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PEANO DIFFERENTIATION VIA INTEGRATION

Abstract

In a little known paper, Haslam-Jones defined a collection of higher order derivatives in terms of an integral and Legendre polynomials. One member of this collection is equivalent to the higher order Peano derivatives. The purpose of this paper is to present a more direct proof of this equivalence.

In a paper that appears to have been overlooked, U. S. Haslam-Jones [3] defined a large collection of higher order derivatives that generalize the usual higher order derivatives. One member of this interesting collection turns out to be equivalent to the Peano derivative. The proof given by Haslam-Jones is hard to follow due to the approach he chooses to take and the omission of some details. It is the purpose of this note to offer an alternate proof of the equivalence of these two higher order derivatives.

We begin with the definitions. Let f be a continuous function defined on some open interval I containing a point c and let n be a positive integer. The function f has an n'th order Peano derivative at c if there exist numbers $f_p^{(1)}(c), f_p^{(2)}(c), \ldots, f_p^{(n)}(c)$ such that

$$f(c+h) = f(c) + f_p^{(1)}(c)\,h + \frac{f_p^{(2)}(c)}{2}\,h^2 + \cdots + \frac{f_p^{(n)}(c)}{n!}\,h^n + \epsilon(h)\,h^n$$

with $\lim\limits_{h\to 0} \epsilon(h) = 0$. The number $f_p^{(n)}(c)$ is the n'th order Peano derivative of f at c. The function f has an n'th order Haslam-Jones derivative at c if the limit

$$\frac{(2n-1)!}{2^n(n-1)!} \lim_{h\to 0} \frac{1}{h^n} \int_{-1}^{1} \big(f(c+h) - f(c+ht)\big)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt$$

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exists, where P_n represents the n'th degree Legendre polynomial. (These polynomials will be discussed in a moment.) We will denote this value by $f_{h\text{-}j}^{(n)}(c)$. The claim made in the opening paragraph is that these two derivative processes are equivalent, that is, $f_p^{(n)}(c)$ exists if and only if $f_{h\text{-}j}^{(n)}(c)$ exists and the values are the same when either exists.

Before giving a proof of the equivalence, we record some properties of Peano derivatives and Legendre polynomials. The Peano derivatives have the following elementary properties:

1. If $f_p^{(n)}(c)$ exists, then $f_p^{(k)}(c)$ exists for $1 \le k < n$.
2. If $f^{(n)}(c)$ exists, then $f_p^{(n)}(c)$ exists and the values are the same.
3. If $f_p^{(1)}(c)$ exists, then $f'(c)$ exists.
4. If (for example) $f(x) = x^5 \sin(1/x^4)$ for $x \neq 0$ and $f(0) = 0$, then $f''(0)$ does not exist, but $f_p^{(4)}(0) = 0$.
5. Peano derivatives satisfy the usual condition of linearity.

For proofs of deeper properties of Peano derivatives (such as the fact that they are Darboux Baire class one functions), see Oliver [4]. For a comprehensive list of properties of Peano derivatives and their generalizations, the reader should consult the survey article by Evans and Weil [2] or perform a web search for more recent results.

There are several ways to obtain the Legendre polynomials, but we will consider Rodrigues' formula. For each nonnegative integer n, the Legendre polynomial P_n is defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\big(3x^2 - 1\big), \quad P_3(x) = \frac{1}{2}\big(5x^3 - 3x\big).$$

We will need the following properties of the Legendre polynomials. The proofs of these facts are not difficult and (if you get stuck verifying them) there are many resources available for these and other properties of this particular collection of orthogonal polynomials.

A. P_n is a polynomial of degree n with leading coefficient $\dfrac{(2n)!}{2^n n!^2}$.

B. P_n is an even function when n is even and an odd function when n is odd.

C. $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ for all nonnegative integers n.

D. $\displaystyle\int_{-1}^{1} w(t)P_n(t)\,dt = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} w^{(n)}(t)(t^2-1)^n\,dt$, assuming that w has a continuous n'th derivative.

E. $\displaystyle\int_{-1}^{1} t^m P_n(t)\,dt = 0$ whenever m is a nonnegative integer less than n.

F. $\displaystyle\int_{-1}^{1} t^n P_n(t)\,dt = \frac{2^{n+1}n!^2}{(2n+1)!}$.

G. $\displaystyle\int_{-1}^{1} \big(Q(c+h) - Q(c+ht)\big)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt = h^n n q_n \cdot \frac{2^n(n-1)!^2}{(2n-1)!}$, if Q is a polynomial of degree n with leading coefficient q_n.

H. $\displaystyle\int_{-1}^{1} Q(t)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt = 2Q(1) - nq_n \cdot \frac{2^n(n-1)!^2}{(2n-1)!}$, assuming that Q is a polynomial of degree n with leading coefficient q_n.

I. $\displaystyle\int_{0}^{1} P_n'(st)P_n'(s)\,ds = P_n'(t)$.

J. $\displaystyle\int_{0}^{1/t} P_n'(st)P_n'(s)\,ds = \frac{P_n'(1/t)}{t}$.

(Many of the proofs of the integral results simply involve integration by parts and properties (B) and (C); no "hidden" properties are required.) Note that property (G) shows that the n'th order Haslam-Jones derivative of a polynomial of degree n gives the usual n'th order derivative of that polynomial. It is somewhat entertaining to use a computer algebra system to evaluate n'th order Haslam-Jones derivatives for common functions.

Using the properties of Legendre polynomials, it is not difficult to show that the existence of a Peano derivative implies the existence of the Haslam-Jones derivative. The details are given in the proof of the next theorem.

Theorem 1. *If $f_p^{(n)}(c)$ exists, then $f_{h\text{-}j}^{(n)}(c)$ exists and the values are the same.*

PROOF. Since $f_p^{(n)}(c)$ exists, we may write

$$f(c+h) = f(c) + f_p^{(1)}(c)\,h + \frac{f_p^{(2)}(c)}{2}\,h^2 + \cdots + \frac{f_p^{(n)}(c)}{n!}\,h^n + \epsilon(h)\,h^n,$$

where $\lim_{h\to 0} \epsilon(h) = 0$. It follows that

$$\begin{aligned} f(c+h) - f(c+ht) &= f_p^{(1)}(c)\,h(1-t) + \frac{f_p^{(2)}(c)}{2}\,h^2(1-t^2) + \cdots \\ &\qquad + \frac{f_p^{(n)}(c)}{n!}\,h^n(1-t^n) + \big(\epsilon(h) - t^n\epsilon(ht)\big)h^n \\ &= Q(t) + \big(\epsilon(h) - t^n\epsilon(ht)\big)h^n, \end{aligned}$$

where Q is the indicated polynomial of degree n. Using property (H) of the Legendre polynomials, we find that

$$\begin{aligned} &\frac{1}{h^n}\int_{-1}^{1} \big(f(c+h) - f(c+ht)\big)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt \\ &\qquad = \frac{2Q(1)}{h^n} + \frac{f_p^{(n)}(c)}{(n-1)!}\cdot\frac{2^n(n-1)!^2}{(2n-1)!} \\ &\qquad\qquad + \int_{-1}^{1}\big(\epsilon(h) - t^n\epsilon(ht)\big)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt. \end{aligned}$$

Since $Q(1) = 0$ and the last integral goes to 0 with h, the result follows once the Haslam-Jones derivative coefficient is included. $\square$

As a prelude to the proof of the converse, we make the following remarks. Some of these may seem trivial, but it is important to understand the simplifying assumptions that will be made. Suppose that $f_{h\text{-}j}^{(n)}(c) = z$ and define a new function $^*f$ by $^*f(x) = f(x+c) - zx^n/n!$. Then (using the linearity of the Haslam-Jones derivative) $^*f_{h\text{-}j}^{(n)}(0) = 0$. Let ϕ and ψ be the even and odd parts of $^*f$, respectively. It is easy to verify that $\phi_{h\text{-}j}^{(n)}(0) = 0$ and $\psi_{h\text{-}j}^{(n)}(0) = 0$. The crucial step is to prove that ϕ and ψ have n'th order Peano derivatives at 0 and that $\phi_p^{(n)}(0) = 0 = \psi_p^{(n)}(0)$. By linearity and the definition of the Peano derivative, it then follows easily that $^*f_p^{(n)}(0) = 0$ and $f_p^{(n)}(c) = z$.

To summarize the discussion of the previous paragraph, to prove the converse, it is sufficient to assume that $f_{h\text{-}j}^{(n)}(0) = 0$ and that f is either an even or an odd function. This leaves four cases to consider: f can be an even or odd function and n can be an even or odd positive integer. If f and n have the same parity, then (recall properties (B) and (C) of the Legendre polynomials)

$$\int_{-1}^{1}\big(f(h)-f(ht)\big)\big(P_n'(t)+P_{n-1}'(t)\big)\,dt = 2\Big(f(h)-\int_0^1 f(ht)P_{n-1}'(t)\,dt\Big).$$

If f and n have opposite parity, then

$$\int_{-1}^{1}\big(f(h)-f(ht)\big)\big(P_n'(t)+P_{n-1}'(t)\big)\,dt = 2\Big(f(h)-\int_0^1 f(ht)P_n'(t)\,dt\Big).$$

We will focus on this second case; the other case is similar. For ease of notation, we will continue to use f rather than ${}^*f$ to represent our function. In other words, for the proofs of the results that follow, the function f is either an even or odd function defined in a neighborhood of 0. Since $f_{h\text{-}j}^{(n)}(0)=0$, we know that

$$f(h)-\int_0^1 f(ht)P_n'(t)\,dt = \epsilon(h)\,h^n,$$

where $\lim_{h\to 0}\epsilon(h)=0$. To show that $f_p^{(n)}(0)=0$, we need to prove that

$$f(h)=b_0+b_1h+\frac{b_2}{2}\,h^2+\cdots+\frac{b_{n-1}}{(n-1)!}\,h^{n-1}+\epsilon_1(h)\,h^n$$

for constants $b_0, b_1, \dots, b_{n-1}$ and a function ϵ_1 such that $\lim_{h\to 0}\epsilon_1(h)=0$. Combining these two facts, we see that the goal is to prove that

$$\int_0^1 f(ht)P_n'(t)\,dt = b_0+b_1h+\frac{b_2}{2}\,h^2+\cdots+\frac{b_{n-1}}{(n-1)!}\,h^{n-1}+\epsilon_2(h)\,h^n,$$

for some function ϵ_2 that satisfies $\lim_{h\to 0}\epsilon_2(h)=0$.

It is the proof of this last fact that requires some effort. Our approach is close to that of Haslam-Jones but avoids some of the complicated machinery he adopts. We begin with three lemmas.

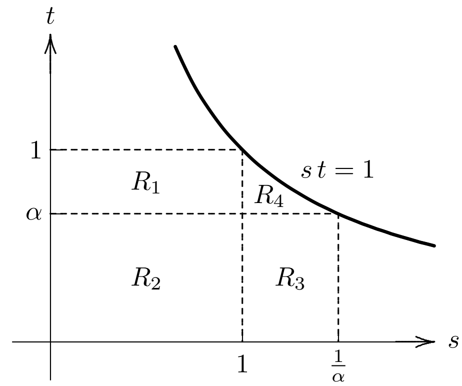
Lemma 2. *Suppose that f is continuous on $[0,\delta]$ for some $\delta<1$ and define a function g on $[0,\delta]$ by*

$$g(h)=f(h)-\int_0^1 f(ht)P_n'(t)\,dt.$$

Then for $0<h\le\delta$ and $0<\alpha<1$, the integral $\int_0^1 f(ht)P_n'(t)\,dt$ is equal to

$$\int_0^{\alpha h}\int_0^{1/(\alpha h)} f(y)P_n'(hx)P_n'(xy)\,dx\,dy+\int_\alpha^1\frac{P_n'(1/v)}{v}\,g(hv)\,dv.$$

PROOF. We begin by converting a single integral into several double integrals; refer to the following graph to find the regions R_i that will be used.



Using property (I) of Legendre polynomials, we note that

$$\begin{aligned}\int_0^1 f(ht)P_n'(t)\,dt &= \int_0^1 f(ht)\Big(\int_0^1 P_n'(s)P_n'(st)\,ds\Big)\,dt\\ &= \iint_{R_1\cup R_2} f(ht)P_n'(s)P_n'(st)\,ds\,dt\\ &= \Big(\iint_{R_2\cup R_3} + \iint_{R_1\cup R_4} - \iint_{R_3\cup R_4}\Big)\Big(f(ht)P_n'(s)P_n'(st)\,dA\Big).\end{aligned}$$

Making the change of variables $s = hx$ and $t = y/h$, the first integral becomes

$$\int_0^{\alpha}\int_0^{1/\alpha} f(ht)P_n'(s)P_n'(st)\,ds\,dt = \int_0^{\alpha h}\int_0^{1/(\alpha h)} f(y)P_n'(hx)P_n'(xy)\,dx\,dy.$$

Using property (J) of Legendre polynomials, the second integral can be written as

$$\int_{\alpha}^1\int_0^{1/t} f(ht)P_n'(s)P_n'(st)\,ds\,dt = \int_{\alpha}^1 f(ht)\,\frac{P_n'(1/t)}{t}\,dt = \int_{\alpha}^1 \frac{P_n'(1/v)}{v}\,f(hv)\,dv.$$

For the third integral, we make the change of variables $s = 1/v$ and $t = uv$ to obtain

$$\begin{aligned}\int_1^{1/\alpha}\int_0^{1/s} f(ht)P_n'(s)P_n'(st)\,dt\,ds &= \int_{\alpha}^1\int_0^1 f(huv)P_n'(1/v)P_n'(u)\,\frac{1}{v}\,du\,dv\\ &= \int_{\alpha}^1 \frac{P_n'(1/v)}{v}\Big(\int_0^1 f(huv)P_n'(u)\,du\Big)\,dv.\end{aligned}$$

Assembling the three pieces and recalling the definition of the function g (along with a change of letters for dummy variables) yields the desired result. $\square$

Lemma 3. *Let ϵ and δ be positive numbers and let $a_1 < a_2 < \cdots < a_n$ be a strictly increasing sequence of nonnegative integers. Consider the function Q defined by $Q(x,y) = \sum_{i=1}^{n} c_i(y)x^{a_i}$, where the coefficients of the powers of x are functions of some other variable y for $0 < y < \epsilon$. If for each fixed $x \in (0,\delta]$, the limit $\lim_{y\to 0^+} Q(x,y)$ exists, then each of the limits $\lim_{y\to 0^+} c_i(y)$ exists.*

PROOF. Choose a positive number d such that $d < \min\{\delta, 1\}$. Letting x take on the values $d, d^2, \ldots, d^n$, the hypotheses of the lemma tell us that each of the functions

$$\begin{aligned} w_1(y) &= c_1(y)d^{a_1} + c_2(y)d^{a_2} + \cdots + c_n(y)d^{a_n}, \\ w_2(y) &= c_1(y)d^{2a_1} + c_2(y)d^{2a_2} + \cdots + c_n(y)d^{2a_n}, \\ &\vdots \\ w_n(y) &= c_1(y)d^{na_1} + c_2(y)d^{na_2} + \cdots + c_n(y)d^{na_n}, \end{aligned}$$

has a limit as $y \to 0^+$. To prove the lemma, it is sufficient to prove that each of the functions $c_i(y)$ is a linear combination of $w_1(y), w_2(y), \ldots, w_n(y)$. Since the system of equations is linear, we simply need to prove that the coefficient matrix

$$\begin{pmatrix} d^{a_1} & d^{a_2} & \cdots & d^{a_n} \\ d^{2a_1} & d^{2a_2} & \cdots & d^{2a_n} \\ \vdots & \vdots & \vdots & \vdots \\ d^{na_1} & d^{na_2} & \cdots & d^{na_n} \end{pmatrix}$$

is nonsingular. Let $z_i = d^{a_i}$ for each i and note that the determinant of this matrix is

$$\begin{vmatrix} z_1 & z_2 & \cdots & z_n \\ z_1^2 & z_2^2 & \cdots & z_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1^n & z_2^n & \cdots & z_n^n \end{vmatrix} = z_1 z_2 \cdots z_n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{vmatrix},$$

a multiple of the well-known Vandermonde determinant. Since the value of this determinant is nonzero (the z_i's are distinct and nonzero), we see that the coefficient matrix is indeed nonsingular. $\square$

Lemma 4. *If the conditions of Lemma 2 are satisfied and if the improper integral* $\int_0^1 \frac{g(u)}{u^n}\,du$ *converges, then there exist constants* $b_0, b_1, \dots, b_{n-1}$ *such that*

$$\int_0^1 f(ht)P_n'(t)\,dt - \int_0^1 \frac{P_n'(1/v)}{v}\,g(hv)\,dv = \sum_{k=0}^{n-1} b_k\,h^k$$

for $0 < h \le \delta$.

Proof. We first prove that the improper integral $\int_0^1 \frac{P_n'(1/v)}{v}\,g(hv)\,dv$ converges for each value of h in the interval $(0,\delta]$. Fix such an h and write $P_n'(t) = p_0 + p_1 t + \cdots + p_{n-1}t^{n-1}$. The substitution $u = hv$ yields

$$\int_0^1 \frac{P_n'(1/v)}{v}\,g(hv)\,dv = \int_0^h \frac{P_n'(h/u)}{u}\,g(u)\,du = \sum_{k=0}^{n-1} p_k h^k \int_0^h \frac{g(u)}{u^{k+1}}\,du.$$

Given the hypothesis concerning the improper integral, each of the improper integrals in the sum exists. Now assume that $0 < \eta < h$ and let $\alpha = \eta/h$ in Lemma 2 to obtain

$$\begin{aligned}
&\int_0^1 f(ht)P_n'(t)\,dt - \int_{\eta/h}^1 \frac{P_n'(1/v)}{v}\,g(hv)\,dv \\
&\qquad = \int_0^{\eta}\int_0^{1/\eta} f(y)P_n'(hx)P_n'(xy)\,dx\,dy \\
&\qquad = \int_0^{\eta}\int_0^{1/\eta} f(y)\Big(\sum_{k=0}^{n-1} p_k (hx)^k\Big)P_n'(xy)\,dx\,dy \\
&\qquad = \sum_{k=0}^{n-1}\Big(p_k \int_0^{\eta}\int_0^{1/\eta} x^k f(y)P_n'(xy)\,dx\,dy\Big)\,h^k \\
&\qquad = \sum_{k=0}^{n-1} c_k(\eta)\,h^k,
\end{aligned}$$

where, as indicated, $c_k(\eta)$ is independent of h. The expression on the left has a limit as $\eta \to 0^+$ and this implies that $\sum_{k=0}^{n-1} c_k(\eta)\,h^k$ has a limit as $\eta \to 0^+$. Since this result is valid for each value of $h \in (0,\delta]$, Lemma 3 implies that each of the functions $c_k(\eta)$ has a limit as $\eta \to 0^+$. Let $b_k = \lim_{\eta\to 0^+} c_k(\eta)$ for each appropriate value of k. The conclusion now follows by taking the limit as $\eta \to 0^+$ of each side of the last displayed equation. □

Theorem 5. *If $f^{(n)}_{h\text{-}j}(c)$ exists, then $f^{(n)}_p(c)$ exists and the values are the same.*

Proof. By the remarks preceding the lemmas, we need only consider the situation in which $f^{(n)}_{h\text{-}j}(0) = 0$ and f is either an even function or an odd function. We will consider the case in which f and n have opposite parity; the case in which they have the same parity is almost identical. When f and n have opposite parity, we know that

$$\int_{-1}^{1} \big(f(h) - f(ht)\big)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt = 2\Big(f(h) - \int_0^1 f(ht)P_n'(t)\,dt\Big).$$

Since $f^{(n)}_{h\text{-}j}(0) = 0$, we can write (with g as in Lemma 2)

$$g(h) = f(h) - \int_0^1 f(ht)P_n'(t)\,dt = \epsilon(h)\,h^n,$$

where $\lim\limits_{h\to 0}\epsilon(h) = 0$. By Lemma 4 (it is easy to verify that the improper integral converges), we have

$$\begin{aligned} f(h) &= g(h) + \int_0^1 f(ht)P_n'(t)\,dt \\ &= g(h) + \sum_{k=0}^{n-1} b_k\,h^k + \int_0^1 \frac{P_n'(1/v)}{v}\,g(hv)\,dv \\ &= \sum_{k=0}^{n-1} b_k\,h^k + h^n\Big(\epsilon(h) + \int_0^1 v^{n-1}P_n'(1/v)\,\epsilon(hv)\,dv\Big). \end{aligned}$$

This expression for $f(h)$ indicates that $f^{(n)}_p(0) = 0$ if the term in parentheses goes to 0 with h. Since $v^{n-1}P_n'(1/v)$ is a polynomial, the integral term does indeed go to 0 with h and the proof is complete. □

Suppose that f has an n'th order Peano derivative on some closed interval I. We have assumed that f is continuous on I, but note that the continuity of f in this case is a consequence of property (3) of Peano derivatives. It then follows that for each positive integer k, the function f_k defined by

$$f_k(x) = \int_{-1}^{1} \Big(f\Big(x + \frac{1}{k}\Big) - f\Big(x + \frac{t}{k}\Big)\Big)\big(P_n'(t) + P_{n-1}'(t)\big)\,dt$$

is continuous on I. (Extend f as a constant so that the function expressions are defined for all values of t and k.) Using the equivalence of the Peano and

Haslam-Jones derivatives, we see that

$$f_p^{(n)}(x) = \frac{(2n-1)!}{2^n(n-1)!} \lim_{k\to\infty} k^n f_k(x)$$

for all $x \in I$, revealing that $f_p^{(n)}(x)$ is a Baire one function on I. This proof is a bit easier than the standard proof of this fact (see [4]). Other properties of Peano derivatives do not seem to fall out as easily from this equivalence, but it might be interesting to delve into this.

We have assumed that f is a continuous function on some interval. It is possible for the derivatives considered here to exist even when f is not so well-behaved; we leave such a study to the interested reader. For the record, Ash [1] has given another equivalent formulation of Peano derivatives involving the existence of a whole class of generalized n'th derivatives involving sums.

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