INROADS

Real Analysis Exchange
Vol. 34(1), 2008/2009, pp. 139–146

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THE POINTWISE LIMIT OF SEPARATELY CONTINUOUS FUNCTIONS

Abstract

The motivation for this paper is due to a question from Z. Piotrowski on whether or not the "salt-and-pepper" function in the plane was the pointwise limit of separately continuous functions. In this paper we answer that question and then go on to investigate the sets D in the plane such that χ_D is the pointwise limit of separately continuous functions. We also look at all pointwise limits of separately continuous functions and their place in the space of Baire Class 2 functions.

The functions we will deal with in this paper will be real functions, but we note here that all the definitions apply in more general metric spaces.

Definition 1. *Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. For a fixed value x, we define the $x-$section of f by the function $f_x(y) = f(x, y)$. Similarly we can define the $y-$section of f. We say a function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is separately continuous if each $x-$section and $y-$section is a continuous function.*

This is *not* the same as continuity in the ordinary sense (referred to as joint continuity) with the first counterexample appearing in the literature in 1873. This example is

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} .$$

Another type of function we will use is the quasi-continuous function.

Mathematical Reviews subject classification: Primary: 26B05; Secondary: 54C30
Key words: pointwise limit, separate continuity, joint continuity, porous sets
Received by the editors May 23, 2007
Communicated by: Paul D. Humke

Definition 2. *We say $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is quasi-continuous at the point (x, y) if for every $\varepsilon > 0$ and any nonempty open sets U and V with $x \in U$ and $y \in V$ there exists open sets $U_0 \subseteq U$ and $V_0 \subseteq V$ with*

$$f(U_0 \times V_0) \subseteq (f(x, y) - \varepsilon, f(x, y) + \varepsilon).$$

Furthermore, f is quasi-continuous if it is quasi-continuous at every point.

We will shortly use the following lemma.

Lemma 3. *If a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is separately continuous, then it is quasi-continuous.*

Our original papers on this subject were joint work with Z. Piotrowski (see [5], [6], and [7]). Early on we defined planar approximable function and looked at some characteristics of them. In the course of this, the following question was asked:

Question: Can something like the "salt-and-pepper" function in the plane be the pointwise limit of separately continuous functions? (By "salt-and-pepper" for $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we mean $\chi_{\mathbb{Q}\times\mathbb{Q}}$.)

In this instance, the answer is, "No." If that was true, then there exists $f_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ each separately continuous such that $f_n(x, y)$ converges to 1 if $x, y \in \mathbb{Q}$ and 0 otherwise. But then if we let $g_n : \mathbb{R} \to \mathbb{R}$ be the restriction of f_n along the line $y = 0$ we have a sequence of continuous functions converging to $\chi_{\mathbb{Q}}$, This is a contradiction since it is well-known that the characteristic function of the rationals is *not* in Baire class one.

The problem above arises from the fact that for any horizontal or vertical line in $\mathbb{R} \times \mathbb{R}$, the intersection of the line and $\mathbb{Q} \times \mathbb{Q}$ is dense and co-dense. There are many sets which are dense and co-dense in the plane yet meet every line in exactly n (a fixed number) points. For more on n–sets see [1] and [2]. Could we construct such a sequence of functions for the characteristic function of a dense n–set? Unfortunately the answer is no.

Theorem 4. *Let D be a countable set dense in the plane. There does not exist a sequence of separately continuous f_n such that f_n converges pointwise to χ_D.*

PROOF. Let us enumerate D as $\{(a_k, b_k)\}$ and assume there exists a sequence of separately continuous (hence quasi-continuous) f_n such that f_n converges pointwise to the characteristic function of D. Look at (a_1, b_1). Since $f_n(a_1, b_1) \to 1$ there exists a natural number n_1 such that for $n > n_1$ we have $f_n(a_1, b_1) > 1/2$. Since f_{n_1} is quasi-continuous there exists an open set E_1 such that for all points (x, y) in the set $f_{n_1}(x, y) > 1/2$.

Now let (a_j, b_j) be the first element of D with $j > 1$ in the set E_1. For this point there exists a natural number n_2 such that $f_{n_2}(a_j, b_j) > 1/2$ and, again, quasi-continuity implies there is an open set $E_2 \subset E_1$ such that for all $(x, y) \in E_2$, $f_{n_2}(x, y) > 1/2$. Continuing in this manner we generate a sequence of open sets E_n where E_n is open and there exists increasing m_n such that $(a_j, b_j) \notin E_n$ if $j < m_n$ and $f(x, y) > 1/2$, if $(x, y) \in E_n$. By the Baire Category Theorem $\cap \overline{E_n}$, where $\overline{A}$ denotes the closure of A, is non-empty. Let $(s, t) = \cap \overline{E_n}$. By our construction $(s, t) \notin D$, but also by construction $f_n(s, t) \nrightarrow 0$, a contradiction. Hence no such sequence exists. □

We can rephrase this as follows. The proof is an application of 5.1.1 in [3].

Theorem 5. *If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the pointwise limit of separately continuous functions, then $D(f)$, the set of points of discontinuity of f is of first category.*

As is well-known ([9]), separately continuous functions are in the first class of Baire ($\mathcal{B}_1$). So the pointwise limit of separately continuous functions that we are looking at must be in Baire class two ($\mathcal{B}_2$). However, we would like to be more precise. If we denote by S the pointwise limit of separately continuous functions, is S really a subset of $\mathcal{B}_2$ or is it possible that we really have a subset of $\mathcal{B}_1$? If $S \subset \mathcal{B}_2$ how big is it in the space of the $\mathcal{B}_2$ functions? Since the original problem dealt with a characteristic function of a set, how are these questions answered if we insist the limit function is χ_A for some set $A \subset \mathbb{R} \times \mathbb{R}$?

Example 6. *There exists a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is the pointwise limit of separately continuous functions, but is not Baire class one.*

Proof. In order to create this function we need to define a few tools. First, define the function $g_r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$g_r(x, y) = \begin{cases} \exp\left(\dfrac{x^2 + y^2}{x^2 + y^2 - r^2}\right) & x^2 + y^2 < r^2 \\ 0 & \textit{otherwise} \end{cases}$$

where $r > 0$. This function g is continuous and reaches it's maximum (one) at the origin. Secondly, let $A \subset \mathbb{R}$ be the set $[-5\pi/6, -2\pi/3] \cup [-\pi/3, -\pi/6] \cup [\pi/6, \pi/3] \cup [2\pi/3, 5\pi/6]$. We'll have C denote the Cantor ternary set in the real line and $K = \{k_n\}$, the set of endpoints of the intervals removed in constructing C. Lastly, define the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows: if $x = 0$ or $y = 0$, then $f(x, y) = 1$, if $\arctan(y/x) \in A$, then $f(x, y) = 0$, on the rest of the plane, continuously connect the graph to previously defined pieces using horizontal and vertical lines. Thus we have a function f which is continuous everywhere except the origin and there it is separately continuous.

Now we create a sequence of functions $F_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in this manner: Let $F_1(x, y) = g_{r_1}(x - k_1, y - k_1) \cdot f(x - k_1, y - k_1)$ where r_1 is chosen so that the support of f_1 does not intersect the $x-$axis. Assume that F_n has been defined. Then

$$F_{n+1}(x, y) = F_n(x, y) + g_{r_{n+1}}(x - k_{n+1}, y - k_{n+1}) \cdot f(x - k_{n+1}, y - k_{n+1})$$

where r_{n+1} is chosen so that the support of $g_{r_{n+1}} \cdot f_{n+1}$ does not intersect the support of F_n. Each F_n is the finite sum of separately continuous functions, hence separately continuous. Define $F(x, y)$ as the limit of $F_n(x, y)$. We claim this function is not separately continuous. We will show this by demonstrating that it is not in Baire class one. A necessary and sufficient condition for a function f to be Baire class one is for any perfect set P, the restriction of the function to P, $f|_P$, has a point of continuity. Let $\widetilde{C} = \{(x, x) \,|x \in C\}$ By our construction $F|_{\widetilde{C}}$ has no point of continuity since all $F(k_n, k_n)$ have value one and all other points have value zero. □

Corollary 7. *There exists a function $\widetilde{F} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is the pointwise limit of separately continuous functions, but is not separately continuous.*

Corollary 8. *There exists a set D so that χ_D is the pointwise limit of separately continuous functions, but χ_D is not Baire class one.*

PROOF. Let $\widetilde{F}_n(x, y) = \sum_{m=1}^{n} g_{r_m/n}(x - k_m, y - k_m) \cdot f(x - k_m, y - k_m)$. Then $D = \{(k_n, k_n)\}$. □

Because separately continuous functions are automatically quasi-continuous, we need the following example to show that these pointwise limits escape the quasi-continuous functions.

Example 9. *The function $\chi_{(0,0)} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is the pointwise limit of continuous functions (hence limit of separately continuous functions) which is not quasi-continuous.*

In the other direction, there is a quasi-continuous function which is not the pointwise limit of separately continuous functions.

Example 10. *Define $g(x, y)$ by*

$$g(x, y) = \begin{cases} 0 & x < 1/2 \\ 1 & x > 1/2 \\ 0 & x = 1/2 \text{ and } y \notin \mathbb{Q} \\ 1 & x = 1/2 \text{ and } y \in \mathbb{Q} \end{cases}.$$

It is easy to see that since it is constant on the half-planes $x > 1/2$ and $x < 1/2$ that it is quasi-continuous and since $g|_{x=1/2}$ is the characteristic function of the rationals g is not the pointwise limit of separately continuous functions.

The "obvious" question about these characteristic functions which are the pointwise limit of separately continuous functions is, "What sets D have χ_D as the pointwise limit of separately continuous functions?" We formalize this below.

Problem 11. *Does there exist a characterization of the sets D in the plane such that χ_D is the pointwise limit of separately continuous functions?*

The answer is not yet known.

We now turn to describing the functions which are the pointwise limit of separately continuous f_n. The plus topology in the plane is found as follows:

Definition 12. *The ε-plus at (a,b) of radius $\varepsilon > 0$ is*

$$B_{\varepsilon}^{+}(a,b) = \{(x,b) : |x-a| < \varepsilon\} \cup \{(a,y) : |y-b| < \varepsilon\}.$$

Definition 13. *A set $B \subset \mathbb{R} \times \mathbb{R}$ is separately open if for each point $(a,b) \in G$ there exists $\varepsilon > 0$ such that $B_{\varepsilon}^{+}(a,b) \subset G$. A set is separately closed if its complement is separately open.*

The canonical example of a set which is separately open, but not open in the usual (Euclidean) metric is the so-called Maltese Cross given by

$$A = (0,0) \cup \{(x,y) : |y| > |3x|\} \cup \{(x,y) : |y| > |x/3|\}.$$

We will do the obvious and refer to a set as separately $\mathcal{G}_\delta$ if the set can be written as the countable intersection of separately open sets and separately $\mathcal{F}_\sigma$ if it can be written as the countable union of separately closed sets. For a discussion on separately $\mathcal{G}_\delta$ versus Euclidean $\mathcal{G}_\delta$ see [8]. Using the standard argument that for a real number a and f the pointwise limit of separately continuous f_n

$$\{x : f(x) < a\} = \cup_{k=1}^{\infty} \left(\cup_{m=1}^{\infty} \left(\cap_{n=m}^{\infty} \{x : f_n(x) \leq a - 1/k\}\right)\right)$$

we can say the following:

Theorem 14. *A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the pointwise limit of separately continuous functions if the inverse image of every open set in $\mathbb{R}$ is a separately $\mathcal{F}_\sigma$ set.*

However, this is a little distasteful since separately open/closed is not a well-known idea. This brings us to the open question

Problem 15. *Is there a way to describe the* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *which are the pointwise limit of separately continuous functions in terms of* $f^{-1}(U)$ *(where* U *is an open set in* $\mathbb{R}$*) in terms of Euclidean open sets?*

Our next result shows that in the space of Baire two functions equipped with the sup norm these limits form a very small set. When we say small we are talking in terms of porosity which we shall now define for any metric space.

Definition 16. *Suppose* (X, d) *is a metric space. The open ball with center* $x \in X$ *and radius* $r > 0$ *will be denoted by* $B(x, r)$*.Let* $M \subset X$*,* $x \in X$*, and* $R > 0$*. Then we denote the supremum of the set of all* $r > 0$ *for which there exists* $z \in X$ *such that*

$$B(z, r) \subset B(x, R) \text{ and } B(z, r) \cap M = \emptyset$$

by $\gamma(x, R, M)$*. The number*

$$p(M, x) = \limsup_{R \to 0^+} \frac{2\gamma(x, R, M)}{R}$$

is called the porosity of M *at* x*. The value of* $p(M, x)$ *is between* 0 *and* 1*. If* $p(M, x) = 0$*, then* M *is non-porous at* x *while if* $p(M, x) = 1$*, then* M *is strongly porous at* x*. At set is (strongly) porous if it is (strongly) porous at each of its points. Sets that are porous are nowhere dense (and measure zero if* X *has a measure on it). For more about porosity see [10].*

Theorem 17. *The set* S *consisting of pointwise limits of separately continuous functions is a porous subset of the set of Baire class two functions,* $\mathcal{B}_2$*.*

PROOF. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Baire class 2 function and let $\varepsilon > 0$ be given. Fix the $y-$value at $y = 0$ and look at $f_0 : \mathbb{R} \to \mathbb{R}$ given by $f_0(x) = f(x, 0)$.

For each $x \in \mathbb{R}$ define $g_0(x)$ to be $\frac{1}{2}\left(\limsup_{t \to x} f_0(t) + \liminf_{t \to x} f_0(t)\right)$ if the value is finite and $\limsup_{t \to x} f_0(t) \geq f(x) \geq \liminf_{t \to x} f_0(t)$, otherwise $g_0(x) = f_0(x)$. Let $A = \{x \in \mathbb{R} | f(x) \geq g(x)\}$ and let $B = \{x \in \mathbb{R} | f(x) \leq g(x)\}$.

Then pick $\widetilde{A}$ and $\widetilde{B}$, both countable and dense in $\mathbb{R}$, such that $\widetilde{A} \subseteq A$ and $\widetilde{B} \subseteq B$. Define $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} f(x,y) & y \neq 0 \\ f_0(x,y) + \frac{\varepsilon}{4} & y = 0,\ x \in \widetilde{A} \\ f_0(x,y) - \frac{\varepsilon}{4} & y = 0,\ x \in \widetilde{B} \\ f_0(x,y) & y = 0 \text{ and } x \notin A \cup B \end{cases}.$$

Then g is Baire class 2 and in the open ball about f with radius ε. However, if we take h in the ball about g with radius $\frac{\epsilon}{4}$ we see that h cannot be in the set S because $h(x,0)$ is not Baire class 1 since $h(x,0)$ has no point of continuity. Thus

$$p(S,f) \geq 2\frac{\varepsilon/4}{\varepsilon} > 0.$$

□

Corollary 18. *The set S is nowhere dense in $\mathcal{B}_2$.*

Finally, we note here, just to contrast this type of sparseness, that since there are c many separately continuous functions [4] and c many Baire class two functions, the cardinality of S is c.

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