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A NEW CHARACTERIZATION OF BAIRE CLASS 1 FUNCTIONS

Abstract

We give a new characterization of the Baire class 1 functions (defined on an ultrametric space) by proving that they are exactly the pointwise limits of sequences of full functions, which are particularly simple Lipschitz functions. Moreover we highlight the link between the two classical stratifications of the Borel functions by showing that the Baire class functions of some level are exactly those obtained as uniform limits of sequences of Delta functions of a corresponding level.

1 Introduction.

If X and Y are metrizable spaces, a function $f\colon X \to Y$ is said to be *continuous* if the preimage of an open set of Y is open with respect to the topology of X; i.e. if $f^{-1}(U) \in \mathbf{\Sigma}^0_1(X)$ for every $U \in \mathbf{\Sigma}^0_1(Y)$. There are two natural generalizations of this definition, namely functions such that $f^{-1}(U) \in \mathbf{\Sigma}^0_{\xi+1}(X)$ for every U open in Y and functions such that $f^{-1}(S) \in \mathbf{\Sigma}^0_\xi(X)$ for every $S \in \mathbf{\Sigma}^0_\xi(Y)$, where ξ is a nonzero countable ordinal: the former are called *Baire class functions (of level ξ)* while the latter are called *Delta functions (of level ξ)*. Each generalization provides a stratification of the Borel functions from X to Y, but if we compare the levels of the two hierarchies, that is if we fix some $\xi < \omega_1$ in the definitions above, they are quite different: for example, each level of the Delta functions is closed under composition, while no level of the Baire class functions, apart from continuous functions, has such a property.

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The Baire class stratification was introduced by Baire in 1899 (with a slightly different definition which, however, turns out to be equivalent to the one proposed here in the relevant cases) and has been extensively studied. Of particular interest are the Baire class 1 functions, or those functions such that the preimage of an open set is a $\mathbf{\Sigma}^0_2$ set. For example, if $f\colon [0,1] \to \mathbb{R}$ is differentiable, where at endpoints we take one-side derivatives, then its derivative f' is of Baire class 1. Moreover Baire class 1 functions, and in particular those from the Baire space ${}^\omega\omega$ or from any compact space X to $\mathbb{R}$, have lots of applications in the theory of Banach spaces — for more on this subject see, for example, [2], [3],[4], [5], [6] and references quoted there.

In this paper we will give a new characterization for the Baire class 1 functions defined from an ultrametric space X, such as the Baire space ${}^\omega\omega$ or the Cantor space ${}^\omega 2$, to any separable metric space Y by showing that they are exactly the pointwise limits of sequences of full functions, which are particular Lipschitz functions (see Definition 2.13). Moreover we will show that the two hierarchies presented before are intimately related by proving that a function is of level ξ in the Baire class stratification just in the case it is the uniform limit of functions of level $\xi + 1$ in the Delta stratification. In particular, this gives another characterization of the Baire class 1 functions when we take $\xi = 1$.

The paper is organized as follows. In Section 2 we give some (old and new) definitions and state the main theorems of the paper. In Section 3 we consider the relations between Baire class and Delta functions, while in Section 4 we prove some theorems about zero-dimensional and ultrametric spaces. The results of these two sections are partially implicit in some classical proofs, but we put them here since we want to highlight the link between the two stratifications of the Borel functions and the special properties of Borel-partitions of completely disconnected spaces. Finally, in Section 5 we give the proof of the new characterization of the Baire class 1 functions.

All the proofs need only a very small fragment of the Axiom of Choice, namely Countable Choice over the Reals ($\mathsf{AC}_\omega(\mathbb{R})$ for short)[1]. It seems not possible to avoid this very weak assumption since it is needed even to prove very basic results in Descriptive Set Theory, e.g. to prove that $\mathbf{\Sigma}^0_2(\mathbb{R})$ is closed under countable unions. Hence we will always work under $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$. All the metrics d considered throughout the paper are always assumed to be such that $d \leq 1$. This condition is needed for the proofs of some of the results,

[1]The fact that we will not use the full Axiom of Choice becomes relevant if one wants to assume other axioms which contradict AC (which however are, in general, consistent with $\mathsf{AC}_\omega(\mathbb{R})$). For example, the Axiom of Determinacy AD is needed to carry out the Wadge's analysis of continuous reducibility, so it could be useful to check that our results hold also in that context.

but it is not a true limitation. In fact, given any metric d on X, it is easy to see that $d' = \frac{d}{1+d}$ is a metric on X compatible with d such that $d' \leq 1$. Moreover, d is an ultrametric if and only if d' is an ultrametric, and one can easily check that all the definitions given in this paper are "invariant" under such a transformation of the metric, e.g. $A \subseteq X$ is a full set with respect to d (see the definition at page 35) just in the case it is a full set with respect to d', although with different constants[2]. Thus all the results hold also when considering *arbitrary* (ultra)metrics. Finally, given any two sets A and B, we will denote by ${}^{A}B$ the set of all the functions from A to B and by ${}^{<\omega}A$ the set of all the *finite* sequences of elements from A. In particular, the set of all the ω-sequences of natural numbers ${}^{\omega}\omega$ will denote the Baire space (endowed with the usual topology), while ${}^{<\omega}\omega$ will denote the set of all the *finite* sequences of natural numbers. For all the other undefined concepts and symbols we will always refer the reader to the standard monograph [1].

Finally, it is the author's pleasure to acknowledge his debt to Sławomir Solecki for his review of the present work at an earlier stage, and for the suggestion of a further generalization of the characterization previously obtained.

2 Preliminaries and Statement of the Main Results.

We start with a few definitions and basic results, following closely the presentation of [1].

Definition 2.1. Let X, Y be metrizable spaces and $\xi < \omega_1$ a nonzero ordinal. A function $f\colon X \to Y$ is of *Baire class* 1 if $f^{-1}(U) \in \mathbf{\Sigma}^0_2(X)$ for every open set $U \subseteq Y$. Recursively, for $1 < \xi < \omega_1$ we define now a function $f\colon X \to Y$ to be of *Baire class* ξ if it is the pointwise limit of a sequence of functions $f_n\colon X \to Y$, where f_n is of Baire class $\xi_n < \xi$.

We denote by $\mathcal{B}_\xi(X,Y)$ the set of Baire class ξ functions from X into Y.

A function f which is of Baire class ξ (for some nonzero countable ordinal ξ) is called a *Baire class function.*

Definition 2.2. Let X, Y be metrizable spaces and let Γ be some collection of subsets of X. We say that $f\colon X \to Y$ is Γ-*measurable* if $f^{-1}(U) \in \Gamma$ for every open set $U \subseteq Y$.

The link between Γ-measurable and Baire class functions is given by the following classical theorem.

[2]In particular, one constant can be obtained from the other one via the bijection $j\colon \mathbb{R}^+ \to (0,1)\colon r \mapsto \frac{r}{1+r}$.

Theorem 2.3 (Lebesgue, Hausdorff, Banach)**.** *Let X, Y be metrizable spaces, with Y separable. Then for $1 \leq \xi < \omega_1$, $f\colon X \to Y$ is of Baire class ξ if and only if f is $\mathbf{\Sigma}^0_{\xi+1}$-measurable.*

By analogy with respect to this theorem, we say that a function f between two metrizable spaces is of *Baire class 0* if and only if it is $\mathbf{\Sigma}^0_1$-measurable; i.e. if and only if f is continuous.

As a consequence of Theorem 2.3, if X and Y are metrizable spaces and Y is separable, then the Baire class ξ functions provide a stratification into ω_1 levels of all the Borel(-measurable) functions (i.e. functions such that $f^{-1}(U)$ is Borel for any $U \in \mathbf{\Sigma}^0_1(Y)$). In fact for every nonzero countable ξ and every $f \in \mathcal{B}_\xi(X,Y)$, f is clearly Borel. Conversely, let U_n be a countable basis for the topology of Y and let f be Borel. Let μ_n be nonzero countable ordinals such that $f^{-1}(U_n) \in \mathbf{\Sigma}^0_{\mu_n}$ and let $\xi = \sup\{\mu_n \mid n \in \omega\}$ (which is again a nonzero countable ordinal). Since $\mathbf{\Sigma}^0_\xi$ is closed under countable unions and $f^{-1}(U_n) \in \mathbf{\Sigma}^0_\xi$ for every $n \in \omega$, we have that $f \in \mathcal{B}_\xi(X,Y)$. Note also that any $\mathcal{B}_\xi(X,X)$ is not closed under composition since, in general, if $f \in \mathcal{B}_\mu(X,Y)$ and $g \in \mathcal{B}_\nu(Y,Z)$ then $g \circ f \in \mathcal{B}_{\mu+\nu}(X,Z)$. This result follows from the fact that if $A \in \mathbf{\Sigma}^0_\nu(Y)$ and $f \in \mathcal{B}_\mu(X,Y)$, then $f^{-1}(A) \in \mathbf{\Sigma}^0_{\mu+\nu}$.

The following is another classical fact[3].

Theorem 2.4 (Lebesgue, Hausdorff, Banach)**.** *Let X, Y be separable metrizable. Moreover, assume that either X is zero-dimensional or $Y = \mathbb{R}^n$ for some $n \in \omega$ (or even $Y = \mathbb{C}^m$ or $Y = [0,1]^m$ for some $m \in \omega$). Then $f\colon X \to Y$ is of Baire class 1 if and only if f is the pointwise limit of a sequence of continuous functions.*

Hence, under the hypotheses of this theorem, $f \in \mathcal{B}_\xi(X,Y)$ if and only if it is the pointwise limit of a sequence of functions in $\bigcup_{\nu<\xi} \mathcal{B}_\nu(X,Y)$, for all $\xi \geq 1$.

There is another stratification of the Borel functions (in the case that Y is separable) which is important because, contrary to the case of Baire class functions, every level is a set of functions closed under composition.

Definition 2.5. Let X, Y be metrizable spaces and $\xi < \omega_1$ be a nonzero ordinal. A function $f\colon X \to Y$ is a $\mathbf{\Delta}^0_\xi$*-function* ($\mathbf{\Delta}^0_\xi$ for short) if $f^{-1}(A) \in \mathbf{\Sigma}^0_\xi(X)$ for every $A \in \mathbf{\Sigma}^0_\xi(Y)$.

[3] In general, if X and Y are metrizable with Y separable and $f\colon X \to Y$ is the pointwise limit of a sequence of continuous functions, then f is of Baire class 1. Nevertheless the converse fails in the general case. For a counterexample, simply take $X = \mathbb{R}$ and $Y = \{0,1\}$ (with the discrete metric) and consider the function such that $f(0) = 1$ and $f(x) = 0$ for every $x \neq 0$.

We denote by $\mathcal{D}_\xi(X, Y)$ the set of such functions.

Proposition 2.6. *For $\xi > 1$ the following are equivalent*[4]*:*

i) f *is* $\mathbf{\Delta}^0_\xi$*;*

ii) $f^{-1}(A) \in \mathbf{\Pi}^0_\xi$ *for every* $A \in \mathbf{\Pi}^0_\xi$*;*

iii) $f^{-1}(A) \in \mathbf{\Delta}^0_\xi$ *for every* $A \in \mathbf{\Delta}^0_\xi$ *(where as usual* $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$*);*

iv) $f^{-1}(A) \in \mathbf{\Sigma}^0_\xi$ *for every* $\nu < \xi$ *and* $A \in \mathbf{\Pi}^0_\nu$*;*

v) $f^{-1}(A) \in \mathbf{\Delta}^0_\xi$ *for every* $\nu < \xi$ *and* $A \in \mathbf{\Sigma}^0_\nu$*.*

PROOF. Since $\mathbf{\Sigma}^0_\xi$ is closed under countable unions, it is easy to see that *i)* $\iff$ *iii)*. *i)* $\iff$ *ii)* is obvious, and also *iii)* $\Rightarrow$ *v)* is trivial (since $\mathbf{\Sigma}^0_\nu \subseteq \mathbf{\Delta}^0_\xi$ for every $\nu < \xi$). *v)* $\Rightarrow$ *iv)* since $\mathbf{\Delta}^0_\xi$ is closed under complementation and is contained by definition in $\mathbf{\Sigma}^0_\xi$. Finally, to see that *iv)* $\Rightarrow$ *i)* recall that, by definition, every $\mathbf{\Sigma}^0_\xi$ set A can be written as a countable union of $\bigcup_{\nu<\xi} \mathbf{\Pi}^0_\nu$ sets. □

As in the case of Baire class functions, a function f which is a $\mathbf{\Delta}^0_\xi$-function (for some nonzero countable ordinal ξ) is called a *Delta function*.

To observe that the Delta functions provide a stratification into ω_1 levels of all the Borel functions it is enough to observe that every open set of Y is in $\mathbf{\Sigma}^0_\xi(Y)$ for every nonzero countable ordinal ξ whenever Y is metrizable (and hence every Delta function is Borel) and that every Baire class function is a Delta function. To see this, let $f \in \mathcal{B}_\nu(X, Y)$ and let ξ be the first additively closed ordinal above ν (that is $\xi = \nu \cdot \omega$): we claim that f is a $\mathbf{\Delta}^0_\xi$-function. In fact, let $S \in \mathbf{\Sigma}^0_\xi$: by definition, $S = \bigcup_n P_n$, where each $P_n \in \mathbf{\Pi}^0_{\mu_n}(Y)$ for some $\mu_n < \xi$. Since $f \in \mathcal{B}_\nu(X, Y)$ we have that $Q_n = f^{-1}(P_n) \in \mathbf{\Pi}_{\nu+\mu_n}(X)$ and hence $f^{-1}(S) = \bigcup_n Q_n$ where each Q_n is in $\mathbf{\Pi}^0_{\nu+\mu_n}(X)$. Since ξ is additively closed and $\nu, \mu_n < \xi$ we have that $\nu + \mu_n < \xi$ for every $n \in \omega$: therefore $f^{-1}(S) \in \mathbf{\Sigma}^0_\xi(X)$ by definition.

Moreover, using again the fact that $\mathbf{\Sigma}^0_1(Y) \subseteq \mathbf{\Sigma}^0_\xi(Y)$, it is easy to check that $\mathcal{D}_{\xi+1}(X, Y) \subseteq \mathcal{B}_\xi(X, Y)$.

[4]For $\xi = 1$ we have in general that *i)* $\iff$ *ii)* $\Rightarrow$ *iii)* $\iff$ *iv)* $\iff$ *v)* but not *iii)* $\Rightarrow$ *i)* (in fact if Y is connected we have that *every* function f satisfies *iii)*), but f is a $\mathbf{\Delta}^0_1$-function if and only if f is continuous). Nevertheless our proposition remains true even for $\xi = 1$ if we require that Y is zero-dimensional.

Definition 2.7. Let X and Y be two metrizable spaces and let $\mathcal{F} \subseteq \mathcal{G}$ be two sets of functions from X to Y. Then $\mathcal{F}$ is a *basis* for $\mathcal{G}$ if every function in $\mathcal{G}$ is the *uniform* limit of a sequence of functions in $\mathcal{F}$.

We will prove in Section 3 that each level of the Delta functions forms a basis for a corresponding level of the Baire class functions. This result is essentially implicit in the proof of Theorem 2.3 (see Theorem 24.3 in [1]), but we will reprove it here for the sake of completeness.

Theorem 2.8. *Let (X, d_X), (Y, d_Y) be two metric spaces and assume that Y is also separable. A function $f\colon X \to Y$ is of Baire class ξ if and only if it is the* uniform *limit of a sequence of $\mathbf{\Delta}^0_{\xi+1}$-functions.*

Corollary 2.9. *Let (X, d_X), (Y, d_Y) be two metric spaces and assume that Y is also separable. A function $f\colon X \to Y$ is in $\mathcal{B}_1(X, Y)$ if and only if it is the* uniform *limit of a sequence of $\mathbf{\Delta}^0_2$-functions.*

From Theorem 2.8 we can also derive the following corollary. It can be seen as an extension of Theorem 2.4: in that case it was proved (under stronger hypotheses) that f is of Baire class 1 if and only if it is the pointwise limit of a sequence of $\mathbf{\Delta}^0_1$-functions, i.e. continuous functions. Here we prove the same result for every level different from 1 (under weaker hypotheses).

Corollary 2.10. *Let X, Y be two metrizable spaces and assume that Y is also separable. Then for every $1 < \xi < \omega_1$, $f\colon X \to Y$ is of Baire class ξ if and only if f is the pointwise limit of a sequence of $\mathbf{\Delta}^0_\xi$-functions.*

By Theorem 2.4, as previously observed, Corollary 2.10 remains true in the case $\xi = 1$ if we require that X is separable and either X is zero-dimensional or Y is one of $\mathbb{R}^n$, $[0,1]^n$ or $\mathbb{C}^n$ for some $n \in \omega$.

Finally, we want to give a new characterization of the Baire class 1 functions. First recall the following definition.

Definition 2.11. Let (X, d_X), (Y, d_Y) be two metric spaces. A function $f\colon X \to Y$ is *Lipschitz (with constant $L \in \mathbb{R}^+$)* if

$$\forall x, x' \in X (d_Y(f(x), f(x')) \leq L \cdot d_X(x, x')).$$

We denote by $\mathrm{Lip}(X, Y; L)$ the set of such functions and put $\mathrm{Lip}(X, Y) = \bigcup_{L \in \mathbb{R}^+} \mathrm{Lip}(X, Y; L)$.

Let now (X, d_X) be an *ultrametric space*, a metric space such that d_X is an ultrametric. A set $A \subseteq X$ is *full (with constant* $r \in \mathbb{R}^+$*)* if

$$\forall x \in A(B(x,r) \subseteq A),$$

where $B(x,r) = \{y \in X \mid d_X(x,y) < r\}$ is the usual open ball.

Proposition 2.12. *Let (X, d_X) be an ultrametric space. Then the full subsets of X form an algebra. Moreover, an arbitrary union of balls with a fixed radius is full (in particular, an arbitrary union of full sets with the same constant is full).*

Proof. Let A and B be full sets with constants r_A and r_B respectively. Then it is easy to check that $A \cup B$ is full with constant $r = \min\{r_A, r_B\}$. Moreover, let $x \notin A$ and assume towards a contradiction that $y \in A$ for some $y \in B(x, r_A)$. By the properties of the ultrametric d_X, we have that $B(y, r_A) = B(x, r_A)$; but since A is full with constant r_A, then $B(y, r_A) \subseteq A$ and hence $x \in A$, a contradiction! Thus $X \setminus A$ is full with constant r_A. The second part follows again from the properties of an ultrametric. □

Definition 2.13. Let (X, d_X) be an ultrametric space and Y be any separable metrizable space. A function $f \colon X \to Y$ is said to be *full* if it has only finitely many values and the preimage of each of these values is a full set.

The function f is said to be ω*-full* if it has at most countably many values and there is some fixed $r \in \mathbb{R}^+$ such that the preimage of each value is a full set with constant r.

It is clear that every full function is ω-full. Moreover, if f is ω-full and $r \in \mathbb{R}^+$ witnesses this, then $f \in \mathrm{Lip}(X, Y; r^{-1})$ with respect to any metric d_Y compatible with the topology of Y such that $d_Y \leq 1$. In fact, let d_Y be such a metric and let $x, x' \in X$: if $d_X(x, x') \geq r$, then

$$d_Y(f(x), f(x')) \leq 1 = r^{-1} \cdot r \leq r^{-1} d_X(x, x'),$$

while if $d_X(x, x') < r$ then $x' \in B(x, r)$, so that $x' \in f^{-1}(f(x))$ and hence $f(x) = f(x')$.

Proposition 2.14. *Let (X, d_X) be an ultrametric space and (Y, d_Y), (Z, d_Z) be two metric spaces. Let $f \colon X \to Y$ be a full function, $g \in \mathrm{Lip}(Y, Z; L)$ and $h \in \mathrm{Lip}(Z, X; L)$. Then $g \circ f$ is full and, if d_Z is an ultrametric, $f \circ h$ is also full.*

The same result holds if we systematically replace “full” with “ω-full”.

Proof. The first part is obvious, since for every $z \in Z$ the set $(g \circ f)^{-1}(z)$ is either empty or the union of finitely many full sets (and the cardinality of range$(g \circ f)$ is less than or equal to the cardinality of range(f)). For the second part, it is enough to show that the preimage via h of a full set $A \subseteq X$ with constant r is a full set with constant $r \cdot L^{-1}$. In fact, let $z \in Z$ be such that $h(z) \in A$ and let $z' \in Z$ be such that $d_Z(z, z') < rL^{-1}$. Then $d_X(h(z), h(z')) \leq L \cdot d_Z(z, z') < LrL^{-1} = r$, and thus $h(z') \in A$. But this implies $B(z, rL^{-1}) \subseteq h^{-1}(A)$ and hence we are done. The case in which f is ω-full is proved in a similar way. □

Now we are ready to state the main theorem of this paper.

Theorem 2.15. *Let (X, d_X) be an ultrametric space and let Y be any separable metrizable space. Then $f : X \to Y$ is of Baire class 1 if and only if f is the pointwise limit of a sequence of full functions.*

By the observations above and since every Lipschitz function is uniformly continuous, we also have the following corollary as a simple consequence of Theorem 2.15 (here we clearly use the fact that every limit of continuous functions is of Baire class 1 — see note 2 on page 32).

Corollary 2.16. *Let (X, d_X), (Y, d_Y) be separable metric spaces and assume that X is an ultrametric space with respect to d_X. For every $f : X \to Y$ the following are equivalent:*

i) f is of Baire class 1;

ii) f is the pointwise limit of a sequence of ω-full functions;

iii) f is the pointwise limit of a sequence of Lipschitz functions;

iv) f is the pointwise limit of a sequence of uniformly continuous functions.

The author first proved Theorem 2.15 but using Lipschitz (in particular ω-full) functions rather than full functions, although the proof was essentially the same presented here in Section 5. The idea to generalize the result to the present form, as well as the definition of fullness, is due to S. Solecki.

3 The Link Between Baire Class and Delta Functions.

We first give some basic definitions.

Definition 3.1. Let X be a topological space and $\Gamma \subseteq \mathscr{P}(X)$ be any pointclass. A Γ-*partition* of a set $C \in \Gamma$ is a family $\langle C_n \mid n < N \rangle$ of nonempty pairwise disjoint sets of Γ such that $C = \bigcup_{n<N} C_n$ and $1 \leq N \leq \omega$ [5].

Definition 3.2. Let X, Y be two metrizable spaces and let $\mathcal{F}$ be some set of functions between X and Y. Let $f\colon X \to Y$ be an arbitrary function and $\langle C_n \mid n < N \rangle$ be some partition of X. We say that f is *(locally) in $\mathcal{F}$ on the partition* $\langle C_n \mid n < N \rangle$ if there is a family of functions $\{f_n \mid n < N\} \subseteq \mathcal{F}$ such that $f \restriction C_n = f_n \restriction C_n$ for every $n < N$.

Moreover, if $\Gamma \subseteq \mathscr{P}(X)$ is any pointclass, we will say that f is *(locally) in $\mathcal{F}$ on a Γ-partition* if there is some Γ-partition such that f is locally in $\mathcal{F}$ on it.

Obviously, if $\mathcal{F}$ and $\mathcal{G}$ are sets of functions and $\mathcal{F} \subseteq \mathcal{G}$, then if f is locally in $\mathcal{F}$ on the partition $\langle C_n \mid n < N \rangle$ we have also that f is locally in $\mathcal{G}$ on the same partition.

Proposition 3.3. *Let X, Y be two metrizable spaces and ξ be some nonzero countable ordinal. Then $f\colon X \to Y$ is in $\mathcal{D}_\xi(X,Y)$ if and only if there is a $\boldsymbol{\Sigma}^0_\xi$-partition of X such that f is locally in $\mathcal{D}_\xi(X,Y)$ on it.*

Proof. The direct implication is trivial, hence we have only to prove that if $\langle C_n \mid n < N \rangle$ is a $\boldsymbol{\Sigma}^0_\xi(X)$-partition on X and $\{f_n \mid n < N\} \subseteq \mathcal{D}_\xi(X,Y)$ witnesses that f is locally in $\mathcal{D}_\xi(X,Y)$ on it, then $f \in \mathcal{D}_\xi(X,Y)$. To see this, let $S \in \boldsymbol{\Sigma}^0_\xi(Y)$. Then $f^{-1}(S) = \bigcup_{n<N}(f^{-1}(S) \cap C_n) = \bigcup_{n<N}(f_n^{-1}(S) \cap C_n)$: but $f_n^{-1}(S) \cap C_n \in \boldsymbol{\Sigma}^0_\xi(X)$ for every $n < N$ and therefore $f^{-1}(S) \in \boldsymbol{\Sigma}^0_\xi(X)$ by closure under countable unions of $\boldsymbol{\Sigma}^0_\xi(X)$. $\square$

We are now ready to prove a theorem from which Theorem 2.8 easily follows. We will use the following standard fact (see e.g. Exercise 24.4 in [1]).

Fact 3.4. Let (X, d_X), (Y, d_Y) be two metric spaces, ξ be a nonzero countable ordinal and let $\langle f_n \mid n \in \omega \rangle$ be a sequence of functions from $\mathcal{B}_\xi(X,Y)$ converging uniformly to some $f\colon X \to Y$. Then $f \in \mathcal{B}_\xi(X,Y)$ as well.

Theorem 3.5. *Let (X, d_X), (Y, d_Y) be two metric spaces and assume that Y is also separable. Then for*[6] $1 \leq \xi < \omega_1$ *and for every function $f\colon X \to Y$ the following conditions are equivalent:*

[5]For the rest of the paper we will always assume without explicitly mentioning it that N is some ordinal less or equal to ω.

[6]If $\xi = 0$ then it is trivially true that $i) \iff iv) \iff v)$, but $ii)$ and $iii)$ are not equivalent to $i)$ unless X is zero-dimensional.

i) *f is of Baire class* ξ*;*

ii) *there is a sequence of functions* $\langle f_k \mid k \in \omega \rangle$ *which converges uniformly to* f *and such that every* f_k *is locally constant on some* $\mathbf{\Sigma}^0_{\xi+1}$*-partition;*

iii) *there is a sequence of functions* $\langle f_k \mid k \in \omega \rangle$ *which converges uniformly to* f *and such that every* f_k *is locally Lipschitz on some* $\mathbf{\Sigma}^0_{\xi+1}$*-partition;*

iv) *there is a sequence of functions* $\langle f_k \mid k \in \omega \rangle$ *which converges uniformly to* f *and such that every* f_k *is locally continuous on some* $\mathbf{\Sigma}^0_{\xi+1}$*-partition;*

v) *there is a sequence of functions* $\langle f_k \mid k \in \omega \rangle$ *which converges uniformly to* f *and such that every* f_k *is a* $\mathbf{\Delta}^0_{\xi+1}$*-function.*

Proof. It is obvious that *ii)* $\Rightarrow$ *iii)* and *iii)* $\Rightarrow$ *iv)*, since every constant function is Lipschitz and every Lipschitz function is also continuous. Moreover, using Proposition 3.3 and the fact that every continuous function is $\mathbf{\Delta}^0_\xi$ for every nonzero $\xi < \omega_1$, we have also *iv)* $\Rightarrow$ *v)*. Also *v)* $\Rightarrow$ *i)* is easy; in fact every $\mathbf{\Delta}^0_{\xi+1}$-function is of Baire class ξ and $\mathcal{B}_\xi(X, Y)$ is closed under uniform limits by Fact 3.4.

Finally we prove *i)* $\Rightarrow$ *ii)*. For every $k \in \omega$, fix some open cover $\langle U^k_n \mid n < N \rangle$ of Y of mesh 2^{-k}, that is a sequence of open sets such that $Y = \bigcup_{n<N} U^k_n$ and $\operatorname{diam}(U^k_n) \leq 2^{-k}$ for every $n < N$, and fix also a sequence $\langle z^k_n \mid n < N \rangle$ of points of Y such that $z^k_n \in U^k_n$ for every $n < N$. One way to do this is to consider an enumeration $\langle y_n \mid n < N \rangle$ of some (at most) countable dense set of points of Y (which exists because Y is separable) and to put $U^k_n = B(y_n, 2^{-(k+1)}) = \{y \in Y \mid d_Y(y, y_n) < 2^{-(k+1)}\}$ and $z^k_n = y_n$. Let $S^k_n = f^{-1}(U^k_n)$. Clearly $\forall n < N(S^k_n \in \mathbf{\Sigma}^0_{\xi+1}(X))$ because $f \in \mathcal{B}_\xi(X, Y)$, and the sets $\langle S^k_n \mid n < N \rangle$ cover X. Since $\mathbf{\Sigma}^0_{\xi+1}$ has the generalized reduction property, we can find for every $k \in \omega$ a sequence $\langle Q^k_n \mid n < N \rangle$ of $\mathbf{\Sigma}^0_{\xi+1}$ sets such that $Q^k_n \subseteq S^k_n$ for every $n < N$, $Q^k_n \cap Q^k_m = \emptyset$ if $n \neq m$ and $\bigcup_{n<N} Q^k_n = \bigcup_{n<N} S^k_n = X$, so that, in particular, the Q^k_n's form a $\mathbf{\Sigma}^0_{\xi+1}$-partition of X. Now define $f_k \colon X \to Y \colon x \mapsto z^k_n$, where $n < N$ is the unique natural number such that $x \in Q^k_n$. Note that f_k is locally constant on $\langle Q^k_n \mid n < N \rangle$. It remains only to prove that the sequence $\langle f_k \mid k \in \omega \rangle$ converges uniformly to f. This follows immediately from the claim below.

Claim 3.5.1. Fix some $k \in \omega$. Then for every $x \in X$

$$d(f_k(x), f(x)) \leq 2^{-k}.$$

Proof of the Claim. Fix some $x \in X$ and let n be such that $x \in Q^k_n$ (so that, in particular, $x \in S^k_n$). Then $f(x) \in U^k_n$, and since $\mathrm{diam}(U^k_n) \leq 2^{-k}$ and $z^k_n \in U^k_n$ we have that $d(f_k(x), f(x)) \leq 2^{-k}$. □ *Claim*

□

Remark 3.6. The same result holds if we consider $\mathbf{\Delta}^0_{\xi+1}$-partitions because it is easy to check that every $\mathbf{\Sigma}^0_{\xi+1}$-partition is actually a $\mathbf{\Delta}^0_{\xi+1}$-partition (the converse is trivially true). We can also consider partitions formed only by sets which are the difference of two $\mathbf{\Pi}^0_\xi$ sets; i.e. 2-$\mathbf{\Pi}^0_\xi$ sets, as every $\mathbf{\Sigma}^0_{\xi+1}$-partition $\langle S_n \mid n < N \rangle$ can be refined to a 2-$\mathbf{\Pi}^0_\xi$-partition. In fact, since $S_n \in \mathbf{\Sigma}^0_{\xi+1}$, by definition there are $\langle P_{n,m} \mid m \in \omega, n < N \rangle$ such that $P_{n,m} \in \mathbf{\Pi}^0_\xi$ and $S_n = \bigcup_{m \in \omega} P_{n,m}$ for every $n < N$ (we are not requiring that the sets $P_{n,m}$ are different for distinct indexes m, hence we can suppose that $P_{n,m}$ is defined for every $m \in \omega$). Fix some bijection $\langle \cdot, \cdot \rangle$ between $\omega \times \omega$ and ω (for example $\langle i, j \rangle = 2^i(2j+1)) - 1$) and let $R_{\langle n,m \rangle} = P_{n,m}$. Inductively put $j_0 = 0$ and $j_{i+1} = \min\{j \mid j > j_i \wedge R_j \setminus \bigcup_{l<j} R_l \neq \emptyset\}$ (in general the sequence j_i is defined for $i < I$ where $I \leq \omega$). Now define

$$Q_i = R_{j_i} \setminus \bigcup_{l<j_i} R_l = R_{j_i} \setminus \bigcup_{l<i} Q_l$$

for every $i < I$. Since the sets S_n cover X, $\langle Q_i \mid i < I \rangle$ is clearly an at most countable partition of X which refines $\langle S_n \mid n < N \rangle$, and each Q_i is the difference of two $\mathbf{\Pi}^0_\xi$ sets because $\mathbf{\Pi}^0_\xi$ is closed under finite unions.

Note also that if d_Y is a compact metric (e.g. it is induced by any metric on a compactification of Y), the partitions above can be taken to be finite.

Finally, as we will see in the next section, if X is zero-dimensional we can strengthen the result a little bit by taking $\mathbf{\Pi}^0_\xi$-partitions instead of $\mathbf{\Sigma}^0_{\xi+1}$-partitions in conditions *ii)-v)*.

Now we restate Corollary 2.10 in the following (slightly) stronger form.

Proposition 3.7. *Let X, Y be two metrizable spaces and assume that Y is also separable. Then for every nonzero $\xi < \omega_1$, if there is a sequence of $\mathbf{\Delta}^0_\xi$-functions that converges pointwise to f then f is of Baire class ξ.*

Conversely, if $\xi > 1$ and $f \colon X \to Y$ is of Baire class ξ, then there is a sequence of $\mathbf{\Delta}^0_\xi$-functions pointwise converging to f.

Proof. Let d be a compatible metric on Y. Recall that every open sphere U of Y can be written as the union of countably many closed spheres each of which is contained in the interior of the following one. In fact let $U =$

$B(y_0, \varepsilon) = \{y \in Y \mid d(y, y_0) < \varepsilon\}$ and let $\langle \varepsilon_m \mid m \in \omega \rangle$ be a strictly increasing sequence of real numbers such that $\varepsilon_m < \varepsilon$ for every $m \in \omega$ and $\lim_m \varepsilon_m = \varepsilon$: then $U = \bigcup_{m\in\omega} \overline{B_m(y_0, \varepsilon_m)} = \bigcup_{m\in\omega}\{y \in Y \mid d(y, y_0) \leq \varepsilon_m\}$. Moreover, since $\varepsilon_n < \varepsilon_{n+1}$, we have also $\overline{B_m} = \overline{B(y_0, \varepsilon_m)} \subseteq B_{m+1} = B(y_0, \varepsilon_{m+1})$, that is $U = \bigcup_{m\in\omega} B_m$.

Now assume that $\langle f_k \mid k \in \omega \rangle$ is a sequence of $\mathbf{\Delta}^0_\xi$-functions pointwise converging to f: it is enough to prove that $f^{-1}(U) \in \mathbf{\Sigma}^0_{\xi+1}(X)$ for every open sphere $U = \bigcup_m \overline{B_m} \subseteq Y$. First note that

$$f^{-1}(U) = \bigcup_{m\in\omega} \bigcup_{n\in\omega} \bigcap_{k\geq n} f_k^{-1}(\overline{B_m}).$$

In fact, if $f(x) \in U$ then there is an m such that $f(x) \in B_m \subseteq \overline{B_m}$ and hence also $f_k(x) \in B_m$ for any k large enough (since f_k converge to f). For the other direction, if there exists m such that $f_k(x) \in \overline{B_m}$ for almost all k, then $f(x) \in \overline{B_m}$ as $\overline{B_m}$ is closed and $f(x)$ is the limit of the points $f_k(x)$.

Since each f_k is a $\mathbf{\Delta}^0_\xi$-function and since $\mathbf{\Pi}^0_1(Y) \subseteq \mathbf{\Pi}^0_\xi(Y)$ for every nonzero countable ξ, we have that $f^{-1}(\overline{B_m}) \in \mathbf{\Pi}^0_\xi(X)$ for every $m \in \omega$ and hence also $\bigcap_{k\geq n} f_k^{-1}(\overline{B_m}) \in \mathbf{\Pi}^0_\xi(X)$ for every $n \in \omega$ by closure under countable intersections of $\mathbf{\Pi}^0_\xi(X)$. But then $f^{-1}(U)$ is a countable union of $\mathbf{\Pi}^0_\xi(X)$ sets; i.e. it is a $\mathbf{\Sigma}^0_{\xi+1}(X)$ set and we are done.

Conversely, if $\xi > 1$ and f is of Baire class ξ then it is the pointwise limit of some sequence f_n of functions such that for every $n \in \omega$ there is a $1 \leq \nu_n < \xi$ such that f_n is of Baire class ν_n. For every $n \in \omega$, construct as in the proof of Theorem 3.5 a sequence $g_{n,m}$ of $\mathbf{\Delta}^0_{\nu_n+1}$-functions converging uniformly to f_n. Note that by Claim 3.5.1 we can assume that $d(g_{n,m}(x), f_n(x)) \leq 2^{-m}$ for every $x \in X$. Moreover, since $\nu_n + 1 \leq \xi$ we have that every $g_{n,m}$ is, in particular, a $\mathbf{\Delta}^0_\xi$-function. Take any diagonal subsequence $\langle h_n \mid n \in \omega \rangle$ of the $g_{n,m}$, e.g. $h_n = g_{n,n}$. It remains only to prove that this sequence converges pointwise to f. To see this, fix some $x \in X$ and $k \in \omega$. Let $j \in \omega$ be such that

$$\forall i \geq j \ (d(f_i(x), f(x)) < 2^{-(k+1)})$$

and put $m = \max\{j, k+1\}$. Clearly, for every $m' \geq m$ we have

$$\begin{aligned} d(h_{m'}(x), f(x)) &\leq d(g_{m',m'}(x), f_{m'}(x)) + d(f_{m'}(x), f(x)) \\ &< 2^{-m'} + 2^{-(k+1)} \leq 2 \cdot 2^{-(k+1)} = 2^{-k}. \end{aligned}$$

□

The same corollary clearly holds if we consider functions which are constant (respectively, Lipschitz, continuous) on a *finite* $\mathbf{\Delta}^0_\xi$-partition.

4 Zero-Dimensional Spaces.

We now prove some theorems on zero-dimensional and ultrametric spaces. In particular, the first one is a simple variation of some classical results — see e.g. [1]. Let $s \in {}^{<\omega}\omega$ be a finite sequence of natural numbers. We will denote the *length of* s by $\mathrm{lh}(s)$ (formally, $\mathrm{lh}(s) = \mathrm{dom}(s)$).

Theorem 4.1. *If (X, d) is a metric, separable and zero-dimensional space, then there is some set $A \subseteq {}^{\omega}\omega$ and an homeomorphism $h\colon A \to X$ such that $h \in \mathrm{Lip}(A, X; 1)$ (with respect to d and the usual metric d' that ${}^{\omega}\omega$ induces on A).*

If moreover d is an ultrametric, then h can be taken bi-Lipschitz, i.e. $h^{-1} \in \mathrm{Lip}(X, A; 2)$ (and $h \in \mathrm{Lip}(A, X; 1)$ as before).

If d is also complete then the set A can be taken to be a closed set.

Proof. The first part is a standard argument: one can construct a Lusin scheme $\langle C_s \mid s \in {}^{<\omega}\omega\rangle$ on X such that

i) $C_\emptyset = X$,

ii) C_s is clopen,

iii) $C_s = \bigcup_{i\in\omega} C_{s^\frown i}$,

iv) $\mathrm{diam}(C_s) \leq 2^{-\mathrm{lh}(s)}$.

From this one can conclude that the induced map h is defined on the set $A = \{y \in {}^{\omega}\omega \mid \bigcap_n C_{y\restriction n} \neq \emptyset\}$ and is an homeomorphism. But condition *iv)* implies also $h \in \mathrm{Lip}(A, X; 1)$. In fact, for every $x, y \in A$ such that $x \neq y$, let $n \in \omega$ be such that $d'(x, y) = 2^{-n}$ and let $s = x \restriction n = y \restriction n$. Clearly we have that $h(x) \in C_s$ and $h(y) \in C_s$. Thus condition *iv)* implies that $d(h(x), h(y)) \leq 2^{-\mathrm{lh}(s)} = 2^{-n} = d'(x, y)$.

If we now assume that d is an ultrametric on X, then we can construct a Lusin scheme $\langle C_s \mid s \in {}^{<\omega}\omega\rangle$ on X such that

i) $C_\emptyset = X$,

ii) either $C_s = \emptyset$ or C_s is a sphere,

iii) $C_s = \bigcup_{i\in\omega} C_{s^\frown i}$,

iv) $\mathrm{diam}(C_s) \leq 2^{-\mathrm{lh}(s)}$.

In fact every nonempty C_s, at least when $s \neq \emptyset$, will be defined as $C_s = B\left(x, 2^{-\mathrm{lh}(s)}\right)$ for some $x \in X$.

Let D be countable and dense in X. We construct the scheme by induction on $\mathrm{lh}(s)$. First put $C_\emptyset = X$. Suppose we have constructed C_s with properties *i)—iv)*. If $C_s = \emptyset$ then put $C_{s^\frown i} = \emptyset$ for every $i \in \omega$, otherwise fix an enumeration $\langle x_k \mid k \in \omega \rangle$ of $C_s \cap D$. Then define $C_{s^\frown 0} = B\left(x_0, 2^{-(\mathrm{lh}(s)+1)}\right)$ and either $C_{s^\frown i+1} = B\left(x_{k_{i+1}}, 2^{-(\mathrm{lh}(s)+1)}\right)$, where k_{i+1} is the smallest $k > k_i$ such that $x_k \notin \bigcup_{j \leq i} C_{s^\frown j}$, or $C_{s^\frown i+1} = \emptyset$ if such a k does not exist.

Clearly $C_{s^\frown i} \subseteq C_s$ because d is an ultrametric, $\mathrm{diam}(C_{s^\frown i}) \leq 2^{-(\mathrm{lh}(s)+1)}$, and $C_s = \bigcup_{i \in \omega} C_{s^\frown i}$ because D is dense, hence we are done. Arguing as before, h is a bijection defined on a set $A \subseteq {}^\omega\omega$ and $h \in \mathrm{Lip}(A, X; 1)$. Now we want to show that $d'(h^{-1}(x), h^{-1}(y)) \leq 2d(x,y)$ for every distinct $x, y \in X$. Put $S_{x,y} = \{s \in {}^{<\omega}\omega \mid x \in C_s \wedge y \in C_s\}$. Clearly $S_{x,y}$ is linearly ordered and admits an element t of maximal length (otherwise $x = y$). Thus $d'(h^{-1}(x), h^{-1}(y)) = 2^{-\mathrm{lh}(t)}$. If $d(x,y) < 2^{-(\mathrm{lh}(t)+1)}$, then, by the construction above and the fact that d is an ultrametric, there would be an $i \in \omega$ such that $x \in C_{t^\frown i}$ and $y \in C_{t^\frown i}$, contradicting the maximality of t. Hence $d(x,y) \geq 2^{-(\mathrm{lh}(t)+1)}$ and

$$d'(h^{-1}(x), h^{-1}(y)) = 2^{-\mathrm{lh}(t)} = 2 \cdot 2^{-(\mathrm{lh}(t)+1)} \leq 2d(x,y)$$

as required.

Finally it is not hard to check by standard arguments that the completeness of d implies that A is a closed set. □

Note that, in particular, this theorem provides also that every separable, metrizable and zero-dimensional space is ultrametrizable; i.e. admits a compatible ultrametric d. Let h be the homeomorphism given by the theorem above and simply put $d(x,x') = d'(h^{-1}(x), h^{-1}(x'))$ for every $x, x' \in X$, where d' is the usual (ultra)metric on ${}^\omega\omega$. Clearly, if X is also Polish we have that the ultrametric d is also complete.

For notational simplicity we put $\mathbf{\Sigma}^0_0 = \mathbf{\Pi}^0_0 = \mathbf{\Delta}^0_0 = \mathbf{\Delta}^0_1$. Moreover, for every countable ordinal ξ, we denote by $\mathbf{\Pi}^0_{<\xi}$ (respectively, $\mathbf{\Sigma}^0_{<\xi}$ and $\mathbf{\Delta}^0_{<\xi}$) the pointclass $\bigcup_{\nu<\xi} \mathbf{\Pi}^0_\nu$ (resp. $\bigcup_{\nu<\xi} \mathbf{\Sigma}^0_\nu$ and $\bigcup_{\nu<\xi} \mathbf{\Delta}^0_\nu$).

Theorem 4.2. *Let X be a separable, metrizable, zero-dimensional space and let A be a subset of X. For every nonzero $\xi < \omega_1$ the following are equivalent:*

i) $A \in \mathbf{\Sigma}^0_\xi$;

ii) *there is a $\mathbf{\Delta}^0_\xi$-partition of A; i.e. there is $\langle C_n \mid n < N \rangle$ such that for every $n, m < N$ we have $C_n \in \mathbf{\Delta}^0_\xi$, $C_n \cap C_m = \emptyset$ for $n \neq m$, and $A = \bigcup_{n<N} C_n$;*

iii) there is a $\mathbf{\Pi}^0_{<\xi}$*-partition of* A*, i.e. there is* $\langle P_n \mid n < N \rangle$ *such that for every* $n < N$ *there is some* $\nu_n < \xi$ *with* $P_n \in \mathbf{\Pi}^0_{\nu_n}$*,* $P_n \cap P_m = \emptyset$ *for* $n \neq m$*, and* $A = \bigcup_{n<N} P_n$*.*

PROOF. The implication $iii) \Rightarrow ii)$ is obvious since every $\mathbf{\Pi}^0_\nu$ set is also $\mathbf{\Delta}^0_\xi$ if $\nu < \xi$. Also $ii) \Rightarrow i)$ is easy since every $\mathbf{\Delta}^0_\xi$ set is by definition a $\mathbf{\Sigma}^0_\xi$ set and the latter pointclass is closed under countable unions. Hence we have only to prove $i) \Rightarrow iii)$ and this will be done by induction on $1 \leq \xi < \omega_1$.

If $\xi = 1$ we have only to note that every open set U can be written as a countable union of pairwise disjoint clopen sets. Since X is separable and zero-dimensional we have that $U = \bigcup_n C_n$ for some sets $C_n \in \mathbf{\Delta}^0_1$. Now define by induction $P_0 = C_0$ and $P_{n+1} = C_{n+1} \setminus \bigcup_{i \leq n} C_i$ and note that each P_n is clopen (since $\mathbf{\Delta}^0_1$ is closed under complementation and finite unions and intersections), $U = \bigcup_n P_n$ and that $P_n \cap P_m = \emptyset$ for $n \neq m$.

If $\xi > 1$ and $S \in \mathbf{\Sigma}^0_\xi$, by definition there are some sets $P_n \in \mathbf{\Pi}^0_{\nu_n}$ such that $S = \bigcup_n P_n$ and $\nu_n < \xi$ for all $n \in \omega$. First define inductively $P'_0 = P_0$ and $P'_{n+1} = P_{n+1} \setminus \bigcup_{i \leq n} P_i$ and note that they form a partition of S. Since $\nu' \leq \nu \Rightarrow \mathbf{\Pi}^0_{\nu'} \subseteq \mathbf{\Pi}^0_\nu$ and $\mathbf{\Pi}^0_\nu$ is closed under finite unions, each P'_n can clearly be seen as the difference of two $\mathbf{\Pi}^0_\nu$ sets where $\nu = \max\{\nu_0, \ldots, \nu_n\} < \xi$, and hence we have only to prove that for all $\nu < \xi$ every set of the form $Q \cap R$ with $Q \in \mathbf{\Pi}^0_\nu$ and $R \in \mathbf{\Sigma}^0_\nu$ admits a $\mathbf{\Pi}^0_\nu$ partition. Using the inductive hypothesis, find a partition $\langle R_n \mid n \in \omega \rangle$ of R such that $R_n \in \mathbf{\Pi}^0_{\mu_n}$ for some $\mu_n < \nu$ and note that $R_n \in \mathbf{\Pi}^0_\nu$ for every $n \in \omega$. Then it is easy to check that the sets $Q_n = Q \cap R_n$ are in $\mathbf{\Pi}^0_\nu$ and that they form a partition of $Q \cap R$, hence we are done. □

In particular, from the previous theorem we get that every $\mathbf{\Sigma}^0_{\xi+1}$ set admits a $\mathbf{\Pi}^0_\xi$-partition. This implies that every $\mathbf{\Sigma}^0_{\xi+1}$-partition of X can be refined to a $\mathbf{\Pi}^0_\xi$-partition and, more generally, every $\mathbf{\Sigma}^0_\xi$-partition can be refined to a $\mathbf{\Pi}^0_{<\xi}$-partition. Therefore, in the case X is separable, metrizable and zero-dimensional, we have the following improvement of Theorem 3.5.

Corollary 4.3. *Let* (X, d_X) *and* (Y, d_Y) *be two metric separable spaces and assume that* X *is also zero-dimensional. Then a function* $f\colon X \to Y$ *is of Baire class* ξ *if and only if there is a sequence of functions converging uniformly to it and such that each such function is locally constant (respectively, Lipschitz, continuous) on a* $\mathbf{\Pi}^0_\xi$*-partition of* X*.*

5 Baire Class 1 and Full Functions.

Let $\boldsymbol{\Gamma} \subseteq \mathscr{P}({}^\omega\omega)$ be a boldface pointclass, or a collection of subsets of ${}^\omega\omega$ closed under continuous preimages. We say that a set $A \in \boldsymbol{\Gamma}$ is $\boldsymbol{\Gamma}$-*complete* if for every $B \in \boldsymbol{\Gamma}$ there is a *reduction of B in A*; i.e. a continuous function $f \colon {}^\omega\omega \to {}^\omega\omega$ such that $B = f^{-1}(A)$.

Recall also that a continuous function from ${}^\omega\omega$ to ${}^\omega\omega$ can be viewed as the function arising from some particular function $\varphi \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$. We say that $\varphi \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ is *continuous* if $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$ for every $s, t \in {}^{<\omega}\omega$ and $\lim_{n\in\omega}(\mathrm{lh}(\varphi(x \restriction n))) = \infty$ for every $x \in {}^\omega\omega$. If φ is continuous it induces in a canonical way the unique function

$$f_\varphi \colon {}^\omega\omega \to {}^\omega\omega \colon x \mapsto \bigcup_{n\in\omega} \varphi(x \restriction n),$$

and it is not hard to see that f_φ is a continuous function.

Conversely, suppose $f \colon {}^\omega\omega \to {}^\omega\omega$ is continuous. For every $s \in {}^{<\omega}\omega$ consider the set $\Sigma_s = \{t \in {}^{<\omega}\omega \mid f(\mathbf{N}_s) \subseteq \mathbf{N}_t\}$, where $\mathbf{N}_s = \{x \in {}^\omega\omega \mid s \subseteq x\}$. The set Σ_s is linearly ordered because if t and t' are incompatible, then $\mathbf{N}_t \cap \mathbf{N}_{t'} = \emptyset$, and hence we can define $\varphi(s) = t_s$ where $t_s \in \Sigma_s$ is such that $\mathrm{lh}(t_s) = \max\{\mathrm{lh}(t) \mid \mathrm{lh}(t) \leq \mathrm{lh}(s) \wedge t \in \Sigma_s\}$. It is not difficult to check that $\varphi \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ is continuous and that $f_\varphi = f$.

By analogy with the previous definitions, if $A, B \subseteq {}^\omega\omega$ and $\varphi \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ is a continuous function such that $f_\varphi^{-1}(A) = B$, we call φ a *reduction of B into A* and we say that φ *reduces* B to A. From the observation above, it is clear that if A is $\boldsymbol{\Gamma}$-complete for some pointclass $\boldsymbol{\Gamma} \subseteq \mathscr{P}({}^\omega\omega)$, then for every $B \in \boldsymbol{\Gamma}$ there is a reduction $\varphi \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ of B in A.

For every $t, s \in {}^{<\omega}\omega$ define $t - s = \emptyset$ if $\mathrm{lh}(t) < \mathrm{lh}(s)$, and $t - s = u \in {}^{<\omega}\omega$, where u is such that $t = (t \restriction \mathrm{lh}(s))^\frown u$, otherwise.

Let $\vec{\varphi} = \langle \varphi_n \mid n < N \rangle$ be a sequence of continuous functions $\varphi_n \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$. Moreover, let $\langle n_k \mid k \in \omega \rangle$ be an enumeration of N with infinite repetitions and, if $N > 1$, such that $n_k \neq n_{k+1}$ for every $k \in \omega$. Define $(\vec{\varphi})^* \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ and $\sigma \colon {}^{<\omega}\omega \to N$ in the following way: first put $(\vec{\varphi})^*(\emptyset) = \emptyset$ and $\sigma(\emptyset) = n_0$. Then suppose we have defined $(\vec{\varphi})^*(s)$ and $\sigma(s) = n_k$, and inductively put

$$(\vec{\varphi})^*(s^\frown i) = (\vec{\varphi})^*(s)^\frown 1$$

if $\varphi_{\sigma(s)}(s^\frown i) - (\vec{\varphi})^*(s)$ does not contain 0, and

$$(\vec{\varphi})^*(s^\frown i) = (\vec{\varphi})^*(s)^\frown 0$$

otherwise. Finally put $\sigma(s^\frown i) = \sigma(s) = n_k$ in the first case and $\sigma(s^\frown i) = n_{k+1}$ in the second one.

The function $(\vec{\varphi})^*$ is clearly continuous (since it is constructed extending at each step the previous value and is such that $\mathrm{lh}((\vec{\varphi})^*(s)) = \mathrm{lh}(s)$ for every $s \in {}^{<\omega}\omega$) and is called the $\mathbf{\Sigma}^0_2$-*control function*[7] of the sequence $\vec{\varphi}$, while the function σ is the *state function* associated to it. Moreover we will say that $\sigma(s) \in N$ is the *state of s with respect to* $(\vec{\varphi})^*$.

Consider now a family $A_n \subseteq {}^{\omega}\omega$ of $\mathbf{\Sigma}^0_2$ sets (for $n < N$) and $S = \{x \in {}^{\omega}\omega \mid \exists n \, \forall m \geq n \, (x(m) \neq 0)\}$. Since S is $\mathbf{\Sigma}^0_2$-complete there are continuous functions $\varphi_n \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ which reduce A_n to S; i.e. $f^{-1}_{\varphi_n}(S) = A_n$. Define $\vec{\varphi} = \langle \varphi_n \mid n < N \rangle$ and let $(\vec{\varphi})^*$ and σ be constructed as above. For notational simplicity we put $\phi = (\vec{\varphi})^*$. We want to prove the following:

Claim 5.0.1. The function $f_\phi \colon {}^{\omega}\omega \to {}^{\omega}\omega$ is a reduction of $\bigcup_{n<N} A_n$ in S; i.e. $f^{-1}_\phi(S) = \bigcup_{n<N} A_n$ (hence, in particular, $\mathbf{\Sigma}^0_2$ is closed under countable unions). Moreover, $x \in \bigcup_{n<N} A_n$ if and only if the sequence $\langle \sigma(x \restriction k) \mid k \in \omega \rangle$ is eventually constant, that is $\exists m \, \forall m' \geq m (\sigma(x \restriction m') = \sigma(x \restriction m))$.

Proof of the Claim. First note that, by the definition of ϕ, for every $x \in {}^{\omega}\omega$ we have $f_\phi(x) \in S$ if and only if the sequence $\langle \sigma(x \restriction m) \mid m \in \omega \rangle$ is eventually constant, since for every $s \in {}^{<\omega}\omega$ and $i \in \omega$ we have that $\phi(s^\frown i) = \phi(s)^\frown 0$ if and only if $\sigma(s^\frown i) \neq \sigma(s)$. So it is enough to prove that $x \in \bigcup_{n<N} A_n \iff \langle \sigma(x \restriction m) \mid m \in \omega \rangle$ is eventually constant.

For every $k \in \omega$ let $o(n_k) = |\{i \leq k \mid n_i = n_k\}|$. Now suppose that $x \in A_n$ for some $n < N$ and let $l = |\{n \in \omega \mid f_{\varphi_n}(x)(n) = 0\}|$. Since φ_n is a reduction of A_n in S we have that $l < \omega$. Let k be such that $n_k = n$ and $o(n_k) = l+1$. If there is no m such that $\sigma(x \restriction m) = n_k$, then the sequence of states of $x \restriction i$ (for $i \in \omega$) with respect to ϕ is eventually constant and we are done. Otherwise, there are $0 < m_0 < m_1 < \ldots < m_{l-1} < m$ such that $\sigma(x \restriction (m_i - 1)) = n$ and $\varphi_n(x \restriction m_i) - \phi(x \restriction (m_i - 1))$ contains a 0 for every $i < l$. Therefore for every $m' \geq m$ we have that $\varphi_{n_k}(x \restriction (m'+1)) - \phi(x \restriction m')$ does not contain any 0 since $\varphi_{n_k} = \varphi_n$, hence $\sigma(x \restriction m') = n_k$ for every $m' \geq m$ and we are done again.

For the other direction, assume $x \notin \bigcup_{n<N} A_n$. Then for every $n < N$ and $m \in \omega$ we have that there is an $m' > m$ such that $\varphi_n(x \restriction m') - \varphi_n(x \restriction m)$ contains some 0. This implies that for every $m \in \omega$ there is an $m' > m$ such that $\sigma(x \restriction m') \neq \sigma(x \restriction m)$ and thus $\langle \sigma(x \restriction m) \mid m \in \omega \rangle$ is not eventually constant. □ *Claim*

[7] The symbol $\mathbf{\Sigma}^0_2$ refers to the $\mathbf{\Sigma}^0_2$-sets which are involved in Claim 5.0.1 and in the other considerations below.

With the notation above, if $x \in \bigcup_{n<N} A_n$, we will call the natural number

$$m_{x,\phi} = \min\{m \in \omega \mid \forall m' \geq m(\sigma(x \restriction m') = \sigma(x \restriction m))\}$$

the *stabilizing point of* ϕ *on* x. Moreover it is not difficult to check that in this case $x \in A_{\sigma(t)}$ (where $t = x \restriction m_{x,\phi}$). In fact, $\varphi_{\sigma(t)}(x \restriction m') - \phi(x \restriction (m'-1))$ does not contain any 0 for every $m' \geq m_{x,\phi}$, and hence $f_{\varphi_{\sigma(t)}}(x) \in S$.

Now we state and prove a theorem which is crucial to obtain Theorem 2.15.

Theorem 5.1. *Let* A *be any subset of* ${}^\omega\omega$ *and* Y *be a separable, metrizable space. If* $f \in \mathcal{B}_1(A,Y)$*, then there is a sequence of full functions* $f_k \colon A \to Y$ *converging pointwise to* f*.*

Proof. Let d_Y be any compatible metric on Y. Let $\langle U_s \mid s \in {}^{<\omega}\omega\rangle$ be an *open scheme on* Y, or a family of sets $U_s \subseteq Y$ such that for every $s \in {}^{<\omega}\omega$ we have[8]:

i) $U_\emptyset = Y$,

ii) U_s is open,

iii) $U_s = \bigcup_{i\in\omega} U_{s^\frown i}$,

iv) $\operatorname{diam}(U_s) \leq 2^{-\mathrm{lh}(s)}$.

One way to do this is to fix an enumeration $\langle y_i \mid i \in \omega\rangle$ of some countable dense subset of Y, and then recursively define $U_\emptyset = Y$ and $U_{s^\frown i} = B(y_i, s^{-(\mathrm{lh}(s)+2)}) \cap U_s$. Note that U_s could be the empty set for some s, but the sequences s such that $U_s \neq \emptyset$ form a pruned tree R on ω. Hence for every $s \in R$ we can fix some $y_s \in U_s$.

Since $f \in \mathcal{B}_1(A,Y)$ and $f^{-1}(U_s) \in \boldsymbol{\Sigma}^0_2(A)$, for every $s \in R$ there is a $V_s \in \boldsymbol{\Sigma}^0_2({}^\omega\omega)$ such that $f^{-1}(U_s) = V_s \cap A$, thus we can consider some reduction $\tilde{\varphi}_s \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ of V_s in $S = \{x \in {}^\omega\omega \mid \exists n \forall m \geq n(x(m) \neq 0)\} \subseteq {}^\omega\omega$. Moreover, for all these s we can consider an enumeration without repetitions $\langle j_i \mid i < I_s\rangle$, where $I_s \leq \omega$, of the $j \in \omega$ such that $s^\frown j \in R$, and define the sequence of continuous functions $\vec{\varphi}_s = \langle \varphi_i \mid i < I_s\rangle$ where $\varphi_i = \tilde{\varphi}_{s^\frown j_i}$. Finally, let $\psi_s = (\vec{\varphi}_s)^*$ be the $\boldsymbol{\Sigma}^0_2$-control function of the sequence $\vec{\varphi}_s$, and σ_s be the state function associated to it.

[8]Note that in general this is not a Lusin scheme since, in this case, we do not require that if s and t are incompatible sequences, then $U_s \cap U_t = \emptyset$. However we can add this condition if Y is also zero-dimensional.

We are now ready to define the functions f_k. Fix some $k \in \omega$ and for every $x \in A$ inductively define for $i < k$:

$$\begin{cases} s_0^{x,k} = \langle \min\{\sigma_\emptyset(x \restriction k), k\}\rangle \\ s_{i+1}^{x,k} = t_i{}^\frown \min\{\sigma_{t_i}(x \restriction k), k\}, \end{cases}$$

where $t_i = s_i^{x,k}$. For notational simplicity, for every $n \in \omega$ we will put $s_n^x = s_n^{x,n}$. Note that by definition of $\vec{\varphi}_s$ one can easily prove by induction on $i \leq k$ that $s_k^x \in R$ for every $x \in A$. Hence we can define

$$f_k \colon A \to Y \colon x \mapsto y_{s_k^x}.$$

Claim 5.1.1. For every $k \in \omega$ the function f_k is full with constant k.

Proof of the Claim. It is clear that $s_k^x \in {}^{k+1}(k+1)$ for every $x \in A$, thus f_k has at most $(k+1)^{k+1}$ values. Moreover, these values depend only on the sequence $x \restriction k$, hence the preimage of each of them is a union of balls with radius 2^{-k} and therefore is a full set by Proposition 2.12. □ *Claim*

Claim 5.1.2. The sequence $\langle f_k \mid k \in \omega\rangle$ converges to f pointwise.

Proof of the Claim. Fix some $x \in A$ and $n \in \omega$. We want to prove that there is an $m \in \omega$ such that $\forall m' \geq m\, (d_Y(f_{m'}(x), f(x)) \leq 2^{-(n+1)})$.

We inductively define a sequence $\langle m_j \mid j \leq n\rangle$ of natural numbers (namely the sequence of the stabilizing points of x) and a sequence $\langle t_j \mid j \leq n\rangle$ of compatible and length increasing sequences of natural numbers. First put $m_0 = m_{x,\psi_\emptyset}$ and $t_0 = \langle \sigma_\emptyset(x \restriction m_0)\rangle$. Then for every $i < n$ define $m_{i+1} = m_{x,\psi_{t_i}}$ and $t_{i+1} = t_i{}^\frown \sigma_{t_i}(x \restriction m_{i+1})$. Finally put $t_{-1} = \emptyset$ by definition.

Now recall that, by Claim 5.0.1 and the observations following it, if $f(x) \in \bigcup_{m\in\omega} U_{s^\frown m}$, then $f(x) \in U_{s^\frown \sigma_s(x \restriction m)}$ for every $m \geq m_{x,\psi_s}$. Therefore the fact that $f(x) \in Y$ implies that $f(x) \in U_{t_0}$. Moreover, using the same argument, one can show that since $f(x) \in U_{t_i}$ then $f(x) \in U_{t_{i+1}}$ for every $i < n$: hence we have $f(x) \in U_{t_n}$.

Recall also that, by definition of the numbers m_i, $t_i = t_{i-1}{}^\frown \sigma_{t_{i-1}}(x \restriction m')$ for every $m' \geq m_i$. Let $m = \max\{m_0, \ldots, m_n, n, k\}$, where k is the smallest natural number such that $t_n \in {}^{<\omega}k$. Again by induction on $i \leq n$, it is not hard to prove that for every $m' \geq m$ and every $i \leq n$ we have $s_i^{x,m'} = t_i$ and hence $s_{m'}^x \supseteq t_n$. Since we have that $f_{m'}(x) = y_{s_{m'}^x} \in U_{s_{m'}^x} \subseteq U_{t_n}$ (as $s_{m'}^x \supseteq t_n$), $\mathrm{lh}(t_n) = n+1$, and $\mathrm{diam}(U_{t_n}) = 2^{-\mathrm{lh}(t_n)}$, we can conclude that $d_Y(f_{m'}(x), f(x)) \leq 2^{-(n+1)}$ and we are done. □ *Claim*

□

We are now ready to prove the characterization of the Baire class 1 functions as pointwise limits of full functions, i.e. Theorem 2.15.

PROOF OF THEOREM 2.15. Since every full function is Lipschitz and every Lipschitz function is continuous, if f is the pointwise limit of a sequence of full functions, then it is in $\mathcal{B}_1(X, Y)$ — see note 2 on page 32.

For the other direction, let X and Y be as in the hypotheses of the Theorem, and let $A \subseteq {}^{\omega}\omega$ and $h \colon A \to X$ be obtained applying the second part of Theorem 4.1 to X. Define

$$g = f \circ h \colon A \to Y.$$

Since h is continuous, if f is of Baire class 1, then g is also of Baire class 1. Let $g_n \colon A \to Y$ be the sequence of full functions converging (pointwise) to g that comes from Theorem 5.1, and define for every $n \in \omega$

$$f_n = g_n \circ h^{-1} \colon X \to Y.$$

Clearly each f_n is a full function by Proposition 2.14, and moreover f is the pointwise limit of the sequence $\langle f_n \mid n \in \omega \rangle$. In fact for every $x \in X$ and every $n \in \omega$ we have that $d(f_m(x), f(x)) = d(g_m(h^{-1}(x)), g(h^{-1}(x))) \leq 2^{-n}$ for m large enough (as $g_m \to g$ pointwise). This completes the proof. $\square$

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