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CONDITIONAL COMPLETENESS OF $C(X, \mathbb{R}_\tau)$ FOR WEAK P-SPACES $\mathbb{R}_\tau$

Abstract

Let τ be a topology on the real numbers which is finer than the usual topology such that $\mathbb{R}_\tau$ is a weak P-space. In this paper, conditional completeness and conditional σ-completeness of $C(X)$ and $C(X, \mathbb{R}_\tau)$ are compared. In particular, it is shown for zero-dimensional spaces that $C(X, \mathbb{R}_\tau)$ is conditionally σ-complete if and only if X is a P-space and that $C(X, \mathbb{R}_\tau)$ is conditionally complete if and only if X is an extremally disconnected P-space.

1 Introduction.

Given two topological spaces X and Y, $C(X, Y)$ denotes the set of continuous functions $f : X \to Y$. When $Y = \mathbb{R}$ we write $C(X)$ instead and this set is a ring under pointwise addition and multiplication. Furthermore $C(X)$ is a lattice, where the least upper bound of two functions $f, g \in C(X)$ is defined as $(f \vee g)(x) = max\{f(x), g(x)\}$ and the greatest lower bound is defined as $(f \wedge g)(x) = \min\{f(x), g(x)\}$.

Let $\mathbb{R}_\tau$ denote the real numbers with topology τ. In this paper we will be investigating completeness properties of $C(X, \mathbb{R}_\tau)$ when $\mathbb{R}_\tau$ is a *weak P-space*, that is when every countable subset of $\mathbb{R}_\tau$ is closed. Recall that a lattice L is *conditionally (σ-)complete* if every (countable) nonempty subset of L which is bounded above has a supremum. These completeness properties can be characterized using the topology on X. Before giving this characterization we will need the following topological definitions.

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In a topological space X the closure of a subset A of X is denoted by $cl_X A$ and the interior of A is denoted by $int_X A$. When a set is both open and closed, it is called clopen. Given $f \in C(X)$, the *zeroset* of f (denoted by $Z(f)$) is the set of $x \in X$ such that $f(x) = 0$. A subset Z of X is called a zeroset if $Z = Z(f)$ for some $f \in C(X)$. The set $X \setminus Z(f)$ is called the *cozeroset* of f and is denoted by $coz(f)$. $Z[X] = \{Z(f) : f \in C(X)\}$ is the set of zerosets of X. We will assume that all domain spaces X in $C(X, Y)$ are Tychonoff, that is completely regular and Hausdorff. In a Tychonoff space the cozerosets form a base for the topology. A space is called *zero-dimensional* if it has a base of clopen sets.

The well-known Stone-Nakano Theorem, stated below, characterizes when $C(X)$ is conditionally (σ-) complete using topological properties of X. A space X is called *basically disconnected* if $cl_X coz(f)$ is open for every $f \in C(X)$ and X is called *extremally disconnected* if $cl_X O$ is open for every open subset O of X. This theorem can be found in [5], [6], and [7].

Theorem 1.1 (Stone-Nakano). *A space X is extremally disconnected if and only if $C(X)$ is conditionally complete. A space X is basically disconnected if and only if $C(X)$ is conditionally σ-complete.*

In this paper we will provide a similar characterization for when $C(X, \mathbb{R}_\tau)$ is conditionally (σ-) complete for any weak P-space $\mathbb{R}_\tau$. One example of a topology on $\mathbb{R}$ which makes $\mathbb{R}_\tau$ a weak P-space is the density topology, which we will now define. Let M be a Lebesgue measurable subset of $\mathbb{R}$ and let $m(M)$ denote the Lebesgue measure of M. Let $M' = \mathbb{R} \setminus M$. A point $p \in \mathbb{R}$ is a *density point* of M if

$$\lim_{h \to 0^+} \frac{m(M \cap (p-h, p+h))}{2h} = 1.$$

We also say that p is a *dispersion point* of M if p is a density point of M'. The set M is called *density open* if every $p \in M$ is a point of density of M. The collection of density open subsets of $\mathbb{R}$ is a topology called the *density topology* and we denote $\mathbb{R}$ with this topology by $\mathbb{R}_d$. We use $\mathbb{R}^d$ to denote the real numbers with the discrete topology and we denote the natural numbers by $\mathbb{N}$. It is known that the density topology is strictly finer than the usual topology on $\mathbb{R}$. As a result we see that $C(X, \mathbb{R}_d) \subseteq C(X)$ for any space X. Elements of $C(X, \mathbb{R}_d)$ will be called *density continuous* functions. A proof that $\mathbb{R}_d$ is a Tychonoff space which is not normal can be found in [2].

Although the set $C(X, \mathbb{R}_d)$ is a lattice, it cannot be assumed that it is a group or a ring. In [3] a space X for which $C(X, \mathbb{R}_d)$ is a group or a ring is called a density P-space. The following theorem from [3] classifies when a space X is a density P-space.

Theorem 1.2. *For a space X, the following are equivalent:*

(1) $C(X, \mathbb{R}_d) = C(X, \mathbb{R}^d)$, *i.e. every density continuous function is locally constant.*
(2) $C(X, \mathbb{R}_d)$ *is a ring.*
(3) $C(X, \mathbb{R}_d)$ *is closed under multiplication.*
(4) $C(X, \mathbb{R}_d)$ *is a group.*
(5) $Z(f)$ *is open for each* $f \in C(X, \mathbb{R}_d)$.

Examples of density P-spaces include pseudocompact spaces and separable spaces. Since $\mathbb{R}$ is connected as well as separable, it follows from the previous theorem that the only elements of $C(\mathbb{R}, \mathbb{R}_d)$ are the constant functions.

Recall that a topological space X is called a *P-space* if $Z(f)$ is open for all $f \in C(X)$. It is clear from the definitions that P-spaces are basically disconnected. Every P-space is a *weak P-space*: a space in which every countable subset is closed. The space $\mathbb{R}_d$ is an example of a weak P-space which is not a P-space. The next theorem gives a useful condition for checking when a topological space X is a P-space.

Theorem 1.3. *[4] A zero-dimensional space X is a P-space if and only if every countable union of clopen sets is again clopen if and only if every countable intersection of clopen sets is again clopen.*

2 Conditional σ-Completeness and Completeness.

Let τ be a topology on $\mathbb{R}$ which is finer than the usual topology. We will use $\mathbb{R}_\tau$ to denote the real numbers equipped with this topology. It follows that $C(X, \mathbb{R}_\tau)$ is a sublattice of $C(X)$. Henceforth, unless stated otherwise, τ is a topology on $\mathbb{R}$ which is finer than the usual topology such that $\mathbb{R}_\tau$ is a weak P-space. In this section we will determine when $C(X, \mathbb{R}_\tau)$ is conditionally complete and when it is conditionally σ-complete.

Observe that it is possible for $C(X, \mathbb{R}_\tau)$ to be conditionally σ-complete without $C(X)$ being conditionally σ-complete. One example is $\mathbb{R}$, which is not basically disconnected so that $C(\mathbb{R})$ is not conditionally σ-complete. However the elements of $C(\mathbb{R}, \mathbb{R}_d)$ are precisely the constant functions. Hence $C(\mathbb{R}, \mathbb{R}_d)$ is conditionally σ-complete. We would like to know if conditional σ-completeness of $C(X)$ implies the same condition for $C(X, \mathbb{R}_\tau)$. The next example illustrates that a countable subset of $C(X, \mathbb{R}_\tau)$ which is bounded above may have a supremum in $C(X)$ without having a supremum in $C(X, \mathbb{R}_\tau)$. In particular it is possible for $C(X)$ to be conditionally σ-complete while $C(X, \mathbb{R}_\tau)$ is not. To construct this example we need a few definitions.

Definition 2.1. A family $\mathcal{B}$ of zerosets of X is called a *z-filter* if the following conditions hold:

(1) $\emptyset \notin \mathcal{F}$;
(2) if $Z_1, Z_2 \in \mathcal{F}$, then there exists $Z_3 \in \mathcal{F}$ such that $Z_3 \subseteq Z_1 \cap Z_2$;
(3) if $Z \in \mathcal{F}$ and $Z' \in Z[X]$ with $Z \subseteq Z'$, then $Z' \in \mathcal{F}$.

A *z-ultrafilter* on X is a z-filter which is not contained in any other distinct z-filter. A straightforward Zorn's Lemma argument shows that every z-filter is contained in a z-ultrafilter.

Example 2.2. This example comes from Problem 4M of [1]. Let $\mathcal{F} = \{S \subset \mathbb{N} : \text{S is cofinite}\}$. It is easy to check that $\mathcal{F}$ is a z-filter on $\mathbb{N}$ which means there exists a z-ultrafilter $\mathcal{U}$ containing $\mathcal{F}$. Let $\Sigma = \mathbb{N} \cup \{\sigma\}$ where $\sigma \notin \mathbb{N}$ and define a topology on Σ as follows: all points of $\mathbb{N}$ are isolated and the open neighborhoods of σ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{U}$. Then according to 4M of [1], Σ is an extremally disconnected topological space. Note that $C(\Sigma)$ is conditionally complete because Σ is extremally disconnected.

For each $n \in \mathbb{N}$ define a function $f_n : \Sigma \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{for } x = 2n \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that each $f_n \in C(\Sigma, \mathbb{R}_\tau) \subset C(\Sigma)$. The space Σ is extremally disconnected, so by the Stone-Nakano Theorem the set $G = \{f_n : n \in \mathbb{N}\}$ has a supremum in $C(\Sigma)$. It is straightforward to check that the function $f : \Sigma \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = 2n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is a continuous function. If G has a supremum in $C(\Sigma, \mathbb{R}_\tau)$, then that supremum must be f. However $f \notin C(\Sigma, \mathbb{R}_\tau)$. Thus G does not have a supremum in $C(\Sigma, \mathbb{R}_\tau)$.

From the previous example we see that $C(X)$ can be conditionally complete without $C(X, \mathbb{R}_\tau)$ being even conditionally σ-complete. The property of conditional σ-completeness of $C(X, \mathbb{R}_\tau)$ is related to P-spaces, as we see in the next theorem.

Theorem 2.3. *If X is a zero-dimensional space, then the following are equivalent:*

(1) X is a P-space.
(2) $C(X, \mathbb{R}_\tau) = C(X)$.

(3) $C(X,\mathbb{R}^d) = C(X,\mathbb{R}_\tau) = C(X)$.
(4) $C(X,\mathbb{R}_\tau)$ *is conditionally* σ*-complete.*
(5) $C(X,\mathbb{R}^d)$ *is conditionally* σ*-complete.*

Proof. Since X is a P-space if and only if $Z(f)$ is open for all $f \in C(X)$, X is a P-space if and only if $C(X,\mathbb{R}^d) = C(X,\mathbb{R}_\tau) = C(X)$. Hence (1) and (3) are equivalent. Clearly (3) implies (2). To show (2) implies (1) we will apply Theorem 1.3. Assume X is not a P-space. Then there exists a sequence $\{C_n\}_{n\in\mathbb{N}}$ of disjoint clopen subsets of X such that $\bigcup_{n=1}^{\infty} C_n$ is not closed. Define $f : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in C_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in C(X)$. Note that $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}_\tau$ because $\mathbb{R}_\tau$ is a weak P-space. However, $f^{-1}(A) = \bigcup_{n=1}^{\infty} C_n$ is not closed in X, so $f \notin C(X,\mathbb{R}_\tau)$. As a result we see that $C(X,\mathbb{R}_\tau) \neq C(X)$ and hence (2) implies (1).

Next suppose (1) holds. We will show (4) and (5). X is zero-dimensional and basically disconnected because it is a P-space. Since $C(X,\mathbb{R}^d) = C(X,\mathbb{R}_\tau) = C(X)$, $C(X,\mathbb{R}^d)$ and $C(X,\mathbb{R}_\tau)$ are also conditionally σ-complete.

Now assume (4) is true. We will use Theorem 1.3 to show (1). Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint clopen subsets of X and for each $n \in \mathbb{N}$ define a function $f_n : X \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in C_n \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $f_n \in C(X,\mathbb{R}^d) \subseteq C(X,\mathbb{R}_\tau)$ for all n. By hypothesis the set $\{f_n : n \in \mathbb{N}\}$ has a supremum and it follows that the supremum must be the function f defined above. The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is closed in $\mathbb{R}_\tau$ since it is a weak P-space. Then continuity of f implies $f^{-1}(A) = \bigcup_{n=1}^{\infty} C_n$ is also closed as needed.

The proof of (5) implies (1) is identical to that of (4) implies (1); simply note that $f_n \in C(X,\mathbb{R}^d)$ for all n. □

Observe that it is necessary to assume X is zero-dimensional in the previous theorem. Consider the space $\mathbb{R}$, which is neither zero-dimensional or a P-space. The only elements of $C(\mathbb{R},\mathbb{R}_\tau)$ are the constant functions so that $C(\mathbb{R},\mathbb{R}_\tau) = C(\mathbb{R},\mathbb{R}^d)$ is conditionally σ-complete.

Corollary 2.4. *If X is zero-dimensional and $C(X,\mathbb{R}_\tau)$ is conditionally σ-complete, then $C(X)$ is conditionally σ-complete.*

Proof. By Theorem 2.3 if X is a zero-dimensional space and $C(X,\mathbb{R}_\tau)$ is conditionally σ-complete, then X is a P-space. Since every P-space is basically disconnected, $C(X)$ is conditionally σ-complete. □

Example 2.5. To see that the converse of Corollary 2.4 fails, consider again the space Σ from Example 2.2. Since Σ is basically disconnected, Σ is zero-dimensional. As we have already seen $C(\Sigma)$ is conditionally σ-complete. But $C(\Sigma,\mathbb{R}_\tau)$ is not conditionally σ-complete because Σ is not a P-space.

For a zero-dimensional space X, we have established that $C(X,\mathbb{R}_\tau)$ is conditionally σ-complete precisely when X is a P-space. The next result should not be surprising.

Theorem 2.6. *For any space X, the following are equivalent:*
(1) X is an extremally disconnected P-space.
(2) X is extremally disconnected and $C(X,\mathbb{R}_\tau)=C(X)$.
(3) $C(X,\mathbb{R}_\tau)$ is conditionally complete and X is zero-dimensional.
(4) $C(X,\mathbb{R}^d)$ is conditionally complete and X is zero-dimensional.

Proof. The equivalence of (1) and (2) follows from Theorem 2.3.

We will now show that (2) implies (3). Every extremally disconnected space is zero-dimensional. Therefore X is zero-dimensional. Since X is extremally disconnected, we know $C(X)$ is conditionally complete by the Stone-Nakano Theorem. Then $C(X,\mathbb{R}_\tau)=C(X)$ where X is extremally disconnected implies $C(X,\mathbb{R}_\tau)$ is conditionally complete.

Next assume (3) holds. If $C(X,\mathbb{R}_\tau)$ is conditionally complete, then $C(X,\mathbb{R}_\tau)$ is conditionally σ-complete. By Theorem 2.3, $C(X,\mathbb{R}^d)=C(X,\mathbb{R}_\tau)=C(X)$ and so $C(X,\mathbb{R}^d)$ is conditionally complete. It follows that (3) implies (4).

Finally assume $C(X,\mathbb{R}^d)$ is conditionally complete and that X is zero-dimensional. It follows from Theorem 2.3 that $C(X,\mathbb{R}_\tau)=C(X)$. Also $C(X)$ is conditionally complete, which implies X is extremally disconnected. Therefore (4) implies (1). □

Note that if X is of nonmeasurable cardinality and satisfies the conditions of Theorem 2.6, then X is discrete. See Problem 12H of [1] for more information on extremally disconnected P-spaces.

Corollary 2.7. *If X is zero-dimensional and $C(X,\mathbb{R}_\tau)$ is conditionally complete, then $C(X)$ is conditionally complete.*

PROOF. According to Theorem 2.6, if X is zero-dimensional and $C(X, \mathbb{R}_\tau)$ is conditionally complete, then X is extremally disconnected. Hence $C(X)$ is conditionally complete. $\square$

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