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A SIMPLE PROOF OF ZAHORSKI'S DESCRIPTION OF NON-DIFFERENTIABILITY SETS OF LIPSCHITZ FUNCTIONS

Abstract

We provide a simplification of Zahorski's argument showing that for every Lebesgue null $G_{\delta\sigma}$ subset G of the line there is a Lipschitz function that is non-differentiable precisely at the points of G.

1 Introduction.

In [4], Zahorski characterized non-differentiability sets of various classes of functions. The main step in the proof of Zahorski's [4] characterization of non-differentiability sets of continuous functions was the construction, for any given G_δ set $G \subset \mathbb{R}$ of measure zero, of a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable at every point outside G and non-differentiable, with particular estimates of non-differentiability, at every point of G. In this note we show that a slightly stronger variant of this statement follows relatively easily from an in-between theorem proved in [2]:

Theorem 1. *Given any G_δ set $G \subset \mathbb{R}$ of measure zero, there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant 1, which is differentiable everywhere outside G and for any $x \in G$,*

$$\limsup_{y\to x} \frac{g(y)-g(x)}{y-x} = 1 \quad \textit{and} \quad \liminf_{y\to x} \frac{g(y)-g(x)}{y-x} = -1.$$

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By a simple argument, which we will reproduce below, Zahorski deduced a characterization of non-differentiability sets of Lipschitz functions. Notice that it is rather easy to see that these sets are $G_{\delta\sigma}$ and Lebesgue's Theorem implies that they are of measure zero.

Theorem 2. *Given any $G_{\delta\sigma}$ set $G \subset \mathbb{R}$ of measure zero, there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ which is differentiable everywhere on $\mathbb{R} \setminus G$ and non-differentiable everywhere on G.*

In the main result of [4], Zahorski proved that a necessary and sufficient condition for a set M to be the set of non-differentiability of a continuous real-valued function is that $M = M_1 \cup M_2$, where M_1 is an arbitrary G_δ set and M_2 is a $G_{\delta\sigma}$ set of measure zero. The main point of the proof is, after observing that M_1, M_2 may be taken disjoint, to add the function from Theorem 2 to a continuous function whose non-differentiability set coincides with M_1. Since we do not contribute anything to this part of Zahorski's argument, we will not reproduce it here. Let us just remark that for any given G_δ set G, Zahorski's construction of a continuous function whose non-differentiability set coincides with G is not simple.

A geometric approach to the proof of the above described results of Zahorski was given by Piranian in [3]. Our approach is quite different: instead of relying on geometric intuition, we replace Zahorski's construction by a topological argument. An advantage of this is the improvement of Theorem 1, where we reached the maximal strength of non-differentiability. Let us also notice that full verification of Piranian's geometric arguments is far from simple, and so the last part of Zahorski's proof is still waiting for a simple argument.

For the convenience of the reader who would like to consult Zahorski's original paper, we should also remark that although he proved Theorem 2, he never stated it. Instead, he stated an analogous result for monotone functions. Of course, since a Lipschitz function becomes monotone after adding a linear function, the monotone functions case follows immediately from Theorem 1.

2 Preliminaries.

The letter $\mathbb{R}$ denotes the set of all real numbers, λ the Lebesgue measure on $\mathbb{R}$. A point $z \in \mathbb{R}$ is called a point of density of a measurable set $M \subset \mathbb{R}$ if

$$\lim_{h \to 0+} \frac{1}{2h} \lambda \left(M \cap (z-h, z+h) \right) = 1.$$

A measurable set $M \subset \mathbb{R}$ is said to be *d-open* if every point of M is a point of density of M. If a set A is a subset of the set of density points of M, we write

$A \subset M$. It is known that the family of d-open sets is a topology on $\mathbb{R}$; this is called the density topology.

Functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous with the density topology on the domain and the ordinary (Euclidean) topology on the range are called approximately continuous. Equivalently, $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at $z \in \mathbb{R}$ if there is a measurable set $M \subset \mathbb{R}$ such that z is a point of density of M and for every $x \in M$,

$$\lim_{x \to z} f(x) = f(z).$$

In this note, any topological notions will be prefixed d- when we refer to the density topology; there is no prefix when we refer to the Euclidean topology. The density interior of a set A will be denoted $\mathring{A}^d$ and the density closure $\overline{A}^d$.

A set in a metric space is called G_δ if it can be written as a countable intersection of open sets and F_σ if it can be written as a countable union of closed sets. A countable union of G_δ sets is called a $G_{\delta\sigma}$ set. A function $f : X \to \mathbb{R}$ defined on a metric space X is called G_δ-measurable if for each $a \in \mathbb{R}$, the sets $\{x \in X \mid f(x) \leq a\}$ and $\{x \in X \mid f(x) \geq a\}$ are G_δ sets in X.

The following results are vital in our construction:

Theorem 3. [1] *Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and measurable and fix $a \in \mathbb{R}$. Define*

$$g(x) = \int_a^x f(t)\ dt.$$

Then for any point $x_0 \in \mathbb{R}$ at which f is approximately continuous, g is differentiable at x_0 and $g'(x_0) = f(x_0)$.

Theorem 4. [2] *Let $E \subset \mathbb{R}$ be a measurable set, $F \subset E$ be closed. Then for every $c > 0$ there is a closed set H such that $F \subset H \subset E$ and*

$$\lambda((x-h, x+h) \cap (E \setminus H)) \leq ch^2$$

whenever $x \in F$ and $h > 0$.

Theorem 5 (In-Between Theorem)**.** [2] *Let $F \subset \mathbb{R}$ and let $t, s : F \to \mathbb{R}$ be bounded functions such that $t(x) \leq s(x)$ for all $x \in F$, t is supposed to be d-upper semicontinuous and s d-lower semicontinuous. If there is a G_δ-measurable function $r : F \to \mathbb{R}$ such that $t(x) \leq r(x) \leq s(x)$ for all $x \in F$, then there is an approximately continuous function $f : F \to \mathbb{R}$ such that $t(x) \leq f(x) \leq s(x)$ for all $x \in F$.*

We now proceed with the construction of the function g described in Theorem 2. In the following section we prove Theorem 1. In the final section, we follow Zahorski's argument to deduce from it Theorem 2.

3 A Lipschitz Function Differentiable Everywhere Except a G_δ Set of Measure Zero.

Let $G \subset \mathbb{R}$ be a G_δ set of measure zero. Then $F = \mathbb{R} \setminus G$ is an F_σ set which has full measure in $\mathbb{R}$.

Our plan of action is to choose suitable functions r, s and t to be able to apply Theorem 5. Then using Theorem 3, we can integrate the resulting approximately continuous function to obtain a function which is differentiable on F. We will ensure, however that this function is still badly behaved enough to be non-differentiable on G, even in the strong sense required in Theorem 1.

Proposition 6. [2] *We can write $F = \bigcup_{k=0}^{\infty} F_k$ where F_k are closed sets such that for each $x \in F_k$, $k \geq 0$ and any $h > 0$,*

$$\lambda\left[(x, x+h) \setminus F_{k+1}\right] \leq \lambda\left[(x-h, x+h) \setminus F_{k+1}\right] \leq h^2.$$

In particular, $F_k \mathrel{\subset\!\cdot} F_{k+1}$ and $F = \bigcup_{k=0}^{\infty} \mathring{F}_k^d$.

Proof. F is an F_σ set, so $F = \bigcup_{n=0}^{\infty} \widetilde{F}_n$, where $\widetilde{F}_n$ is closed for each n. We set $F_0 = \widetilde{F}_0$ and define F_k inductively as follows: By hypothesis, F is measurable, F_k is closed and $F_k \subset F$, so the conditions for Theorem 4 are satisfied. Therefore, there exists a closed set H_k such that $F_k \subset H_k \subset F$ for each $k \geq 0$; for each $x \in F$ and any $h > 0$,

$$\lambda\left[(x-h, x+h) \cap (F \setminus H_k)\right] \leq h^2.$$

Therefore, $\lambda\left[(x-h, x+h) \setminus H_k\right] \leq h^2$, since F has full measure in $\mathbb{R}$. Set

$$F_{k+1} = \widetilde{F}_k \cup H_k.$$

Then $F_k \mathrel{\subset\!\cdot} H_k \subset F_{k+1}$, so that $F_k \mathrel{\subset\!\cdot} F_{k+1}$ for each $k \geq 0$ and since $H_k \subset F_{k+1}$, we have

$$\lambda\left[(x-h, x+h) \setminus F_{k+1}\right] \leq \lambda\left[(x-h, x+h) \setminus H_k\right] \leq h^2$$

as required. $\square$

Define the following sets:

$$\begin{array}{lll}
A_0 = \mathring{F}_1^d, & A_k = (\mathring{F}_{4k+1}^d \setminus F_{4k-2})\ (k \geq 1), & A = \bigcup_{k=0}^{\infty} A_k \\
 & B_k = (\mathring{F}_{4k+3}^d \setminus F_{4k})\ (k \geq 1), & B = \bigcup_{k=1}^{\infty} B_k \\
 & R_k^- = (F_{4k-2} \setminus F_{4k-4})\ \ (k \geq 1), & R^- = \bigcup_{k=1}^{\infty} R_k^- \\
 & R_k^+ = (F_{4k} \setminus F_{4k-2})\ \ (k \geq 1), & R^+ = F_0 \cup \left(\bigcup_{k=1}^{\infty} R_k^+\right).
\end{array}$$

Observe that A and B are d-open sets. It is also clear that $R^- \cup R^+ = F$ and $R^- \cap R^+ = \emptyset$. Then since F_n is closed for every n, R^+ and R^- are F_σ sets. Therefore $R^- = F \setminus R^+$ and $R^+ = F \setminus R^-$ are G_δ sets in F. We use these sets to define functions s, t and r from F to $\mathbb{R}$ as follows:

$$s(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in F \setminus A \end{cases}$$

$$t(x) = \begin{cases} -1 & \text{if } x \in B \\ 1 & \text{if } x \in F \setminus B \end{cases}$$

$$r(x) = \begin{cases} -1 & \text{if } x \in R^- \\ 1 & \text{if } x \in R^+. \end{cases}$$

By careful consideration of the various subsets of A, B, R^- and R^+, it is fairly straightforward to prove that $t(x) \le r(x) \le s(x)$ for every $x \in F$. We call $t(x)$ the lower function and $s(x)$ the upper function. We will prove that s is d-upper semicontinuous, t is d-lower semicontinuous and r is G_δ-measurable. Recalling that $F_k \subset F_{k+1}$ for each k, the diagram below gives an idealized view of what the sets and functions we have defined above may look like:

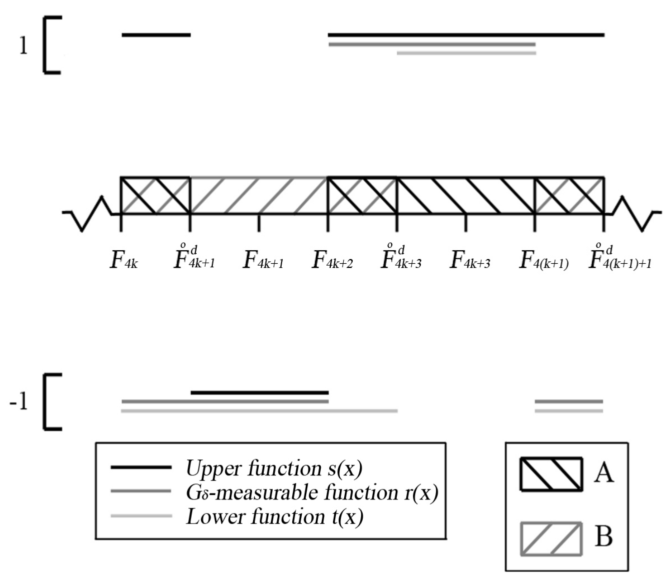

Figure 1: Diagram of the sets and functions defined above

Consider the sets

$$E_c(s) = \{x \in F \,|s(x) > c\}, \; E^c(t) = \{x \in F \,|t(x) < c\}.$$

There are three distinct cases for each set:

$$E_c(s) : \begin{cases} c \geq 1 \Rightarrow E_c(s) = \emptyset \\ -1 \leq c < 1 \Rightarrow E_c(s) = A \\ c < -1 \Rightarrow E_c(s) = F \end{cases}$$

$$E^c(t) : \begin{cases} c \geq 1 \Rightarrow E^c(t) = F \\ -1 \leq c < 1 \Rightarrow E^c(t) = B \\ c < -1 \Rightarrow E^c(t) = \emptyset \end{cases} .$$

In every case, $E_c(s)$ and $E^c(t)$ are d-open, so s is d-lower semicontinuous on F and t is d-upper semicontinuous on F. Consider now the sets

$$\tilde{E}_c(r) = \{x \in F|r(x) \geq c\}, \; \tilde{E}^c(r) = \{x \in F|r(x) \leq c\}.$$

Again, there are three distinct cases for each set:

$$\tilde{E}_c(r) : \begin{cases} c \geq 1 \Rightarrow \tilde{E}_c(r) = \emptyset \\ -1 \leq c < 1 \Rightarrow \tilde{E}_c(r) = R^+ \\ c < -1 \Rightarrow \tilde{E}_c(r) = F \end{cases}$$

$$\tilde{E}^c(r) : \begin{cases} c \geq 1 \Rightarrow \tilde{E}^c(r) = F \\ -1 \leq c < 1 \Rightarrow \tilde{E}^c(r) = R^- \\ c < -1 \Rightarrow \tilde{E}^c(r) = \emptyset \end{cases} .$$

In every case, $\tilde{E}_c(r)$ and $\tilde{E}^c(r)$ are G_δ subsets of F, so r is G_δ measurable on F.

Applying Theorem 5 using the functions t, r and s as defined, we obtain an approximately continuous function $f : F \to \mathbb{R}$ such that $t(x) \leq f(x) \leq s(x)$ for all $x \in F$. So f is defined almost everywhere on $\mathbb{R}$ and the way we defined s and t forces f to take value -1 on $(F \setminus A)$ and 1 on $(F \setminus B)$. We now fix $a \in \mathbb{R}$ and define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_a^x f(t)\, dt.$$

f is bounded and measurable, so since f is approximately continuous on F, it follows from Theorem 3 that g is differentiable and $g' = f$ on F. We also have

$$\begin{aligned} |g(y) - g(x)| &= \left| \int_a^y f(t)\, dt - \int_a^x f(t)\, dt \right| = \left| \int_x^y f(t)\, dt \right| \\ &\leq \int_x^y |f(t)|\, dt \leq \int_x^y dt = |y - x| \end{aligned}$$

so g is Lipschitz with constant 1.

Proposition 7. *For any $x \in G$,*

$$\limsup_{y \to x+} \frac{g(y)-g(x)}{y-x} = 1, \ \liminf_{y \to x+} \frac{g(y)-g(x)}{y-x} = -1.$$

PROOF. Recall that

$$F = (\mathbb{R} \setminus G) = \bigcup_{n=0}^{\infty} F_n,$$

where F_n is closed for each n. For $x \in G$, we know that $x \notin F_k$ for any k and moreover, since G has measure zero, we know that F is dense in $\mathbb{R}$.

Therefore, for each sufficiently large k we can choose h_k to be the minimal positive real number which satisfies

$$x + h_k \in F_k \text{ and } (x, x+h_k) \cap F_k = \emptyset,$$

and by Proposition 6,

$$\lambda\left[(x, x+h_k) \setminus F_{k+1}\right] \leq \lambda\left[(x-h_k, x+h_k) \setminus F_{k+1}\right] \leq h_k^2.$$

We also have that $g(x) = f'(x) = 1$ for $x \in (F \setminus B)$. Thus, if we take values of k of the form $k = 4t-1$ for sufficiently large t, then $\left(F_{k+1} \setminus \mathring{F}_k\right) \subset (F \setminus B)$ and

$$\begin{aligned} g\left(x+h_k\right) - g\left(x\right) &= \int_{F_{k+1} \cap (x, x+h_k)} f(y)\, dy + \int_{(x, x+h_k) \setminus F_{k+1}} f(y)\, dy \\ &\geq \lambda\left[F_{k+1} \cap (x, x+h_k)\right] - \lambda\left[(x, x+h_k) \setminus F_{k+1}\right] \\ &\geq h_k - 2\lambda\left[(x, x+h_k) \setminus F_{k+1}\right] \geq h_k - 2h_k^2. \end{aligned}$$

Therefore,

$$\frac{g\left(x+h_k\right) - g\left(x\right)}{h_k} \geq 1 - 2h_k \to 1 \text{ as } h_k \to 0+,$$

so

$$\limsup_{y \to x+} \frac{g\left(y\right) - g(x)}{y-x} = \limsup_{h \to 0+} \frac{g\left(x+h\right) - g(x)}{h} = 1,$$

as required.

The argument for the limit inferior is similar. We choose h_k as before, and note that $g(x) = f'(x) = -1$ for $x \in (F \setminus A)$. So if we take values of k of the form $k = 4t+1$ for sufficiently large t, $\left(F_{k+1} \setminus \mathring{F}_k\right) \subset (F \setminus A)$ and in a similar way to the previous case,

$$g\left(x + h_k\right) - g\left(x\right) \leq -h_k + 2h_k^2.$$

Therefore,

$$\frac{g\left(x + h_k\right) - g\left(x\right)}{h_k} \leq -1 + 2h_k \to -1 \text{ as } h_k \to 0+,$$

so

$$\liminf_{y \to x+} \frac{g\left(y\right) - g(x)}{y - x} = \liminf_{h \to 0+} \frac{g\left(x + h\right) - g(x)}{h} = -1,$$

as required. □

In fact, adding an argument symmetric to the one used above, we can even achieve that for every $x \in G$,

$$\limsup_{y \to x+} \frac{g(y) - g(x)}{y - x} = \limsup_{y \to x-} \frac{g(y) - g(x)}{y - x} = 1$$

and

$$\liminf_{y \to x+} \frac{g(y) - g(x)}{y - x} = \liminf_{y \to x-} \frac{g(y) - g(x)}{y - x} = -1.$$

The function g is therefore Lipschitz on $\mathbb{R}$, differentiable everywhere except the G_δ set G and its limit superior and inferior satisfy the requirements of Theorem 1.

4 A Lipschitz Function Differentiable Everywhere Except a $G_{\delta\sigma}$ Set of Measure Zero.

Here we will follow Zahorski's deduction of Theorem 2 from Theorem 1. The only difference between what follows and what Zahorski, as well as Piranian did, is that thanks to the slightly improved statement of Theorem 1, we can choose a more natural geometric sequence to control the Lipschitz constant of functions whose sum will be g.

Let G now be a $G_{\delta\sigma}$ set of measure zero. Then

$$G = \bigcup_{n=1}^{\infty} G_\delta^{(n)}$$

where $\left\{G_\delta^{(n)}\right\}_{n=1}^{\infty}$ is a sequence of G_δ sets in $\mathbb{R}$. Since the union has measure zero, each G_δ set in the sequence must also have measure zero. Consider the sequence of F_σ sets $\left\{F_\sigma^{(n)}\right\}_{n=1}^{\infty}$ given by $F_\sigma^{(k)} = \mathbb{R} \setminus G_\delta^{(k)}$ for each $k \in \mathbb{N}$. Using the results of the previous section, for each $k \in \mathbb{N}$, we can find a function $g_k : \mathbb{R} \to \mathbb{R}$ which is differentiable on $F_\sigma^{(k)}$, non-differentiable on $G_\delta^{(k)}$ and satisfies

$$\limsup_{y \to x} \frac{g(y) - g(x)}{y - x} = 1 \quad \text{and} \quad \liminf_{y \to x} \frac{g(y) - g(x)}{y - x} = -1$$

for every $x \in G_\delta^{(k)}$. We define

$$g\,(x) = \sum_{n=1}^{\infty} \frac{g_n(x)}{3^n}.$$

It is easy to show (mimicking the argument following Proposition 7 above) that for each $n \in \mathbb{N}$, the function $\hat{g}_n := g_n 3^{-n}$ is Lipschitz with constant $M_n = 3^{-n}$. We use this fact to prove that g is Lipschitz on the whole of $\mathbb{R}$. The proof is given in Zahorski [4] but we repeat it here for completeness.

Proposition 8. *$g : \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz.*

Proof. Set $S_m(x) := \sum_{n=1}^{m} \hat{g}_n(x)$. Then S_m is a sequence of partial sums; $S_m \to \infty$ as $m \to \infty$. For each $m \in \mathbb{N}$, S_m is Lipschitz, with

$$|S_m(y) - S_m(x)| \leq \frac{1}{2}\,|x - y|\,,$$

since

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

Now choose $k > 0$. Since $S_m(x)$ converges to $g(x)$ as $m \to \infty$, there exists $M \in \mathbb{N}$ such that

$$|g(y) - S_M(y)| < \frac{|y - x|}{k} \text{ and } |g(x) - S_M(x)| < \frac{|y - x|}{k}.$$

Therefore,

$$\begin{aligned} |g(y) - g(x)| &\leq |g(y) - S_M(y)| + |S_M(y) - S_M(x)| + |S_M(y) - g(x)| \\ &< \frac{2\,|y - x|}{k} + \frac{1}{2}\,|y - x|\,. \end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$|g(y) - g(x)| \leq \frac{1}{2}|y - x|$$

as required. □

Finally, we establish the differentiability properties of our function g.

Proposition 9. *If $x \notin G$, then g is differentiable at x.*

PROOF. The sum of the Lipschitz constants for the functions $\hat{g}_n$ is given by

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

Therefore, for any $\epsilon > 0$, we can find N such that

$$\sum_{n=N+1}^{\infty} 3^{-n} < \epsilon.$$

By the Lipschitz property of $\hat{g}_n(x)$, we also have that for $x \notin G$, $\hat{g}_n'(x)$ exists and $|\hat{g}_n'(x)| \leq 3^{-n}$. Therefore,

$$\begin{aligned}\left|\frac{g(x+h) - g(x)}{h} - \sum_{n=1}^{\infty} \hat{g}_n'(x)\right| &= \left|\sum_{n=1}^{N} \left(\frac{g(x+h) - g(x)}{h} - \hat{g}_n'(x)\right)\right. \\ &\qquad \left. + \sum_{n=N+1}^{\infty} \frac{g(x+h) - g(x)}{h} - \sum_{n=N+1}^{\infty} \hat{g}_n'(x)\right| \\ &< \left|\sum_{n=1}^{N} \left(\frac{g(x+h) - g(x)}{h} - \hat{g}_n'(x)\right)\right| + 2\epsilon.\end{aligned}$$

Taking the limit as $h \to 0$, we get

$$\limsup_{h \to 0} \left|\frac{g(x+h) - g(x)}{h} - \sum_{n=1}^{\infty} \hat{g}_n'(x)\right| \leq 2\epsilon.$$

Since this inequality holds for every $\epsilon > 0$, the limit

$$\lim_{h \to 0} \left|\frac{g(x+h) - g(x)}{h} - \sum_{n=1}^{\infty} \hat{g}_n'(x)\right|$$

exists and is equal to zero. Therefore,

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\sum_{n=1}^{\infty}\hat{g}_n'(x)=\sum_{n=1}^{\infty}\frac{g_n'(x)}{3^n}=: g'(x)$$

where

$$|g'(x)|=\left|\sum_{n=1}^{\infty}\frac{g_n(x)}{3^n}\right|\le\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}.$$

□

Proposition 10. *If $x\in G$, then g is not differentiable at x.*

PROOF. Let $x\in G$ and suppose that $G_\delta^{(p)}$ is the first G_δ set for which $x\in G_\delta^{(p)}$. Then $x\notin G_\delta^{(1)},\cdots,G_\delta^{(p-1)}$ and $x\in G_\delta^{(p)}$. Taking this into account, we have

$$g\left(x\right)=\sum_{n=1}^{\infty}\frac{g_n(x)}{3^n}=\sum_{n=1}^{p-1}\frac{g_n(x)}{3^n}+\frac{g_p\left(x\right)}{3^p}+\sum_{n=p+1}^{\infty}\frac{g_n(x)}{3^n}.$$

If we now define

$$\varphi\left(x\right)=\sum_{n=1}^{p-1}\frac{g_n(x)}{3^n},$$

then by the proof of the preceding proposition, $x\notin\bigcup_{n=1}^{p-1}G_\delta^{(n)}$ implies that φ is differentiable at x.

By Proposition 7, $x\in G_\delta^{(p)}$ implies that

$$\limsup_{y\to x}\frac{g_p(y)-g_p(x)}{y-x}=1$$

and

$$\liminf_{y\to x}\frac{g_p(y)-g_p(x)}{y-x}=-1.$$

Finally, we have

$$\sum_{n=p+1}^{\infty}\frac{1}{3^n}\frac{(g_n(y)-g_n(x))}{y-x}\le\sum_{n=p+1}^{\infty}\frac{1}{3^n}=\frac{3^{-p}}{2}.$$

Therefore,

$$\limsup_{y\to x}\frac{g(y)-g(x)}{y-x}\ge\varphi'(x)+3^{-p}-\frac{3^{-p}}{2}=\varphi'(x)+\frac{3^{-p}}{2}$$

and

$$\liminf_{y\to x} \frac{g(y)-g(x)}{y-x} \leq \varphi'(x) - 3^{-p} + \frac{3^{-p}}{2} = \varphi'(x) - \frac{3^{-p}}{2}$$

for some $p \in \mathbb{N}$.

Hence g is not differentiable at $x \in G$. $\square$

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