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ALMOST EVERYWHERE CONVERGENCE OF ERGODIC AVERAGES

Abstract

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1 Introduction.

A transformation $T : X \to X$ of the measure space $(X, \mathcal{B}, \mu)$ is measure preserving if $\mu(T^{-1}A) = \mu(A)$, for all $A \in \mathcal{B}$. A dynamical system $(X, \mathcal{B}, \mu, T)$ is aperiodic if the set of periodic points is of measure zero. (The periodic case for most of the questions discussed in this talk is obvious, and hence most of the time one should think of the aperiodic case.)

The Birkhoff Ergodic Theorem (see for example [31]) states the following:

Theorem 1.1. *Assume that $(X, \mathcal{B}, \mu)$ is a probability space, $T : X \to X$ is invertible and measure preserving and $f \in L^1(X, \mathcal{B}, \mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^k x) = \overline{f}(x) \tag{1}$$

exists μ almost everywhere, and $\overline{f}$ is T invariant, that is, $\overline{f}(Tx) = \overline{f}(x)$, μ almost everywhere.

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From (1) it follows that the tail of the ergodic averages, $\frac{f(T^N x)}{N}$, converges to 0, μ almost everywhere.

The study of the pointwise convergence of ergodic averages shares several tools with Harmonic Analysis where pointwise convergence of Fourier series is investigated. One important common tool is a weak $(1,1)$ maximal inequality which, for the ergodic averages, comes from the Maximal Ergodic Theorem. Suppose $\lambda > 0$ then with the conditions of Theorem 1.1 we have

$$\mu\left\{x : \sup_{N>0} \frac{1}{N}\sum_{k=1}^{N} f(T^k x) > \lambda\right\} \leq \frac{\int |f| d\mu}{\lambda}. \tag{2}$$

Usually the maximal function $\sup_N \frac{1}{N}\sum_{k=1}^{N} f(T^k x)$ does not belong to L^1 and hence one cannot estimate its L^1 norm by the L^1 norm of f. Since (2) holds for $|f|$ one can use $\sup\limits_N \frac{1}{N}\sum\limits_{k=1}^{N} |f(T^k x)|$, or $\sup\limits_N \frac{1}{N}|\sum\limits_{k=1}^{N} f(T^k x)|$ in (2) as well.

Recall that $A \in \mathcal{B}$ is *T-invariant* if $0 = \mu(T^{-1}A\Delta A) = \mu((T^{-1}A \setminus A) \cup (A \setminus T^{-1}A))$. The function $\overline{f}$ in Theorem 1.1 is the conditional expectation of f with respect to the sigma algebra of the T invariant sets. The transformation T is *ergodic* if for any T-invariant set A we have $\mu(A) = 0$ or 1. In this case the T invariant sigma algebra is trivial. Therefore, if T is ergodic then in the above Theorem 1.1 we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(T^k x) = \int_X f d\mu \text{ for } \mu \text{ a.e. } x.$$

2 Non L^1 Results.

There are many ways one can generalize Birkhoff's Ergodic theorem. First I discuss one direction when functions not belonging to L^1 are considered. I worked for a long time with Henstock-Kurzweil integrals and, as a Ph.D. student, got interested in ergodic averages of non-L^1 functions.

However, working with non-L^1 functions one needs to be cautious, since answering one of my questions, P. Major proved the following theorem:

Theorem 2.1. *There exists a function $f : X \to \mathbb{R}$, and $S, T : X \to X$ two ergodic transformations on a probability space (X, μ) such that*

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} f(S^k x) = 0, \ \mu \text{ a.e.}$$

and

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) = a \neq 0,\ \mu \text{ a.e.}$$

Clearly, by Theorem 1.1, f cannot belong to $L^1(X,\mu)$, and this is bad news for our "possible generalized integration process" because the generalized integral suitable for f in the ergodic theorem would have to take the values 0 and a simultaneously.

My thesis advisor M. Laczkovich asked whether in the above result the two transformations S, and T can be irrational rotations of the unit circle, $\mathbb{T}$, equipped with the Lebesgue measure. In Major's construction the two transformations were conjugate and hence to answer Laczkovich's question I had to find a different approach, since different rotations have different rotation number and hence cannot be conjugate. Following the suggestions of a referee I stated the result about the two rotations in a slightly more general setting. Assume that a $\mathbb{Z}^2$ action is generated by S and T on a finite nonatomic Lebesgue measure space $(X, \mathcal{S}, \mu)$ and $T^j S^k$ for all $(j,k) \in \mathbb{Z}^2$ is a measure preserving transformation on X. We say that the group action generated by T and S is free if $T^j S^k x \neq x$ for $(j,k) \neq (0,0)$ and μ a.e. x. In [17] I have managed to prove the following theorem:

Theorem 2.2. *Assume that $(X, \mathcal{S}, \mu)$ is a finite non-atomic Lebesgue measure space and $S, T : X \to X$ are two μ-ergodic transformations which generate a free $\mathbb{Z}^2$ action on X. Then for any $c_1, c_2 \in \mathbb{R}$ there exists a μ-measurable function $f : X \to \mathbb{R}$ such that*

$$M_N^S f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(S^j x) \to c_1,$$

and

$$M_N^T f(x) = \frac{1}{N+1} \sum_{j=0}^{N} f(T^j x) \to c_2,$$

μ almost every x as $N \to \infty$.

Two different irrational rotations generate a free $\mathbb{Z}^2$ action on $\mathbb{T}$ and hence Theorem 2.2 answers Laczkovich's question.

There are some interesting recent results with respect to ergodic averages of non L^1 functions and rotations by Y. Sinai and C. Ulcigrai. In [37] the authors consider trigonometric sums

$$\frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{1 - e^{2\pi i (k\alpha + x)}}, \qquad (x, \alpha) \in (0,1) \times (0,1).$$

The product space $(0,1)\times(0,1)$ is endowed with the uniform probability distribution. It is proved that such trigonometric sums have a non-trivial joint limiting distribution in x and α as N tends to ∞. This result also applies to Birkhoff sums of a function with a singularity of type $1/x$ over a rotation. This limiting distribution is determined by results from [36].

Motivated by these results I have managed to solve during the Summer of 2008 one of my open questions (see [7], [16]) concerning ergodic averages of rotations of non-L^1 functions.

Theorem 2.3 from [16] states the following:

Theorem 2.3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a given measurable function, periodic by 1. For an $\alpha\in\mathbb{R}$ put*

$$M_n^\alpha f(x)=\frac{1}{n+1}\sum_{k=0}^{n} f(x+k\alpha).$$

Let Γ_f denote the set of those α's in $(0,1)$ for which $M_n^\alpha f(x)$ converges for almost every $x\in\mathbb{R}$. Then from $|\Gamma_f|>0$ it follows that f is integrable on $[0,1]$.

In the above theorem $|\Gamma_f|$ denotes the Lebesgue measure of Γ_f. By the Birkhoff Ergodic Theorem if f is Lebesgue integrable on $[0,1]$ then $M_n^\alpha f(x)\to\int_0^1 f$ for any irrational number α and for almost every $x\in\mathbb{R}$. Hence, when $|\Gamma_f|>0$ by the ergodic theorem all these limits $M_n^\alpha f(x)$ should be of the same value, namely, $\int_0^1 f$. On the other hand, in [16] the following result is also verified:

Theorem 2.4. *For any sequence of independent irrationals $\{\alpha_j\}_{j=1}^\infty$ there exists a function $f:\mathbb{R}\to\mathbb{R}$, periodic by 1 such that $f\notin L^1[0,1]$ and $M_n^{\alpha_j} f(x)\to 0$ for almost every $x\in[0,1]$.*

This result implies that Γ_f can be dense for non-integrable functions. In [39] R. Svetic improves this result by showing that there exists a non-integrable $f:\mathbb{T}\to\mathbb{R}$ such that Γ_f is c-dense in $\mathbb{T}$. (A set $S\subset\mathbb{T}$ is *c-dense* if the cardinality of $S\cap I$ equals continuum for every nonempty open interval $I\subset\mathbb{T}$.)

It was not known whether Γ_f can be of Hausdorff dimension one. I advertised this question at several places ([7], [16]), and Svetic's paper contained a partial solution. Finally in [19] a non-L^1 function is constructed for which Γ_f is of Hausdorff dimension one, but of course, it is of zero Lebesgue measure.

3 The Squares and Good Sequences of Zero Banach Density.

Before turning to the L^1 results we recall *Banach's principle*, see for example p. 91 of [31].

Theorem 3.1. *Let $1 \leq p < \infty$ and let T_n be a sequence of bounded linear operators on L^p. If $\sup_n |T_n f| < \infty$ almost everywhere for all $f \in L^p$ then the set of f for which $T_n f$ converges almost everywhere is closed in L^p.*

In the ergodic setting one can think of the operators

$$T_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f(T^k x).$$

In the Harmonic Analysis setting the partial sums of Fourier series can be considered. The weak $(1,1)$ inequality in (2) holds for all L^1 functions. Letting $\lambda \to +\infty$ one can see by using the remark after (2) that $\sup_n |T_n f| < \infty$ almost everywhere for all $f \in L^1$. Hence it is sufficient to verify the almost everywhere convergence of $T_n f(x)$ for some nice set (like bounded functions, or continuous functions) of functions which is dense in L^1 and then by Banach's principle the almost everywhere convergence follows for all functions in L^1.

Research related to almost everywhere convergence of ergodic averages along the squares was initiated by questions of A. Bellow (see [8]) and of H. Furstenberg, [25]. Results of Bourgain [10], [11], [12] imply that if $f \in L^p(\mu)$, for some $p > 1$ then the non-conventional ergodic averages

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n^2}(x)) \tag{3}$$

converge almost everywhere. Bourgain also asked in [10], [14] whether this result is true for $p = 1$, that is, for L^1 functions. In Section 6 of [9], V. Bergelson writes the following about it: "The case p = 1 is still open and is perhaps one of the central open problems in that branch of ergodic theory which deals with almost everywhere convergence". On p. 64 of [27] R. L. Jones writes the following: "There is an important open problem associated with subsequences. At this time, there is no known example of a subsequence that is good for a.e. convergence for all $f \in L^1$, and has successive gaps increasing to infinity. In particular, the question of a.e. convergence for $f \in L^1$ along the squares is open, and probably very difficult. The techniques used in Section 4 and Section 6, including the Calderón-Zygmund decomposition, do not seem to apply."

Definition 3.2. A sequence $\{n_k\}_{k=1}^{\infty}$ is L^1-*universally bad* if for all ergodic aperiodic dynamical systems there is some $f \in L^1$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^{n_k} x)$$

fails to exist for all x in a set of positive measure.

By the Conze principle and the Banach principle of Sawyer ([35], [38]) a sequence $\{n_k\}_{k=1}^{\infty}$ is *not* L^1-universally bad if and only if there exists a constant $C < \infty$ such that for all systems (X, Σ, μ, T) and all $f \in L^1(\mu)$ we have the following weak $(1,1)$ inequality for all $t > 0$:

$$\mu\left(\left\{x : \sup_{N\geq 1}\left|\frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x)\right| > t\right\}\right) \leq \frac{C}{t}\int |f|d\mu. \tag{4}$$

In [21] we prove that

Theorem 3.3. *The sequence $\{k^2\}_{k=1}^{\infty}$ is L^1-universally bad.*

This theorem is proved by showing that there is no constant C such that the weak $(1,1)$ inequality in (4) holds. The proof is quite complicated. In [20] we try to make it more accessible by discussing its heuristic background. We started to work on this paper in 2003 during my one semester visit to University of North Texas and the paper [21] went through several revisions and phases.

An infinite set $A \subset \mathbb{N}$ is of zero Banach density if

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\frac{\#(A\cap[n,n+k])}{k+1} = 0.$$

I learned from M. Keane in 2003 that at that time it was not known whether there exists a sequence (n_k) such that $n_{k+1} - n_k \to \infty$ and for any $f \in L^1(\mu)$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x) \tag{5}$$

converges μ almost everywhere.

A sequence satisfying $n_{k+1} - n_k \to \infty$ is of zero Banach density.

J. Rosenblatt and M. Wierdl ([33] Conjecture 4.1) had the following conjecture:

Conjecture 3.4. Suppose that the sequence (n_k) has zero Banach density and let (X, Σ, μ, T) be an aperiodic dynamical system. Then for some $f \in L^1(\mu)$ the averages (5) do not converge almost everywhere.

In [18] the following theorem is proved.

Theorem 3.5. *There exists a sequence* (n_k) *satisfying* $n_{k+1} - n_k \to \infty$ *(and hence of zero Banach density) which is universally* L^1*-good, that is, for any invertible aperiodic ergodic dynamical system* (X, Σ, μ, T) *and* $f \in L^1(\mu)$ *we have*

$$\lim_{N\to\infty} A(f,x,N) = \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} f(T^{n_k}x) = \int_X f d\mu, \tag{6}$$

for μ *almost every* $x \in X$.

Theorem 3.5 implies that Conjecture 3.4 is false. It also provides an explanation why it was so difficult to obtain the result that $n_k = k^2$ is L^1-universally bad.

In a recent paper [40], R. Urban and J. Zienkiewicz showed the following:

Theorem 3.6. *The sequence* $\lfloor k^\alpha \rfloor$, $1 < \alpha < 1.001$ *is universally* L^1 *good.*

4 Assani's L^1 Counting Problem in Ergodic Theory.

Learning about our work in [21], I. Assani suggested to look at his L^1 Counting Problem in Ergodic Theory as well.

Suppose that $(X, \mathcal{B}, \mu)$ is a probability measure space, T is an invertible measure preserving transformation and f belongs to $L^1_+(\mu)$, that is, f is non-negative and belongs to $L^1(\mu)$. As we remarked it after Theorem 1.1, $\frac{f(T^n x)}{n}$ converges to 0, μ almost everywhere. Therefore,

$$\mathbf{N}_n(f)(x) = \#\left\{k : \frac{f(T^k x)}{k} > \frac{1}{n}\right\}$$

is finite μ almost everywhere. Assani's counting problem was originally mentioned in [1], [2], later also discussed by R. Jones, J. Rosenblatt, D. Rudolph and M. Wierdl in [28] and [34].

Problem 4.1 (Counting Problem I)**.** Given $f \in L^1_+(\mu)$ do we have
$\sup_n \frac{\mathbf{N}_n(f)(x)}{n} < \infty$, μ a.e.?

In Assani [1] and [2] the maximal operator $\sup_n \frac{\mathbf{N}_n(f)(x)}{n}$ is used to study the pointwise convergence of $\frac{\mathbf{N}_n(f)(x)}{n}$.

If $f \in L^p_+$ for $p > 1$, or $f \in L^+ \log L^+$ and the transformation T is ergodic, then $\frac{\mathbf{N}_n(f)(x)}{n}$ converges almost everywhere to $\int f d\mu$.

If T is not ergodic, then the limit is the conditional expectation of the function f with respect to the σ field of the invariant sets for T. Hence, the limit is the same as the limit of the ergodic averages $\frac{1}{N}\sum_{n=1}^{N} f(T^n x)$.

It is natural to ask whether $\frac{\mathbf{N}_n(f)(x)}{n}$ also converges a.e., when $f \in L^1(\mu)$.

By using a generalized version of the Stein-Sawyer result from Assani [1] one can state the following equivalent problem to the counting problem.

Problem 4.2 (Counting Problem II.)**.** Does there exist a finite positive constant C such that for all measure preserving systems and all $\lambda > 0$

$$\mu \left\{ x : \sup_n \frac{\mathbf{N}_n(f)(x)}{n} > \lambda \right\} \leq \frac{C}{\lambda} \|f\|_1 ?$$

In our joint paper [6] with I. Assani and D. Mauldin we proved the following theorem which gives a negative answer to both counting problems.

Theorem 4.3. *In any nonatomic, invertible ergodic system $(X, \mathcal{B}, \mu, T)$ there exists $f \in L^1_+$ such that $\sup_n \frac{\mathbf{N}_n(f)(x)}{n} = \infty$ almost everywhere.*

Definition 4.4. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. The Return Times for the Tail Property holds in $L^r(\mu)$, $1 \leq r \leq \infty$ if for each $f \in L^r(\mu)$ we can find a set X_f of full measure such that for all $x \in X_f$ for all measure preserving systems $(Y, \mathcal{G}, \nu, S)$ and each $g \in L^1(\nu)$ the sequence $\frac{f(T^n x) \cdot g(S^n y)}{n}$ converges to zero for almost every y.

Using Theorem 4.3 and results from [1] it is not difficult to show that the next corollary holds. The details of its verification can be found in [7].

Corollary 4.5. *The Return Times for the Tail Property does not hold for $r = 1$.*

There are several other interesting consequences of Theorem 4.3. For example, it implies that Bourgain's Return Time Theorem does not hold for pairs of (L^1, L^1) functions, for the details we refer to [6] and [7].

5 The Bilinear Hardy–Littlewood Maximal Function.

First we recall the definition of the original Hardy–Littlewood maximal function:

$$H^* : f \in L^1 \to H^* f(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x+u) du.$$

It maps L^1 functions into weak L^1, that is, by a theorem of F. Riesz [32] it satisfies the following weak type $(1,1)$ inequality:

$$m\{x : H^* f(x) > s\} \leq \frac{\int |f| dm}{s}.$$

Calderón introduced in the 1960's the bilinear Hardy–Littlewood maximal function. For f, g measurable we put

$$M^*(f,g)(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x+s)g(x+2s)ds.$$

Conjecture 5.1 (Calderón's conjecture)**.** *M^* is integrable as soon as f and g are in L^2.*

After the results by M. Lacey and C. Thiele in [30], finally M. Lacey in [29] proved a theorem which implies the following:

Theorem 5.2. *Let $1 < p, q < \infty$, and set $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $\frac{2}{3} < r \leq 1$ then M^* extends to a bounded map from $L^p \times L^q$ into L^r.*

This theorem settles Calderón's conjecture.

If $r > 1$ then Hölder's inequality used for the x variable implies that M^* maps into L^r. Hölder's inequality and the weak type $(1,1)$ property of H^* implies that $M^*(f,g)$ is almost everywhere finite if $f \in L^p$, $g \in L^q$ and $\frac{1}{p} + \frac{1}{q} \leq 1$. It is a very interesting fact that Theorem 5.2 goes beyond the range of exponents used for usual Hölder duality statements.

Problem 5.3. It is not known what happens if $3/2 < \frac{1}{p} + \frac{1}{q} \leq 2$? Lacey's method does not work for $r \leq 2/3$.

Lacey's method was reproved and generalized for maximal multilinear averages by C. Demeter, T. Tao and C. Thiele in [24].

One can consider the Ergodic Theory/discrete version of the above continuous Hardy–Littlewood maximal functions and as the Correspondance Principle proved in the Appendix of [24] shows, quite often the results for the discrete Ergodic version are equivalent to the continuous version.

Suppose $f_1, ..., f_k$ are measurable functions and (X, Σ, μ, T) a dynamical system. The Furstenberg averages are defined as

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{k} f_j(T^{jn}x).$$

In the Ergodic Theory proof of Szemerédi's theorem Furstenberg showed that the weak-lim inf of certain multilinear averages (which became later the Furstenberg averages) is positive.

We consider the ergodic version of the Bilinear Hardy–Littlewood function, which is a very special case of the symmetric Furstenberg averages. Suppose T

is an ergodic measure preserving transformation on a non-atomic probability measure space. We consider the maximal function:

$$\mathcal{M}(f,g)(x) \stackrel{\text{def}}{=} \sup_N \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^n x) g(T^{2n} x) \tag{7}$$

for functions $f \in L^p$ and $g \in L^q$. The equivalent problem to Problem 5.3 in this setting is to find the range of values $p, q \geq 1$ for which $\mathcal{M}(f,g)(x) < \infty$, μ almost everywhere.

While for Szemerédi's theorem the Furstenberg averages are interesting for large values of k, for our problem the smallest possible multilinear case, the bilinear case, seems to be different and in many ways more challenging. The trilinear Hardy–Littlewood maximal function can be defined the following way:

$$R^*(f,g,h)(x) = \sup_t \frac{1}{2t} \int_{-t}^{t} f(x+s) g(x+2s) h(x+3s) ds.$$

From the dependence of the monomials $x+s$, $x+2s$, and $x+3s$, C. Demeter in [23] deduced some negative results for the trilinear Hardy–Littlewood maximal function. For example the following theorem is valid for the ergodic version of these averages (see [23]):

Theorem 5.4. *Define* $p_0 = 1 + \frac{\log_6 2}{1+\log_6 2}$ *and consider* $p < p_0$. *In every ergodic dynamical system* (X, Σ, μ, T) *there are three functions* $f, g, h \in L^p(X)$ *such that*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) g(T^{2n} x) h(T^{3n} x) = \infty$$

for μ *a.e.* $x \in X$.

For the bilinear Hardy–Littlewood M^*, the monomials $x+s$ and $x+2s$ are "independent" and no negative result are known close to L^1.

A good indicator of the behavior of the ergodic averages is given by the tail of the averages $\dfrac{f(T^N x) g(T^{2N} x)}{2N+1}$. The maximal function associated with the tail of the ergodic averages $\sup_n \dfrac{f(T^n x)}{n}$, satisfies similar weak type inequalities as the maximal function for the ergodic averages. Since $\mathcal{M}(f,g)(x) \geq \sup_N \dfrac{f(T^N x) g(T^{2N} x)}{2N+1}$, I. Assani suggested that we should first try to find out what happens to the maximal function

$$R^*(f,g)(x) = \sup_n \frac{f(T^n x) g(T^{2n}(x))}{n}.$$

In [3] we prove the following theorem:

Theorem 5.5. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving transformation on a finite non-atomic measure space. Then for all $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} < 2$, R^* maps $L^p \times L^q$ into L^r as soon as $0 < r < 1/2$.*

In [5] we also show the following:

Theorem 5.6. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving transformation on a finite non-atomic measure space. Then there exist functions f, g both in $L^1_+(X)$ for which the maximal function*

$$R^*(f,g)(x) = \sup_n \frac{f(T^n x) g(T^{2n} x)}{n}$$

is not finite a.e.

Since Theorem 5.5 implies that $R^*(f, g)$ is finite almost everywhere for $(f, g) \in L^p \times L^q$ when $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} < 2$ the above two theorems give a complete characterization of the range of values (p, q) for which $R^*(f, g)$ is finite almost everywhere.

Theorem 5.6 solves an open problem in Ergodic Theory. The averages

$$F_N(f,g)(x) = \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^n x) g(T^{2n} x)$$

are special (bilinear and symmetric) Furstenberg averages. A deep result of J. Bourgain, [15], showed that these averages converge almost everywhere as soon as the Hölderian duality is respected, (that is, $\frac{1}{p} + \frac{1}{q} \leq 1$). Theorem 5.6 shows that these averages do not converge for pairs of (L^1, L^1) functions as the tail of these averages does not converge a.e. to zero for some functions $f, g \in L^1$.

In [3] Theorem 5.5 is deduced from the following maximal inequality:

Theorem 5.7. *Given $p > 1$ there exists a universal finite constant C_p^* such that if $(X, \mathcal{B}, \mu, T)$ is any invertible ergodic dynamical system on a non-atomic finite measure space $(X, \mathcal{B}, \mu)$ then the following holds. For every function $f \in L^p$, $|f| > 1$, for every $g \in L^1$, $|g| > 1$, and for each $s > 0$ we have*

$$\mu\left\{x : \sup_{0<l} \frac{f(T^l x) g(T^{2l} x)}{l} \geq s\right\} \leq C_p^* \sqrt{\frac{||f||_p^p ||g||_1}{s}}. \tag{8}$$

Therefore, for such functions f and g we have

$$\frac{f(T^l x)g(T^{2l}x)}{l} \to 0 \text{ as } l \to \infty. \tag{9}$$

Furthermore, for $1 < p < 2$ there exists a universal constant C such that $C_p^* \le \dfrac{C}{p-1}$.

A similar maximal inequality can be obtained if one considers instead $f \in L^1$ and $g \in L^p$.

By Theorem 5.5 for the maximal function R^* there is nothing magic about 2/3, like in Problem 5.3. For R^* one can go beyond $\dfrac{1}{p} + \dfrac{1}{q} < 3/2$. In fact, R^* maps functions in $L^p \times L^q$ into any of the L^r spaces as long as $1 \le \frac{1}{p} + \frac{1}{q} < 2$ and $0 < r < 1/2$.

Remark 5.8. The maximal inequality in Theorem 5.7 is good enough to derive Theorem 5.5 but it is not homogeneous with respect to f, or g. During the 2007 Ergodic Theory workshop at University of North Carolina at Chapel Hill, J.P. Conze asked if this inequality could be made homogeneous with respect to f and g.

In [4] we prove the following version of Theorem 5.7:

Theorem 5.9. *For each $1 < p < \infty$ there exists a finite constant C_p such that for each $f, g \ge 0$ and for all $\lambda > 0$ we have*

$$\mu\left\{x : \sup_{n\ge 1} \frac{f(T^n x)g(T^{2n}x)}{n} > \lambda\right\} \le C_p \left(\frac{\|f\|_p \|g\|_1}{\lambda}\right)^{1/2}, \tag{10}$$

and there exists $\widetilde{C}$ such that for any $1 < p < 2$ we have

$$C_p \le \frac{\widetilde{C}}{p-1}. \tag{11}$$

At the same meeting a question was raised about the almost everywhere finiteness of $R^*(f,g)$ for pairs of functions in $(L \log L, L^1)$. In [4] we could prove:

Theorem 5.10. *If $\alpha > 2$ and the pair of non-negative functions (f,g) belongs to $(L(\log L)^{2\alpha}, L^1)$ then $R^*(f,g) = \sup_n \dfrac{f(T^n x)g(T^{2n}x)}{n}$ is a.e. finite.*

The problem whether this Theorem is true for functions in $(L \log L, L^1)$ seems to be still unsolved.

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