$. (We are using the construction from Corollary 3.9 in [4].) Clearly each $K_{2^i,2^{i-1}}^{(A)}$ has μ-shade $\frac{a}{2^i}$. To show this set satisfies the theorem we only need a variation of Theorem 3.8 of [4] and to show all of the $K_{2^i,2^{i-1}}$'s are pairwise disjoint. (Theorem 3.8 says that if a countable union of disjoint μ-shadings exhausts A, and if the sum of their μ-shades is equal to a, then any subcollection of that union is itself a μ-shading with μ-shade equal to the sum of the μ-shades in the subcollection.) Suppose $x \in K_{2^i,2^{i-1}} \bigcap K_{2^j,2^{j-1}}$. Then we can let $x = \gamma_1 + t_1 \left(2^i j + 2^{i-1}\right) + t_2 k = \gamma_2 + t_1 \left(2^l m + 2^{l-1}\right) + t_2 n$ for positive integers i, j, k, l, m, n. But then $\gamma_1 = \gamma_2$, since otherwise $\gamma_1 - \gamma_2 \in t_1\mathbb{Z} + t_2\mathbb{Z}$. This implies that $k = n$ and $2^i j + 2^{i-1} = 2^l m + 2^{l-1}$, since it follows that $\frac{t_1}{t_2}$ is rational. But this is impossible unless $i = l$. We conclude all of the $K_{2^i,2^{i-1}}^{(A)}$'s are pairwise disjoint. Hence B has μ-shade $\sum_{i \in M}^{\infty} d_i \left(\frac{a}{2^i}\right) = a \left(\frac{b}{a}\right) = b$. □

Corollary 2.2. *For sets B and A as set forth in Theorem 2.1 we have that*

$$\frac{\mathrm{sh}_\mu (cB)}{\mathrm{sh}_\mu (cA)} = \frac{\mathrm{sh}_\mu B}{\mathrm{sh}_\mu A}$$

for any nonzero real number c and any Banach measure μ.

PROOF. As before, we have $B = \bigcup_{i \in M} K_{2^i,2^{i-1}}^{(A)} \Rightarrow cB = \bigcup_{i \in M} \left(cK_{2^i,2^{i-1}}^{(A)}\right)$. It can be shown using the ideas from the previous proof that $\mathrm{sh}_\mu \left(cK_{2^i,2^{i-1}}^{(A)}\right) = \frac{\mathrm{sh}_\mu (cA)}{2^i}$. (We note that $\mathrm{sh}_\mu (cA)$ exists because cA is Archimedean.) Hence, by Theorem 3.8 of [4],

$$\frac{\mathrm{sh}_\mu(cB)}{\mathrm{sh}_\mu(cA)} = \frac{\mathrm{sh}_\mu\left(\bigcup_{i\in M}\left(cK^{(A)}_{2^i,2^{i-1}}\right)\right)}{\mathrm{sh}_\mu(cA)} = \frac{\sum_{i=1}^{\infty} d_i \mathrm{sh}_\mu\left(cK^{(A)}_{2^i,2^{i-1}}\right)}{\mathrm{sh}_\mu(cA)}$$

$$= \frac{\sum_{i=1}^{\infty}\left(\frac{d_i}{2^i}\mathrm{sh}_\mu(cA)\right)}{\mathrm{sh}_\mu(cA)} = \sum_{i=1}^{\infty}\frac{d_i}{2^i} = \frac{\mathrm{sh}_\mu B}{\mathrm{sh}_\mu A}.$$

□

In [1] it is shown that the outer and inner Lebesgue measures of sets that exhibit certain invariant properties take on only certain values. More specifically, if $C + t = C$ for densely many $t \in \mathbb{R}$ or if $sC = C$ for densely many $s \in \mathbb{R}$, then the outer measure of a set of the form $C \cap B$, where B is a Borel set, is always either 0 or $\lambda(B)$. The same is true for the inner measure of such a set. Mabry has already shown that Archimedean sets are μ-shadings. We will now show that sets S that satisfy $cS = S$ for densely many $c \in \mathbb{R}$ are also μ-shadings for certain μ's. We will also show that a subset result similar to Theorem 2.1 can be proved for such a set. The two proofs that follow require Corollary 11.5 of [7], which guarantees the existence of a Banach measure μ satisfying $\mu(cA) = |c|\mu(A)$ for any nonzero constant $c \in \mathbb{R}$ and any set $A \subset \mathbb{R}$. We define $M(S)$ to be the set of numbers c satisfying $cS = S$.

Theorem 2.3. *Let S be a set satisfying $cS = S$ for densely many $c \in \mathbb{R}$, and let μ be a Banach measure satisfying $\mu(cA) = |c|\mu(A)$ for any nonzero constant $c \in \mathbb{R}$ and any set $A \subset \mathbb{R}$. Then S is a μ-shading.*

Proof. First we show that if $c_1, c_2 \in M(S)$, where $c_2 > c_1 \geq 0$, then $\mu(S\bigcap[c_1,c_2]) = (c_2 - c_1)\mu(S\bigcap[0,1])$. (This shows $\frac{\mu(S\bigcap I)}{\mu(I)} = \mu\left(S\bigcap[0,1]\right)$ for $I = [c_1, c_2]$.) We have

$$\begin{aligned}(c_2 - c_1)\mu\left(S\bigcap[0,1]\right) &= c_2\mu\left(S\bigcap[0,1]\right) - c_1\mu\left(S\bigcap[0,1]\right)\\ &= \mu\left(c_2S\bigcap c_2[0,1]\right) - \mu\left(c_1S\bigcap c_1[0,1]\right)\\ &= \mu\left(S\bigcap[0,c_2]\right) - \mu\left(S\bigcap[0,c_1]\right)\\ &= \mu\left(S\bigcap[c_1,c_2]\right).\end{aligned}$$

The cases where $c_1 < c_2 \leq 0$ and $c_1 < 0, c_2 > 0$ can be proven similarly, so in all cases, $\mu\left(S \bigcap [c_1, c_2]\right) = \mu\left([c_1, c_2]\right) \mu\left(S \bigcap [0,1]\right)$. If the endpoints c_1, c_2 are not in $M(S)$, then we can choose endpoints that *are* in $M(S)$ that are close to c_1 and c_2 and make a limiting argument to show that for any finite interval I, $\mu\left(S \bigcap I\right) = \mu(I)\, \mu\left(S \bigcap [0,1]\right)$. □

Theorem 2.4. *Let μ be a Banach measure satisfying $\mu(cA) = |c|\mu(A)$ for every nonzero constant c and every set $A \subset \mathbb{R}$. Also let S be a set satisfying $cS = S$ for densely many $c \in \mathbb{R}$, and assume there exist $m_1, m_2 \in M(S)$ satisfying $m_1 > 0, m_2 < 0$, and $m_1^q \neq |m_2|$ for each $q \in \mathbb{Q}$. If $a = \mathrm{sh}_\mu S$, then for every b in the interval $(0, a)$, there exists a subset B of A that has μ-shade b.*

PROOF. It is easy to verify that $x \sim y \Leftrightarrow \dfrac{x}{y} \in m_1^{\mathbb{Z}} m_2^{\mathbb{Z}}$ for $x, y \in S$ is an equivalence relation. (Here $m_i^{\mathbb{Z}} = \{m_i^z | z \in \mathbb{Z}\}$.) As in the proof of Theorem 2.1 we choose one element γ from each equivalence class to form the set Γ. It follows that $S = m_1^{\mathbb{Z}} m_2^{\mathbb{Z}} \Gamma$.

We now show that the set $m_1^{\mathbb{Z}} m_2^{\mathbb{Z}}$ is dense. Since $m_1^{\mathbb{Z}} (m_2)^{2\mathbb{Z}}$ is a set of positive numbers, we can say $\ln\left(\mathrm{m}_1^{\mathbb{Z}} (\mathrm{m}_2)^{2\mathbb{Z}}\right) = \mathbb{Z}\ln(\mathrm{m}_1) + 2\mathbb{Z}\ln|\mathrm{m}_2|$. This is dense if $\dfrac{\ln(m_1)}{\ln|m_2|} \notin \mathbb{Q}$, which is true by assumption. So $m_1^{\mathbb{Z}} (m_2)^{2\mathbb{Z}}$ is dense in $\mathbb{R}^+$, which implies $m_1^{\mathbb{Z}} (m_2)^{2\mathbb{Z}+1}$ is dense in $\mathbb{R}^-$. This implies $m_1^{\mathbb{Z}} m_2^{\mathbb{Z}}$ is dense in $\mathbb{R}$.

Now let $S_{2^n} = (m_1)^{2^n \mathbb{Z}} (m_2)^{\mathbb{Z}} \Gamma$. Clearly $S = \bigcup_{i=0}^{2^n - 1} c_i S_{2^n}$, where c_i is any number in the dense set $(m_1)^{2^n \mathbb{Z} + i} m_2^{\mathbb{Z}}$. Thus for any finite interval I,

$$\begin{aligned} \mu\left(S \bigcap I\right) &= \mu\left(\bigcup_{i=0}^{2^n-1} c_i S_{2^n} \bigcap I\right) = \sum_{i=0}^{2^n-1} \mu\left(c_i S_{2^n} \bigcap I\right) \\ &= \sum_{i=0}^{2^n-1} c_i \mu\left(S_{2^n} \bigcap \frac{I}{c_i}\right) \end{aligned}$$

Since each c_i can be made as close to 1 as we like, for any $\varepsilon > 0$, we can choose the c_i so that $\mu\left(S_{2^n} \bigcap I\right) - \dfrac{\varepsilon}{2^n} < c_i \mu\left(S_{2^n} \bigcap \dfrac{I}{c_i}\right) < \mu\left(S_{2^n} \bigcap I\right) + \dfrac{\varepsilon}{2^n}$ for all i, which implies $2^n \mu\left(S_{2^n} \bigcap I\right) - \varepsilon < \mu\left(S \bigcap I\right) < 2^n \mu\left(S_{2^n} \bigcap I\right) + \varepsilon$.

Since ε can be made arbitrarily small, we have $\dfrac{\mu\left(S_{2^n}\bigcap I\right)}{\mu\left(S\bigcap I\right)} = \dfrac{1}{2^n}$. Now let $\displaystyle\sum_{i=1}^{\infty}\frac{d_i}{2^i}$ be a binary expansion of $\dfrac{b}{a}$, where $d_i = 0$ or 1 for all i, and define $S_{2^n,2^{n-1}} = (m_1)^{2^n\mathbb{Z}+2^{n-1}}(m_2)^{\mathbb{Z}}\Gamma$. Finally, choose $B = \displaystyle\bigcup_{i\in M} S_{2^i,2^{i-1}}$, where $M = \{i|d_i = 1\}$. The rest of the proof is similar to the last part of the proof of Theorem 2.1. □

3 The μ-Shade of the Difference Set of an Archimedean Set.

The next theorem involves estimating the μ-shade of $A - A$, where A is Archimedean. We note that $A - A$ will *have* a μ-shade because $A - A$ is also Archimedean. The proof is similar to the proof of Proposition 1 in [2, p. 126], where it is proved that if A is a (nonmeasurable) set satisfying $\mu\left(A\bigcap I\right) > \dfrac{1}{2}\mu(I)$ on some interval I, then $A - A$ contains an interval about 0. Hence if A is Archimedean with this property, $\mathrm{sh}_\mu A > 1/2$, and so $\mathrm{sh}_\mu(A - A) = 1$. We will weaken this assumption to prove a more general theorem, although our result will be an inequality instead of an equality. But first, we need a lemma. (The original proof of this lemma was a bit longer; the proof that follows is due to Mabry.)

Lemma 3.1. *Let μ be a Banach measure and let H be an Archimedean set with $\mathrm{sh}_\mu(H) > \frac{k-1}{k}$, where $k \geq 2$ is an integer. Then there exist distinct $h_1, h_2, \cdots, h_k \subset \mathbb{R}$ such that $\mathrm{sh}_\mu\left(\bigcap_{i=1}^{k}(H - h_i)\right) > 0$.*

Proof. For any $h_1, h_2, \cdots, h_k$, one has

$$\mathrm{sh}_\mu\left(\bigcap_{i=1}^{k}(H-h_i)\right) = 1 - \mathrm{sh}_\mu\left(\bigcup_{i=1}^{k}(H-h_i)^c\right) \geq 1 - \sum_{i=1}^{k}\mathrm{sh}_\mu(H-h_i)^c$$
$$= 1 - k\left(1 - \mathrm{sh}_\mu(H)\right) = k\,\mathrm{sh}_\mu(H) - (k-1).$$

Thus $\mathrm{sh}_\mu\left(\bigcap_{i=1}^{k}(H-h_i)\right) > 0$ if $\mathrm{sh}_\mu H > \frac{k-1}{k}$. For $1 \leq n \leq k$ it is also clear that $\mathrm{sh}_\mu\left(\bigcap_{i=1}^{n}(H-h_i)\right) > \frac{k-n}{k} > 0$. The h_i can therefore be chosen recursively so that they are distinct. Specifically, let h_1 be arbitrary and take $h_n \in \bigcap_{i=1}^{n-1}(H-h_i)$ for $1 < n \leq k$, such that $h_n \notin \{h_1, h_2, \cdots, h_{n-1}\}$. This is possible because h_n is chosen from a set of positive μ-shade, which must be (uncountably) infinite. $\square$

Theorem 3.2. *Let A be an Archimedean set satisfying* $\mathrm{sh}_\mu A > \frac{1}{k+1}$ *for an integer* $k \geq 1$. *Then* $\mathrm{sh}_\mu (A - A) \geq \frac{1}{k}$.

PROOF. Assume to the contrary that $\mathrm{sh}_\mu (A - A) < \frac{1}{k}$ and let $H = (A - A)^c$. Clearly H is Archimedean and $\mathrm{sh}_\mu (H) > \frac{k-1}{k}$. Choose distinct $h_1, h_2, \cdots, h_k$ as per Lemma 3.1 and then take $h_{k+1} \in \bigcap_{i=1}^{k} (H - h_i) \setminus \{h_1, h_2, \cdots, h_k\}$, this being possible since the latter intersection has positive μ-shade. It follows that the sets $A + h_1, A + h_2, \cdots, A + h_k$ are pairwise disjoint. (To see this, note that if $x \in (A + h_j) \cap (A + h_i)$ for $j \neq i$, then $h_j - h_i \in A - A$, which is impossible.) But the sum of μ-shades of disjoint μ-shadings cannot exceed unity, so $1 \geq \sum_{i=1}^{k+1} \mathrm{sh}_\mu (A + h_i) = (k+1)\,\mathrm{sh}_\mu (A)$, which implies that $\mathrm{sh}_\mu A \leq \frac{1}{k+1}$, a contradiction. $\square$

4 An Intersection Result.

In his paper, Mabry proved that if $f : \mathbb{R} \to [0,1]$ is a continuous function, then there exists a point set F such that $\lim_{\mu(I_x)\to 0} \dfrac{\mu\left(F \bigcap I_x\right)}{\mu(I_x)} = f(x)$ for all Banach μ and for all $x \in \mathbb{R}$, where I_x is a closed interval about x. (A. B. Kharazishvili constructs something similar in [3].) Mabry also proved ([4, Example 5.4]) that for any finite collection $v_1, v_2, ..., v_n$ of real numbers in $(0,1)$, there exist shadings $C_1, C_2, ..., C_n$ with the property that for any set M of distinct integers in $\{1, 2, \cdots, n\}$, $\mathrm{sh}\left(\bigcap_{j\in M} C_j\right) = \prod_{j\in M} v_j$. We will combine these results to prove that this intersection property holds for countably many continuous functions.

Theorem 4.1. *Let $\{f_i\}_{i=1}^{\infty}$ be a set of continuous functions, $f_i : \mathbb{R} \to [0,1]$. Then there exist subsets $\{F_i\}_{i=1}^{\infty}$ of $\mathbb{R}$ such that for each finite subset M of $\mathbb{N}$,*

$$\lim_{\mu(I_x)\to 0} \frac{\mu\left(\left(\bigcap_{i\in M} F_i\right) \bigcap I_x\right)}{\mu(I_x)} = \prod_{i\in M} f_i(x), \tag{1}$$

where $x \in \mathbb{R}$ is arbitrary and I_x is a closed interval centered at x.

Before proving the theorem, we need a few lemmas.

Lemma 4.2. *For $i = 1, 2, \cdots, t$, let $\{p_i\}$ be distinct primes and let $\{m_i, a_i\}$ be pairs of nonnegative integers. Then $(x_1, x_2, \cdots, x_t)$ is an integer solution of the equation $p_1^{m_1}x_1 + a_1 = p_2^{m_2}x_2 + a_2 = \cdots = p_t^{m_t}x_t + a_t$ if and only if $x_i = \left(\prod_{j \neq i} p_j^{m_j}\right) k + c_i$ for all i, where $k \in \mathbb{Z}$ and $(c_1, c_2, \cdots, c_t)$ is any single integer solution of the equation.*

Proof. Fix a solution $(c_1, c_2, \cdots, c_t)$. Let x_0 denote the common value of $p_i^{m_i}c_i + a_i$. By the Chinese Remainder Theorem (see, e.g., [5]), x is a solution of the set of congruences

$$x \equiv a_1 \, (\mathrm{mod} p_1^{m_1}), \quad x \equiv a_2 \, (\mathrm{mod} p_2^{m_2}), \quad \ldots \quad , x \equiv a_t \, (\mathrm{mod} p_t^{m_t})$$

if and only if $x = x_0 + km$, where k is an integer and $m = p_1^{m_1}p_2^{m_2} \cdots p_t^{m_t}$. Clearly x is a solution of the above congruences if and only if $x = a_1 + k_1p_1^{m_1} = a_2 + k_2p_2^{m_2} = \cdots = a_t + k_tp_t^{m_t}$ for integers k_i. Thus we can say that $(k_1, k_2, \cdots, k_t)$ is a solution of the equation mentioned in the theorem if and only if there exists a $k \in \mathbb{Z}$ such that $x_0 + km = a_i + k_ip_i^{m_i}$ for all i. After a little algebra, this is seen to be equivalent to the conditions $k_i = k\dfrac{m}{p_i^{m_i}} + c_i$. □

Lemma 4.3. *Let x be a real number written in base p, where $p > 1$ is an integer. Assume that the base-p representation of x never ends in an infinite string of $p - 1$'s.* (For example, if $x = 0.23\overline{4}$ in base-5, we write $x = 0.24$.) *Then for any $N \in \mathbb{N}$, there exists an $\varepsilon > 0$ such that the base-p representation of every number in $(x - \varepsilon, x)$ begins with the same N digits after the radix point and the base-p representation of every number in $[x, x + \varepsilon)$ begins with the same N digits after the radix point.*

The radix point in base 10 is the decimal point. From now on, we will refer to the n^{th} digit after the radix point as the digit in the n^{th} *radix place*. We omit the obvious proof of Lemma 4.3, but note that the N digits corresponding to $(x - \varepsilon, x)$ are, in general, different than the N digits corresponding to $[x, x + \varepsilon)$ whenever x terminates in base p. We use the notation x^- to represent the rational number in base p whose only nonzero digits after the radix point are the N digits corresponding to $(x - \varepsilon, x)$. The notation x^+ has a similar meaning.

Proof of Theorem 4.1. Consider the set $K_{p^k, lp^{k-1}} = \Gamma + h\left(p^k\mathbb{Z} + lp^{k-1}\right) + \mathbb{Z}$, where Γ is the same selector set mentioned in the introduction, h is the same irrational constant, and $1 \le l \le p-1$. It is easy to show that if $k_1 \ne k_2$ or if $l_1 \ne l_2$, then $K_{p^{k_1},l_1 p^{k_1-1}} \cap K_{p^{k_2},l_2 p^{k_2-1}} = \emptyset$. Let $\{p_i\}$ denote the usual sequence $2, 3, 5, \cdots$ of primes, and let $C_k^{(l)}(i) = K_{p_i^k, l p_i^{k-1}}$, where $k \in \mathbb{N}$ and $1 \le l \le p_i - 1$. Then for each fixed i the sets $C_k^{(l)}(i)$ are pairwise disjoint shadings (for distinct pairs (k,l)) with shade $1/p_i^k$. We will associate these shadings with the nonzero digits $l = 1, 2, \cdots, p_i - 1$ in the k^{th} radix place of a number expressed in base p_i.

We will now construct the point set F_i using the i^{th} prime p_i. We assume $f_i(x_0)$ is written in base p_i and also we make the same assumption about numbers written in base p_i that we made in Lemma 4.3: The base-p_i representation of a number never ends with an infinite string of $p_i - 1$'s. Let $S_k^{(j)}(i)$ be the set of x-values such that $f_i(x)$ has a j in its k^{th} radix place, where $0 \le j < p_i, k \in \mathbb{N}$. (Notice that $S_k^{(j)}(i)$ is Lebesgue measurable, being the inverse image of a finite union of intervals under the continuous function f_i.) Let $F_i = \bigcup_{j,k} \left[S_k^{(j)}(i) \bigcap \left(\bigcup_{1 \le l \le j} C_k^{(l)}(i) \right) \right]$ for $i \in \mathbb{N}$, and let $M \subset \mathbb{N}$ be finite. (For $j = 0$, the expression $S_k^{(j)}(i) \bigcap \left(\bigcup_{1 \le l \le j} C_k^{(l)}(i) \right)$ is understood to be empty.) Now fix $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. We will show that the limit in (1) holds for this arbitrary x_0. For now, assume $f_i(x_0) > 0$ for all $i \in M$, and choose $N \in \mathbb{N}$ large enough so that $\prod_{i \in M} f_i(x_0) - \prod_{i \in M} \left(f_i(x_0) - \frac{1}{p_i^N} \right) < \varepsilon$, $\frac{|M|}{2^N} < \varepsilon$, and $f_i(x_0) - \frac{1}{p_i^N} > 0$ for all $i \in M$. From Lemma 4.3 we know $\exists\ \varepsilon' > 0$ such that the base-p_i representation of every number in $(f_i(x_0) - \varepsilon', f_i(x_0))$ begins with the same N digits after the radix point and the base-p_i representation of every number in $[f_i(x_0), f_i(x_0) + \varepsilon')$ begins with the same N digits after the radix point. (In the above statement ε' depends, in general, on i, but we can always set $\varepsilon' = \min\{\varepsilon_i'\}$ and use the same ε' for every i.) Let $f_i(x_0)^-$ and $f_i(x_0)^+$ have meanings similar to x^- and x^+ mentioned after Lemma 4.3. Now define $I_{x_0}^+(f_i) = \{x \in I_{x_0} | f_i(x) \in [f_i(x_0), f_i(x_0) + \varepsilon')\}$ and $I_{x_0}^-(f_i) = \{x \in I_{x_0} | f_i(x) \in (f_i(x_0) - \varepsilon', f_i(x_0))\}$, where I_{x_0} is an interval centered at x_0 satisfying $f_i(I_{x_0}) \subset (f_i(x_0) - \varepsilon', f_i(x_0) + \varepsilon')$ for all $i \in M$.

Let the k^{th} digit after the radix point of $f_i(x_0)^+$ be denoted $m_k^+(i)$. If $j = m_k^+(i), k \le N$, then $S_k^{(j)}(i) \bigcap I_{x_0}^+(f_i) = I_{x_0}^+(f_i)$; otherwise $S_k^{(j)}(i) \bigcap I_{x_0}^+(f_i) =$

$\emptyset$. Hence for $k \leq N$, $\bigcap_{i \in M} \left(S_k^{(j)}(i) \bigcap I_{x_0}^+(f_i) \right) = \bigcap_{i \in M} I_{x_0}^+(f_i)$ if $j = m_k^+(i)$ and

$$\bigcap_{i \in M} \left(S_k^{(j)}(i) \bigcap I_{x_0}^+(f_i) \right) = \emptyset$$

otherwise. (For each i in the intersection above we fix the j, k pair, but each j, k pair is, in general, different for each i.) Now let $I_1 = \bigcap_{i \in M} I_{x_0}^+(f_i)$ and let $G = \bigcap_{i \in M} \left[\left(\bigcup_{k \leq N, j} \left[S_k^{(j)}(i) \bigcap \left(\bigcup_{1 \leq l \leq j} C_k^{(l)}(i) \right) \right] \right) \bigcap I_1 \right]$. Also let

$$x \in \left(\left(\bigcap_{i \in M} F_i \right) \bigcap I_1 \right) \setminus G.$$

Then x is contained in $\bigcap_{i \in M} \left[\left(\bigcup_{j,k} \left[S_k^{(j)}(i) \bigcap \left(\bigcup_{1 \leq l \leq j} C_k^{(l)}(i) \right) \right] \right) \bigcap I_1 \right]$. But x is not in G, so x must be in some set of the form

$$\left[S_k^{(j)}(i) \bigcap \left(\bigcup_{1 \leq l \leq j} C_k^{(l)}(i) \right) \right] \bigcap I_1$$

for $k > N$. This means the set $\left(\left(\bigcap_{i \in M} F_i \right) \bigcap I_1 \right) \setminus G$ is contained in

$$\left(\bigcup_{i \in M, k > N} \left[\bigcup_{1 \leq l \leq p_i - 1} C_k^{(l)}(i) \right] \right) \bigcap I_1.$$

But the measure of this set is less than $|M| \left(\frac{1}{(\min\{p_i | i \in M\})^N} \right) \mu(I_1)$. We conclude that $\mu \left(\left(\bigcap_{i \in M} F_i \right) \bigcap I_1 \right) \leq \mu(G) + \frac{|M|}{2^N} \mu(I_1)$. Using the intersections mentioned at the beginning of the paragraph and the fact that for $k \leq N, S_k^{(j)}(i) \bigcap I_1 = \emptyset$ unless $j = m_k^+(i)$, we can write

$$G = \bigcap_{i \in M} \left[\bigcup_{k \leq N} \left(I_1 \bigcap \left(\bigcup_{1 \leq l \leq m_k^+(i)} C_k^{(l)}(i) \right) \right) \right]. \tag{2}$$

We now want to show $\mu(G) = \left(\prod_{i\in M} f_i(x_0)^+\right)\mu(I_1)$. To do this, we think of each $f_i(x_0)^+$ as a sum of terms of the form $\frac{1}{p_i^k}$, where k is a positive integer. For each k, there are $m_k^+(i)$ of these terms and each $\frac{1}{p_i^k}$ corresponds to exactly one $C_k^{(l)}(i)$ in (2). So we need to show that the shade of any set of the form $\bigcap_i C_k^{(l)}(i)$ is equal to the product of all of the individual shades of the $C_k^{(l)}(i)$'s. This is where Lemma 4.2 is used. Since $C_k^{(l)}(i) = K_{p_i^k, lp_i^{k-1}} = \Gamma + h\left(p_i^k\mathbb{Z} + lp_i^{k-1}\right) + \mathbb{Z}$, by the construction of Γ, our intersection requires that $p_1^{k_1}x_1 + l_1p_1^{k_1-1} = p_2^{k_2}x_2 + l_2p_2^{k_2-1} = \cdots = p_{|M|}^{k_{|M|}}x_{|M|} + l_{|M|}p_{|M|}^{k_{|M|}-1}$ for $\{x_i\} \subset \mathbb{Z}$. From Lemma 4.2, we know that any number of the form $\left(\prod_{s\neq i} p_s^{k_s}\right)z + c_i$ can be used for x_i, where $z \in \mathbb{Z}$ is arbitrary and $c_i \in \mathbb{Z}$ is fixed. This implies the intersection set can be written in the form $\Gamma + h\left(\left(\prod_{i\in M} p_i^{k_i}\right)\mathbb{Z} + d\right) + \mathbb{Z}$ for some integer d. But this set has shade $\frac{1}{\prod_{i\in M} p_i^{k_i}}$, the product of all the shades of the $C_k^{(l)}(i)$'s in the intersection. We conclude that $\mu(G) = \left(\prod_{i\in M} f_i(x_0)^+\right)\mu(I_1)$. We note that I_1 is Lebesgue measurable, so the last equation also follows from Mabry's Theorem 3.11, which says that shadings are evenly distributed on Lebesgue measurable sets and not just intervals. Thus we have $\prod_{i\in M} f_i(x_0)^+\mu(I_1) \leq \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_1\right) \leq \prod_{i\in M} f_i(x_0)^+\mu(I_1) + \frac{|M|}{2^N}\mu(I_1)$. Using $f_i(x_0) - \frac{1}{p_i^N} \leq f_i(x_0)^+ \leq f_i(x_0)$ and the assumptions on the size of ε, we can write $\left(\prod_{i\in M} f_i(x_0) - \varepsilon\right)\mu(I_1) \leq \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_1\right) \leq \left(\prod_{i\in M} f_i(x_0) + \varepsilon\right)\mu(I_1)$. (The inequality $f_i(x_0) - \frac{1}{p_i^N} \leq f_i(x_0)^+ \leq f_i(x_0)$ holds if we again assume that any number written in base p_i that might end with an infinite string of $p_i - 1$'s is written in terminating form.) We proved $\left(\prod_{i\in M} f_i(x_0) - \varepsilon\right)\mu\left(I_1\right) \le \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_1\right) \le \left(\prod_{i\in M} f_i(x_0) + \varepsilon\right)\mu\left(I_1\right)$ for $I_1 = \bigcap_{i\in M} I^+_{x_0}(f_i)$, but a similar process can be used to prove it for any intersection of the sets $\{I^+_{x_0}(f_i), I^-_{x_0}(f_i)\}$, where for each i either $I^+_{x_0}(f_i)$ or $I^-_{x_0}(f_i)$ is chosen. (We need to use both $f_i(x_0) - \frac{1}{p_i^N} \le f_i(x_0)^+ \le f_i(x_0)$ and $f_i(x_0) - \frac{1}{p_i^N} \le f_i(x_0)^- \le f_i(x_0)$ in the general case.) There are $2^{|M|}$ such sets, and each one is Lebesgue measurable. If we add up all $2^{|M|}$ of these inequalities and use the finite additivity of μ, we can write $\left(\prod_{i\in M} f_i(x_0) - \varepsilon\right)\mu\left(I_{x_0}\right) \le \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_{x_0}\right) \le \left(\prod_{i\in M} f_i(x_0) + \varepsilon\right)\mu\left(I_{x_0}\right)$. Dividing both sides by $\mu\left(I_{x_0}\right)$ and using the arbitrary smallness of ε gives us the desired result.

We now consider the case where $f_t(x_0) = 0$ for some $t \in M$. Besides $\frac{|M|}{2^N} < \varepsilon$ and the one involving Lemma 4.3, the other assumptions on N are not used. Everything in the proof is the same until we get to the inequality $\prod_{i\in M} f_i(x_0)^+\mu\left(I_1\right) \le \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_1\right) \le \prod_{i\in M} f_i(x_0)^+\mu\left(I_1\right) + \frac{|M|}{2^N}\mu(I_1)$, or $0 \le \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_1\right) \le \varepsilon\mu(I_1)$. Since $f_t\left(x_0\right) = 0$ and $f_t\left(x\right) \ge 0$ for all $x \in \mathbb{R}$, $I^-_{x_0}\left(f_t\right) = \emptyset$. (Hence $f_t\left(x_0\right)^-$ does not exist.) This last case then gives us fewer than $2^{|M|}$ inequalities to add together, since there are fewer than $2^{|M|}$ nonempty intervals to consider. Their sum, nevertheless, is still $0 \le \mu\left(\left(\bigcap_{i\in M} F_i\right)\bigcap I_{x_0}\right) \le \varepsilon\mu(I_{x_0})$. □

We should mention that the F_i sets above can be made to be subsets of arbitrary Archimedean sets satisfying the conditions of Theorem 2.1, if we use $\Gamma_A + t_1\mathbb{Z} + t_2\mathbb{Z}$ in place of $\Gamma + h\mathbb{Z} + \mathbb{Z}$ in the proof.

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