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UNILATERAL $\mathbb{I}$-APPROXIMATE LIMITS OF REAL FUNCTIONS

Abstract

We consider sets of generalized discontinuity of real functions with respect to local systems fulfilling the intersection condition. We give a sufficient condition for countability of such set. This result is used to prove its $\mathbb{I}$-density analogue.

1 Local Systems.

Let $\mathcal{T}_{nat}$ stand for the natural topology on $\mathbb{R}$.

Definition 1.1. (After [5]) Any family $\mathcal{S} = \{S(x)\}_{x\in\mathbb{R}}$, where each $S(x)$ is a collection of subsets of $\mathbb{R}$, will be called *a local system* if it satisfies the following four conditions:

1. $\{x\} \notin S(x) \neq \varnothing$ for any $x \in \mathbb{R}$,
2. If $S \in S(x)$ then $x \in S$,
3. If $S_1 \in S(x), S_1 \subset S_2$ then $S_2 \in S(x)$,
4. If $x \in A \in \mathcal{T}_{nat}, S \in S(x)$ then $S \cap A \in S(x)$.

The system of collections of all the neighborhoods in the natural topology, denoted by $\mathcal{S}_0 := \{S_0(x)\}_{x\in\mathbb{R}}$, serves as a good and simple example of a local system. Since we wish to investigate unilateral limits of real functions with respect to local systems, we focus on local systems which are *bilateral at every point*, namely on such systems that whenever $S \in S(x)$, both intersections $S\cap(x,\infty)$ and $S\cap(-\infty,x)$ are nonempty. Otherwise the notions of such limits wouldn't make any sense. Assuming $\mathcal{S}$ is bilateral at every point, we define

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$S^+(x)$ as the collection of all the supersets of the sets of the form $S \cap [x, \infty)$, with S belonging to $S(x)$. Exchanging $[x, \infty)$ with $(-\infty, x]$ we define $S^-(x)$.

Lemma 1.2. *Provided $\mathcal{S}$ is bilateral at every point, the collections $\mathcal{S}^-$ and $\mathcal{S}^+$ established by the formulae $\mathcal{S}^- := \{S^-(x) : x \in \mathbb{R}\}$ and $\mathcal{S}^+ := \{S^+(x) : x \in \mathbb{R}\}$ are well defined local systems.*

Local systems with the property that for any $x \in \mathbb{R}$, the collections $S(x)$ are closed under taking intersections are called the *filterings*. Obviously $\mathcal{S}_0$ is a bilateral filtering while $\mathcal{S}_\infty = \{S_\infty(x)\}_{x\in\mathbb{R}}$ (the system of sets having x as a point of accumulation) is neither bilateral nor a filtering.

Corollary 1.3. *For any bilateral local system $\mathcal{S} = \{S(x)\}_{x\in\mathbb{R}}$, $S(x) \subset S^+(x) \cap S^-(x)$. In addition, if $\mathcal{S}$ is a filtering, then*

$$S(x) = S^+(x) \cap S^-(x). \tag{1.1}$$

PROOF. The first statement is obvious. To prove the second one it suffices to fix $x \in \mathbb{R}$ and S from $S^-(x) \cap S_+(x)$. Then $S \supset S_0 \cap [x, \infty)$ and $S \supset S_1 \cap (-\infty, x]$ for some $S_0, S_1 \in S(x)$. The proof will be complete when we find $S_2 \subset S$ such that $S_2 \in S(x)$. To this end set $S_2 := (S_0 \cap [x, \infty)) \cup (S_1 \cap (-\infty, x])$. Note that $S_2 = (S_0 \cup (-\infty, x]) \cap ([x, \infty) \cup S_1)$. Both of the intersected sets belong to $S(x)$ and therefore, so does S_2. $\square$

The assumption of filtering is essential for the equality (1.1). This is a simple observation in the light of the following example. Consider the local system such that each $S \in S(x)$ is a superset of $\{x_n^-\}_{n\in\mathbb{N}} \cup [x, x+\varepsilon)$ or $(x-\varepsilon, x] \cup \{x_n^+\}_{n\in\mathbb{N}}$ for some sequences $\{x_n^-\}_{n\in\mathbb{N}}$ and $\{x_n^+\}_{n\in\mathbb{N}}$, convergent to x from the left and the right, respectively, and for positive ε. In other words, when $S \in S(x)$ then S has a subset which simultaneously belongs to $R(x)$, where

$$\begin{aligned} R(x) := &\left\{\{x_n^-\}_{n\in\mathbb{N}} \cup [x, x+\varepsilon) : x_n^- \to x^-, \varepsilon > 0\right\} \\ &\cup \left\{(x-\varepsilon, x] \cup \{x_n^+\}_{n\in\mathbb{N}} : x_n^+ \to x^+, \varepsilon > 0\right\}. \end{aligned}$$

Set $S_+ := \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup [0, +\infty)$ and $S_- := (-\infty, 0] \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. It is clear that both S_+ and S_- belong to $S(0)$. Thus $S_- \cap [0, +\infty) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \in S^+(0)$ and $S_+ \cap (-\infty, 0] = \{0\} \cup \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \in S^-(0)$. Therefore $S := (S_- \cap [0, +\infty)) \cup (S_+ \cap (-\infty, 0])$ belongs to $S^+(0) \cap S^-(0)$. On the other hand $S = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \notin S(0)$.

Definition 1.4. Let $\mathcal{S}$ be a local system and let $f \in \mathbb{R}^\mathbb{R}$ and $x \in \mathbb{R}$ be fixed. If there is exactly one $c \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that for any neighborhood

$U_c \in \mathcal{T}_{nat}$ of c, $\{x\} \cup f^{-1}(U_c) \in S(x)$. This unique element is called the *$\mathcal{S}$-limit of f at x*, denoted $c = (\mathcal{S})\text{-}\lim_{y\to x} f(y)$. Note that in general the above mentioned c may not exist or be unique. This is evident when one considers $S_0(x)$ and $S_\infty(x)$. It suffices to take the function $f = \mathrm{sgn}$ and $x = 0$ to see that there are two candidates for $\mathcal{S}_\infty$-limit at 0 and no candidate for $\mathcal{S}_0$-limit at 0. In fact, for $\mathcal{S}_0$, the notions of standard limits and $\mathcal{S}$-limits coincide. When $\mathcal{S}$ is bilateral at each point, limits with respect to $\mathcal{S}^-$ and $\mathcal{S}^+$ are called *unilateral $\mathcal{S}$-limits of f at x*, left and right, respectively. Thus the following notations are justified: $(\mathcal{S})\text{-}\lim_{y\to x+} f(y) = (\mathcal{S}^+)\text{-}\lim_{y\to x} f(y)$ and $(\mathcal{S})\text{-}\lim_{y\to x-} f(y) = (\mathcal{S}^-)\text{-}\lim_{y\to x} f(y)$.

The system of sets having x as a point of bilateral accumulation and the Dirichlet function serve as an example illustrating the possibility of absence (because of ambiguity) of the $\mathcal{S}$-limit in the case of a bilateral local system. A function $f \in \mathbb{R}^{\mathbb{R}}$ is said to be *unilaterally $\mathcal{S}$-continuous at x* (*left-* and *right-*, respectively) if $f(x) = (\mathcal{S})\text{-}\lim_{y\to x^-} f(y)$ and resp. $f(x) = (\mathcal{S})\text{-}\lim_{y\to x^+} f(y)$. If both the above conditions are fulfilled simultaneously, f is *$\mathcal{S}$-continuous at x*. By $D_{\mathcal{S}}f$ we shall denote the set on which the equality $f(x) = (\mathcal{S})\text{-}\lim_{y\to x} f(y)$ does not hold.

Definition 1.5. (After [2]) The family $\mathcal{S} = \{S(x)\}_{x\in\mathbb{R}}$, where each $S(x) \neq \varnothing$ is a collection of sets, satisfies

- the *strong intersection condition* of the form "$S_x \cap S_y \cap (x, y) \neq \varnothing$" ($\mathcal{S} \in$ SIC for short) if there exists $\delta : (x, S_x) \mapsto \delta^x_{S_x} > 0$, with $S_x \in S(x)$, such that

$$\mathop{\forall}_{x,y\in\mathbb{R}} \mathop{\forall}_{\substack{S_x\in S(x)\\ S_y\in S(y)}} \left(|y-x| < \min\{\delta^x_{S_x}, \delta^y_{S_y}\} \Rightarrow S_x \cap S_y \cap (x,y) \neq \varnothing \right), \quad (1.2)$$

- *the intersection condition* of the same form ($\mathcal{S} \in$ IC) if for any collection $\{S_x\}_{x\in\mathbb{R}}$ such that $S_x \in S(x)$, there exists $\delta : x \mapsto \delta(x) > 0$ such that

$$\mathop{\forall}_{x_1,x_2\in\mathbb{R}} \left(|x_1 - x_2| < \min\{\delta(x_1), \delta(x_2)\} \Rightarrow S_{x_1} \cap S_{x_2} \cap (x_1, x_2) \neq \varnothing \right). \quad (1.3)$$

It is quite clear that SIC $\subset$ IC.

For $\mathcal{S}$ bilateral at each point, consider the class $(\mathcal{S})$-*Reg* of unilaterally $\mathcal{S}$-continuous functions such that for any $x \in \mathbb{R}$ both $\mathcal{S}$-unilateral limits do exist and are finite.

Theorem 1.6. *Assume $\mathcal{S} \in$ IC is a filtering and that $f \in (\mathcal{S})$-Reg. Then* $\operatorname{card} D_{\mathcal{S}}f \leqslant \aleph_0$.

PROOF. Suppose

$$A := \left\{ x : (\mathcal{S})\text{-}\lim_{t \to x-} f(t) < (\mathcal{S})\text{-}\lim_{t \to x+} f(t) \right\}$$

is not countable. The same reasoning applies to the set on which the opposite inequality holds. For any $x \in A$, there exists $a(x) \in \mathbb{Q}$ such that

$$(\mathcal{S})\text{-}\lim_{t \to x-} f(t) < a(x) < (\mathcal{S})\text{-}\lim_{t \to x+} f(t).$$

By assumption $A_a := \{x \in A : a = a(x)\}$ is not countable for at least one $a \in \mathbb{Q}$. Write

$$S_x := \{x\} \cup \big((-\infty, x) \cap f^{-1}(-\infty, a)\big) \cup \big((x, \infty) \cap f^{-1}(a, \infty)\big)$$

and note that $S_x \in S(x)$ provided $x \in A_a$. This is quite easy to see as

$$\{x\} \cup \big((-\infty, x) \cap f^{-1}(-\infty, a)\big) \in S^-(x),$$
$$\{x\} \cup \big((x, \infty) \cap f^{-1}(a, \infty)\big) \in S^+(x),$$

and by (1.1) and the fact that local systems are stable under taking supersets. Now it suffices to take $x < y \in A_a$ close enough to see a contradiction with (1.3) for the collection $\{S_x\}$ defined above. This is evident by virtue of the following. We may assume that $\delta(x)$ from (1.3) is rational, so by *the pigeonhole principle* there exists a $\delta_0 \in \delta(A_a)$ such that for $A_a(\delta_0) := \{z \in A_a : \delta(z) = \delta_0\}$, we have $\operatorname{card} A_a(\delta_0) > \aleph_0$. This makes possible the choice of $x < y \in A_a(\delta_0) \subset A_a$, for which $|x - y| < \min\{\delta(x), \delta(y)\} = \delta_0$, since $A_a(\delta_0)$ must contain its condensation point. Therefore

$$S_x \cap S_y \cap (x, y) = (x, y) \cap f^{-1}(a, \infty) \cap f^{-1}(-\infty, a) = \varnothing.$$

□

Remark. Note that the concept of local system seems to be unnecessary and too abstract since we needed the assumption of bilateral filtering. Nevertheless this approach illustrates the fact that our assumptions are in a way minimal.

2 The $\mathbb{I}$-density.

Define $\mathbb{S}$ as the collection of all strictly increasing sequences of positive integers. The upper limit of a sequence of sets will be referred to as ls. Fix $\mathbb{B}$, invariant under similarities (compositions of translation and multiplication by a nonzero number) σ-algebra of subsets of $\mathbb{R}$ and fix $\mathbb{I} \subset \mathbb{B}$, invariant under similarities σ-ideal of subsets of $\mathbb{R}$. Given any set $A \subset \mathbb{R}$, by its $\mathbb{B}$-kernel we understand a set $A_0 \in \mathbb{B}$ such that whenever $A_0 \subset D \subset A$ for $D \in \mathbb{B}$, it must be the case that $D \setminus A_0 \in \mathbb{I}$. When A_0 is a $\mathbb{B}$-kernel of A we write $A_0 \in A_{\mathbb{B}}$. For $A \subset \mathbb{R}$ the collection $\{A' \subset A : A' \in \mathbb{B}\}$ will be denoted by $A^*_{\mathbb{B}}$. Obviously $A_{\mathbb{B}} \subset A^*_{\mathbb{B}}$. Moreover, the symmetric difference of any two $\mathbb{B}$-kernels of a fixed set remains in $\mathbb{I}$. It is worth mentioning that the statement: *"any set has its $\mathbb{B}$-kernel"* is not true in the general case. Nevertheless, there is a class of pairs $(\mathbb{B}, \mathbb{I})$ (σ-algebra, σ-ideal) for which the above is true. Namely, the class of pairs fulfilling the c.c.c., the countable chain condition, by which we mean that any pairwise disjoint subfamily of $\mathbb{B}$ can have at most countably many members from $\mathbb{B} \setminus \mathbb{I}$. To see this, fix $B \subset \mathbb{R}$. Consider the collection $B^*_{\mathbb{B}} \setminus \mathbb{I}$. If it is empty, then $\varnothing \in B_{\mathbb{B}}$. If it is not the case, let $B_0 \in B^*_{\mathbb{B}} \setminus \mathbb{I}$. We use transfinite recursion to construct the collection $\mathcal{B} := \{B_\alpha : B_\alpha \in \mathbb{B}\}$ such that for each α, $B_\alpha \subset B \setminus \bigcup_{\beta<\alpha} B_\beta$ and $B_\alpha \in B^*_{\mathbb{B}} \setminus \mathbb{I}$. Obviously this construction must stop at some countable α_0. Thus $\bigcup \mathcal{B} \in \mathbb{B}$ and consequently $\bigcup \mathcal{B} \in B_{\mathbb{B}}$. We shall use the standard notation $A \pm x := \{a \pm x : a \in A\}$ and $xA := \{xa : a \in A\}$ for $A \subset \mathbb{R}$ and $x \in \mathbb{R}$.

For a fixed $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ let us consider the following conditions (compare [1], pp 22–23, Theorem 2.2.2):

$$\exists_{A_1 \in A_{\mathbb{B}}} \ \forall_{\{n_k\} \in \mathbb{S}} \ \exists_{\{k_p\} \in \mathbb{S}} \ \operatorname*{ls}_{p \in \mathbb{N}} \big((-1,1) \setminus n_{k_p}(A_1 - x)\big) \in \mathbb{I} \tag{2.1}$$

$$\forall_{A_2 \in A_{\mathbb{B}}} \ \forall_{\{n_k\} \in \mathbb{S}} \ \exists_{\{k_p\} \in \mathbb{S}} \ \operatorname*{ls}_{p \in \mathbb{N}} \big((-1,1) \setminus n_{k_p}(A_2 - x)\big) \in \mathbb{I} \tag{2.2}$$

$$\forall_{\{n_k\} \in \mathbb{S}} \ \exists_{\{k_p\} \in \mathbb{S}} \ \exists_{A_3 \in A_{\mathbb{B}}} \ \operatorname*{ls}_{p \in \mathbb{N}} \big((-1,1) \setminus n_{k_p}(A_3 - x)\big) \in \mathbb{I} \tag{2.3}$$

$$\forall_{\{n_k\} \in \mathbb{S}} \ \exists_{\{k_p\} \in \mathbb{S}} \ \forall_{A_4 \in A_{\mathbb{B}}} \ \operatorname*{ls}_{p \in \mathbb{N}} \big((-1,1) \setminus n_{k_p}(A_4 - x)\big) \in \mathbb{I} \tag{2.4}$$

$$\exists_{A_0 \in A^*_{\mathbb{B}}} \ \forall_{\{n_k\} \in \mathbb{S}} \ \exists_{\{k_p\} \in \mathbb{S}} \ \operatorname*{ls}_{p \in \mathbb{N}} \big((-1,1) \setminus n_{k_p}(A_0 - x)\big) \in \mathbb{I}. \tag{2.5}$$

We will show that assuming the c.c.c., all the above conditions are equivalent.

Let A' and A'' be any $\mathbb{B}$-kernels of A and let $A_0 \in A^*_{\mathbb{B}}$. Then $A'' = (A' \setminus I_1) \cup I_2$ for some $I_1, I_2 \in \mathbb{I}$ and $A'' \supset A_0 \setminus I_0$ for some $I_0 \in \mathbb{I}$. Thus for any subsequence $\{n_{k_p}\}$ of any $\{n_k\} \in \mathbb{S}$,

$$\begin{aligned} n_{k_p}(A'' - x) &= n_{k_p}\left(((A' \setminus I_1) \cup I_2) - x\right) \\ &= \big(n_{k_p}(A' - x) \setminus n_{k_p}(I_1 - x)\big) \cup n_{k_p}(I_2 - x) \end{aligned}$$

and

$$n_{k_p}(A'' - x) \supset n_{k_p}\left((A_0 \setminus I_0) - x\right) = \big(n_{k_p}(A_0 - x) \setminus n_{k_p}(I_0 - x)\big)\,.$$

Furthermore,

$$\operatorname*{ls}_{p\in\mathbb{N}}\big((-1,1) \setminus n_{k_p}(A'' - x)\big) = \bigcap_{q\in\mathbb{N}}\bigcup_{p>q}\big((-1,1) \setminus n_{k_p}(A'' - x)\big) \tag{2.6}$$

$$= \bigcap_{q\in\mathbb{N}}\bigcup_{p>q}\big[\big[\big((-1,1) \setminus n_{k_p}(A' - x)\big) \cup n_{k_p}(I_1 - x)\big] \setminus n_{k_p}(I_2 - x)\big] \tag{2.7}$$

$$\subset \bigcap_{q\in\mathbb{N}}\bigcup_{p>q}\big[\big((-1,1) \setminus n_{k_p}(A' - x)\big) \cup n_{k_p}(I_1 - x)\big] \tag{2.8}$$

$$= \bigcap_{q\in\mathbb{N}}\Big(\bigcup_{p>q}\big((-1,1) \setminus n_{k_p}(A' - x)\big) \cup \bigcup_{p>q} n_{k_p}(I_1 - x)\Big) \tag{2.9}$$

$$\subset \bigcap_{q\in\mathbb{N}}\Big(\bigcup_{p>q}\big((-1,1) \setminus n_{k_p}(A' - x)\big) \cup \bigcup_{p\in\mathbb{N}} n_{k_p}(I_1 - x)\Big) \tag{2.10}$$

$$= \Big(\bigcap_{q\in\mathbb{N}}\bigcup_{p>q}\big((-1,1) \setminus n_{k_p}(A' - x)\big)\Big) \cup \bigcup_{p\in\mathbb{N}} n_{k_p}(I_1 - x) \in \mathbb{I}, \tag{2.11}$$

provided the left term of the last line is in $\mathbb{I}$. By a very similar argument, $\operatorname{ls}_{p\in\mathbb{N}}\big((-1,1) \setminus n_{k_p}(A'' - x)\big) \in \mathbb{I}$ provided $\operatorname{ls}_{p\in\mathbb{N}}\big((-1,1) \setminus n_{k_p}(A_0 - x)\big) \in \mathbb{I}$. It follows that (2.1) is equivalent to (2.5), (2.1) implies (2.2), and that (2.3) implies (2.4). Thus we have the equivalences

$$(2.5) \;\Leftrightarrow\; (2.1) \;\Leftrightarrow\; (2.2) \quad \text{and} \quad (2.3) \;\Leftrightarrow\; (2.4). \tag{2.12}$$

Note that (2.4) yields

$$\forall_{\{n_k\}\in\mathbb{S}}\ \forall_{A_4\in A_{\mathbb{B}}}\ \exists_{\{k_p\}\in\mathbb{S}}\ \operatorname*{ls}_{p\in\mathbb{N}}\big((-1,1) \setminus n_{k_p}(A_4 - x)\big) \in \mathbb{I},$$

which is equivalent to (2.2). Similarly (2.1) gives

$$\forall_{\{n_k\}\in\mathbb{S}}\ \exists_{A_1\in A_{\mathbb{B}}}\ \exists_{\{k_p\}\in\mathbb{S}}\ \operatorname*{ls}_{p\in\mathbb{N}}\big((-1,1) \setminus n_{k_p}(A_1 - x)\big) \in \mathbb{I}$$

and consequently (2.3). Thus

$$(2.4) \Rightarrow (2.2) \quad \text{and} \quad (2.1) \Rightarrow (2.3).$$

We conclude from (2.12) that the operator $\Phi_{\mathbb{I}}$ established by: $x \in \Phi_{\mathbb{I}}(A) : \{$if and only if (2.1) is fulfilled for x and $A\}$ may be as well defined by any formula among (2.1)–(2.5). From now on we are assuming that $\Phi_{\mathbb{I}}$ satisfies the following condition:

$$\underset{A\in\mathbb{B}\setminus\mathbb{I}}{\forall} A \cap \Phi_{\mathbb{I}}(A) \neq \varnothing. \tag{2.13}$$

Now let us define the left- and right-hand operators $\Phi_{\mathbb{I}}^{-}$ and $\Phi_{\mathbb{I}}^{+}$, which are nothing more than unilateral versions of $\Phi_{\mathbb{I}}$:

$$\Phi_{\mathbb{I}}^{-}(A) := \{x : x \in \Phi_{\mathbb{I}}(A \cup [x,\infty))\} \text{ and } \Phi_{\mathbb{I}}^{+}(A) := \{x : x \in \Phi_{\mathbb{I}}(A \cup (-\infty,x])\}.$$

It is worth mentioning that these operators can be defined as well by versions of (2.1)–(2.5) with $(-1,1)$ replaced with $(-1,0)$ and with $(0,1)$, respectively. Indeed fix $A' \in \mathbb{B}$, a subset of A and $\{n_k\} \in \mathbb{S}$. Set $A'' := A' \cup [x,\infty) \in \mathbb{B}$. We have

$$\operatorname*{ls}_{p\in\mathbb{N}} \big((-1,1) \setminus n_{k_p}(A''-x)\big) \in \mathbb{I}$$

for some $\{k_p\} \in \mathbb{S}$. The following formula shall accomplish the proof.

$$\begin{aligned}
(-1,1) \setminus n_{k_p}(A''-x) =&(-1,1) \setminus n_{k_p}\left((A' \cup [x,\infty)) - x\right)\\
=&(-1,1) \setminus n_{k_p}\left((A'-x) \cup [0,\infty)\right)\\
=&\big((-1,1) \setminus n_{k_p}(A'-x)\big) \setminus [0,\infty)\\
=&(-1,0) \setminus n_{k_p}(A'-x).
\end{aligned}$$

Definition 2.1. We define g to be the $\mathbb{I}$-*approximate left-hand limit of* $f \in \mathbb{R}^{\mathbb{R}}$ *at* $x \in \mathbb{R}$ when for any positive ε

$$x \in \Phi_{\mathbb{I}}^{-}\left(f^{-1}(B(g,\varepsilon)\right),$$

where $B(g,\varepsilon)$ is:

- the interval of the length 2ε with centre at g when $g \in \mathbb{R}$ and
- the half-lines $(\frac{1}{\varepsilon},\infty)$ and $(-\infty,-\frac{1}{\varepsilon})$, for $g=\infty$ and $g=-\infty$, respectively.

Right-hand limit is defined in the same manner with $\Phi_{\mathbb{I}}^{+}$ in place of $\Phi_{\mathbb{I}}^{-}$.

Definition 2.2. A function f is

- *unilaterally* $\mathbb{I}$-*approximately continuous at* x when $f(x)$ is equal to at least one of f's $\mathbb{I}$-approximate unilateral limits at x and
- $\mathbb{I}$-*approximately continuous at* x provided f is simultaneously unilaterally $\mathbb{I}$-approximately continuous at x at either side.

Lemma 2.3 ([6])**.** *Assume f is a real function with the property that for any $E \in \mathbb{B}\setminus\mathbb{I}$ and for any $\varepsilon > 0$ there exists $D \subset E$ such that $D \in \mathbb{B}\setminus\mathbb{I}$ and $\operatorname{osc}_D f \leqslant \varepsilon$. Then f is $\mathbb{B}$-measurable.*

PROOF. Let $\mathcal{D}_m$ be a maximal disjoint family of sets from $\mathbb{B} \setminus \mathbb{I}$ such that $\operatorname{osc}_D f \leqslant \frac{1}{m}$ for each $D \in \mathcal{D}_m$. Due to the c.c.c., $\mathcal{D}_m$ is countable and by the assumption, $\mathbb{R} \setminus \bigcup \mathcal{D}_m \in \mathbb{I}$. Define f_m for $m \in \mathbb{N}$ as $\chi_{D_{m,n}} \sup_{D_{m,n}} f$ on each $D_{m,n}$, where $\mathcal{D}_m = \{D_{m,n}\}_{n\in\mathbb{N}}$. All f_ms are $\mathbb{B}$-measurable and $f_m \to f$ on $\bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} D_{m,n}$. The complement of this set belongs to $\mathbb{I}$. □

Theorem 2.4. *Any function such that for each $x \in \mathbb{R}$ both the unilateral $\mathbb{I}$-approximate limits of f at x do exist and are finite must be $\mathbb{B}$-measurable.*

PROOF. Take any $E \in \mathbb{B} \setminus \mathbb{I}$. Let $a \in E \cap \Phi_{\mathbb{I}}(E)$ (see (2.13)). For every $\varepsilon > 0$, a is a left $\mathbb{I}$-density point of $f^{-1}(B(g,\varepsilon))$, where g is the left $\mathbb{I}$-approximate limit of f at a. Let A be the union of $\{a\}$ and a $\mathbb{B}$-kernel of $f^{-1}(B(g,\varepsilon))$. Note that A is a $\mathbb{B}$-kernel for itself and that $a \in A \cap \Phi_{\mathbb{I}}(A)$. Consequently $a \in (E \cap A) \cap \Phi_{\mathbb{I}}(E \cap A)$. Now it suffices to note that $A \cap E \in \mathbb{B} \setminus \mathbb{I}$, for if $A \cap E \in \mathbb{I}$, then $\Phi_{\mathbb{I}}^{-}(A \cap E) = \varnothing$. The application of Lemma 2.3 completes the proof. □

3 Category Case.

From now on we assume that $\mathbb{B}$ and $\mathbb{I}$ are the σ-algebra of subsets of $\mathbb{R}$ with the Baire property and the σ-ideal of the first-category subsets of $\mathbb{R}$, respectively.

Assuming A is open in the natural topology, from the fact that $x \in \Phi_{\mathbb{I}}(\mathbb{R} \setminus A)$ it follows that

$$\underset{\substack{k(x,A)\in\mathbb{N}\\ \delta(x,A)>0}}{\exists} \quad \underset{h\in(0,\delta(x,A))}{\forall} \quad \underset{\substack{i_l\in\{1,\dots,k(x,A)\}\\ i_r\in\{1,\dots,k(x,A)\}}}{\exists} \begin{pmatrix} (A-x)\cap\frac{h}{k(x,A)}(-i_l,-i_l+1)=\varnothing \\ \wedge\ (A-x)\cap\frac{h}{k(x,A)}(i_r-1,i_r)=\varnothing \end{pmatrix}, \tag{3.1}$$

which we shall compare (after [4], Lemma 1, and the proof of Theorem 1) with

$$x \in \Phi_{\mathbb{I}}(A) \Rightarrow \underset{(n\in\mathbb{N})}{\forall} \quad \underset{\delta_n(x,A)>0}{\exists} \quad \underset{\substack{h\in(0,\delta_n(x,A))\\ i\in\{-n+1,\dots,n\}}}{\forall} (A-x)\cap\tfrac{h}{n}[i-1,i]\neq\varnothing. \tag{3.2}$$

Let $\mathcal{S}_{\mathbb{I}} := \{S_{\mathbb{I}}(x) : x \in \mathbb{R}\}$ be the local system derived from the $\mathbb{I}$-density by requiring that S belongs to $S_{\mathbb{I}}(x)$ only when $x \in S$ is an $\mathbb{I}$-density point of S. From the proof of Theorem 1 [4], one can deduce (compare [2]) that

Lemma 3.1. *For $\mathcal{S}_{\mathbb{I}}$ the* SIC *is fulfilled.*

Let us recall the argument. For $x \in \mathbb{R}$ and $A_x \in S_{\mathbb{I}}(x)$ let

- $A_x^* = G_x \triangle P_x^1$, where $A_x^* \in \mathbb{B}$ is any $\mathbb{B}$-kernel of A_x,

- G_x be open in the natural topology with P_x^1 of the first category, and
- $x \in \Phi_{\mathbb{I}}(G_x)$.

This is reasonable due to the fact that the considered operator $\Phi_{\mathbb{I}}$ is a special case of the one defined by (2.1), so it sees only $\mathbb{B}$-kernels of sets and it doesn't distinguish the sets which differ by a first category set. Now we shall define $\delta^x_{A_x}$ so that

$$(x, y) \cap A_x \cap A_y \neq \varnothing \tag{3.3}$$

whenever $|x - y| < \min\{\delta^x_{A_x}, \delta^y_{A_y}\}$, as required in (1.2) for $A_x \in S_{\mathbb{I}}(x), A_y \in S_{\mathbb{I}}(y)$. To this end set $\delta^x_{A_x} := \min\{\delta(x, G_x), \delta_{2k(x,G_x)}(x, G_x)\}$, where:

- $\delta(x, G_x)$ is taken from (3.1) with $\operatorname{int}(\mathbb{R} \setminus G_x)$ in place of A,
- $2k(x, G_x)$ from (3.1) (where $\operatorname{int}(\mathbb{R} \setminus G_x)$ replaces A) is substituted in place of n (see (3.2)) and
- $\delta_{2k(x,G_x)}(x, G_x)$ is from (3.2) with A replaced by G_x.

In order to make (3.3) evident, note the decomposition of (x, y), with $|x - y| < \min\{\delta^x_{A_x}, \delta^y_{A_y}\}$, into $k := \min\{k(x, G_x), k(y, G_y)\}$ intervals J_j of equal lengths combined with the formulae (3.1) and (3.2) with appropriate substitutions described above. It is clear that among J_j, $j \in \{1, \ldots, k\}$ there is such a J that $J \setminus G_x \in \mathbb{I}$ if $k = k(x, G_x)$, which assumption involves no loss of generality. Simultaneously $G_y \cap J \neq \varnothing$. This is a consequence of the fact that $J'_p \subset J$ for at least one J'_p, $1 \leqslant p \leqslant k_0$ related to the decomposition of (x, y) into $k_0 := 2k(y, G_y)$parts of equal length. Thus $G_x \cap G_y \cap J$ is of the second category and therefore the proof of (3.3) is complete.

Now let us summarize these considerations with three corollaries which conjoin the $\mathbb{I}$-density and the local-system approach.

Let $\mathbb{I}$ap-*Reg* be the class of the functions which are simultaneously $\mathbb{I}$-approximately unilaterally continuous and such that for any $x \in \mathbb{R}$ both unilateral $\mathbb{I}$-approximate limits do exist and are finite. For any $f \in \mathbb{R}^{\mathbb{R}}$ the symbol $D_{\mathbb{I}ap} f$ stands for the set of all points of the $\mathbb{I}$-approximate discontinuity of f.

Corollary 3.2. *The classes* $\mathbb{I}$ap-*Reg and* $(\mathcal{S}_{\mathbb{I}})$-*Reg coincide. So do the classes of* $\mathcal{S}_{\mathbb{I}}$-*continuous and* $\mathbb{I}$-*approximately continuous functions.*

Corollary 3.3. *For any function* $f \in \mathbb{R}^{\mathbb{R}}$, $D_{\mathbb{I}ap} f = D_{\mathcal{S}_{\mathbb{I}}} f$.

Corollary 3.4. *Assume* $f \in \mathbb{I}$ap-*Reg*. *Then* $\operatorname{card} D_{\mathbb{I}ap} f \leqslant \aleph_0$.

Note 3.5. See [3] for *measure density* versions of Theorem 2.4 and of Corollary 3.4. In fact, this article [3] inspired the hereby considerations.

References

[1] Krzysztof Ciesielski, Lee Larson, and Krzysztof Ostaszewski, *$\mathcal{I}$-density Continuous Functions,* Mem. Amer. Math. Soc., **107(515)** (1994).

[2] Tomasz Filipczak, *Intersection conditions for some density and I-density local systems,* Real Anal. Exchange, **15(1)** (1989-90), 170–192.

[3] Marcin Grande, *On the sums of unilaterally approximately continuous and approximately jump functions,* Real Anal. Exchange, **28(2)** (2002/2003), 623–630.

[4] Ewa Łazarow and Władysław Wilczyński, *I-Approximate derivatives,* Radovi Matematički, **5** (1989), 15–27.

[5] Brian Thomson, *Real Functions,* Lecture Notes in Math., **1170** Springer-Verlag, New York, 1980.

[6] Władysław Wilczyński, *Separate I-approximate continuity implies the Baire property,* Zeszyty Naukowe Politechniki Śląskiej, Seria: Matematyka–Fizyka, **48(853)** (1986).