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DESCRIPTIOVE CHARACTER OF SETS OF DENSITY AND $\mathcal{I}$-DENSITY POINTS. A CORRECTION

Abstract

We give corrected proofs of Proposition 2.1 and Theorem 2.3 from [BP].

By our oversight, the proof of Proposition 2.1 in [BP] contains several misprints and technical flaws. Consequently, the argument for Theorem 2.3 is not completely correct. In this note we would like to repeat the both proofs with necessary changes.

First of all let us observe that the results of [BP] can be formulated for $X = \mathbb{R}$. This framework seems more elegant and simpler than that for $X = [a, b]$ since we overcome a problem if one or both endpoints a and b are one-sided density (or $\mathcal{I}$-density) points of the respective section of a plane set. (In fact, that case has not been considered in [BP] separately which could result in difficulties for the reader.) So, we let $X = \mathbb{R}$.

Recall that for $A \subset X^2$ we let $D(A)$ be the set of points $\langle x, y\rangle \in X^2$ such that the section $A_x = \{t \in X : \langle x, t\rangle \in A\}$ is Lebesgue measurable and y is a density point of A_x. Let $\mathbb{Q}$ denote the set of rationals and λ denote linear Lebesgue measure. By $\mathrm{pr}_Z(E)$ we mean the projection of the set $E \subset Z \times W$ on Z. If Y is a metric space, $\mathcal{K}(Y)$ denotes the hyperspace of all compact subsets of Y equipped with the Vietoris topology (or, equivalently with the Hausdorff metric).

Lemma 1. *[Ke, Th. 29.27] Let Z, W be Polish spaces and $H \subset Z \times W$ be closed. If μ is a Borel probability measure on Z and for some $a \in \mathbb{R}$, $\mu(pr_Z(H)) > a$, then there is a compact set $K \subset H$ such that $\mu(pr_Z(K)) > a$.*

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Proposition 1. *[BP, Prop. 2.1] If $A \subset X^2$ is analytic and $h > 0, a \in \mathbb{R}$, then*

$$T = \{\langle x, y\rangle \in X^2 : \ \lambda(A_x \cap [y-h, y+h]) \geq a\}$$

is analytic.

Proof. Observe that

$$T = \bigcap_{p\in\omega} \bigcup_{s\in\mathbb{Q}} \left(T(p,s) \times \left\{ y \in X : |y-s| < \frac{1}{p+1} \right\} \right)$$

where

$$T(p,s) = \left\{ x \in X : \ \lambda(A_x \cap [s-h, s+h]) > a - \frac{1}{p+1} \right\}.$$

It suffices to show that $T(p,s)$ is analytic. So, fix $p \in \omega$ and $s \in \mathbb{Q}$. Since A is analytic, there exists a closed set $E \subset X^2 \times \omega^\omega$ such that $A = \mathrm{pr}_{X^2}(E)$. It is easy to check that for a fixed $x \in X$ we have

$$A_x \cap [s-h, s+h] = \mathrm{pr}_X \Big(E_x \cap ([s-h, s+h] \times \omega^\omega) \Big).$$

Obviously $E_x \cap ([s-h, s+h] \times \omega^\omega)$ is closed. Then by Lemma 1 we infer that

$$\begin{aligned} \lambda(A_x \cap [s-h, s+h]) > a - \frac{1}{p+1} \quad &\Leftrightarrow \\ \lambda(\mathrm{pr}_X(E_x \cap ([s-h, s+h] \times \omega^\omega))) > a - \frac{1}{p+1} \quad &\Leftrightarrow \\ \Big(\exists K \in \mathcal{K}(X \times \omega^\omega)\Big) \ \Big(K \subset E_x \cap ([s-h, s+h] \times \omega^\omega) \quad &\& \\ \lambda(\mathrm{pr}_X(K)) > a - \frac{1}{p+1} \Big). & \end{aligned} \tag{1}$$

Consider the sets

$$\begin{aligned} M_1 &= \{\langle x, K\rangle \in X \times \mathcal{K}(X \times \omega^\omega) : \ K \subset E_x \cap ([s-h, s+h] \times \omega^\omega)\}, \\ M_2 &= X \times \left\{ K \in \mathcal{K}(X \times \omega^\omega) : \ \lambda(\mathrm{pr}_X(K)) > a - \frac{1}{p+1} \right\}. \end{aligned}$$

The set M_1 is closed since from

$$K \subset E_x \cap (X \times [s-h, s+h] \times \omega^\omega) \Leftrightarrow \{x\} \times K \subset E \cap (X \times [s-h, s+h] \times \omega^\omega)$$

it follows that $M_1 = f^{-1}[W]$ where:

• the mapping $f : X \times \mathcal{K}(X \times \omega^\omega) \to \mathcal{K}(X^2 \times \omega^\omega)$ given by $f(x, K) = \{x\} \times K$ is continuous [Ke, p.27];

• the set $W = \{F \in \mathcal{K}(X^2 \times \omega^\omega) : \ F \subset E \cap (X \times [s-h, s+h] \times \omega^\omega)\}$ is closed.

The set M_2 is of type F_σ. Indeed, for each $c \in \mathbb{R}$, the set $S(c)$, given by $S(c) = \{F \in \mathcal{K}(X) : \ \lambda(F) < c\}$, can be expressed as

$$\bigcup \{V(G) : \ G \text{ open } \ \& \ \lambda(G) < c\}$$

where $V(G) = \{F \in \mathcal{K}(X) : \ F \subset G\}$ is a set from the subbasis of the Vietoris topology. Hence $S(c)$ is open, and therefore

$$\left\{F \in \mathcal{K}(X) : \ \lambda(F) > a - \frac{1}{p+1}\right\} = \bigcup_{n \in \omega} \left(\mathcal{K}(X) \setminus S\left(a - \frac{1}{p+1} + \frac{1}{n+1}\right)\right)$$

is of type F_σ. Consequently, M_2 is of type F_σ since $\mathrm{pr}_X : \mathcal{K}(X \times \omega^\omega) \to \mathcal{K}(X)$ is continuous.

Now, from (1) it follows that the set $T(p, s)$ is the projection of a Borel set $M = M_1 \cap M_2$ on X. Thus $T(p, s)$ is analytic. □

Theorem 1. *[BP, Th. 2.3] If $A \subset X^2$ is analytic (coanalytic), so is $D(A)$.*

PROOF. Let A be analytic. We can express

$$D(A) = \bigcap_{n \in \omega} \bigcup_{m \in \omega} \bigcap_{q \in (0, \frac{1}{m+1}) \cap \mathbb{Q}} T(n, q) \tag{2}$$

where

$$T(n, q) = \{\langle x, y \rangle \in X^2 : \ \lambda(A_x \cap [y-q, y+q]) \geq 2q(1 - 1/(n+1))\}.$$

(See [BP, Lemma 2.1].) Then the assertion follows from (2) and Proposition 1.

Now let A be coanalytic. Recall that y is a density point of A_x if and only if y is a dispersion point of $X \setminus A_x = (X^2 \setminus A)_x$. Therefore we can express $D(A)$ by (2) where $T(n, q)$ is given by

$$\begin{aligned} T(n, q) &= \{\langle x, y \rangle \in X^2 : \ \lambda((X^2 \setminus A)_x \cap [y-q, y+q]) < 2q(n+1)\} \\ &= X^2 \setminus \{\langle x, y \rangle \in X^2 : \ \lambda((X^2 \setminus A)_x \cap [y-q, y+q]) \geq 2q(n+1)\}. \end{aligned}$$

We apply Proposition 1 to the analytic set $X^2 \setminus A$ and infer that $T(n,q)$ is coanalytic. Then the assertion follows from (2). $\square$

Finally, note that the statement "Observe that if A is open, then $D(A)$ and $D_{\mathcal{I}}(A)$ are open" written in Remark 1 in [BP] is false. Indeed, let

$$B = \bigcup_{n=1}^{\infty} \Big((-b_n, -a_n) \cup (a_n, b_n) \Big)$$

where $0 < b_{n+1} < a_n < b_n$, for $n = 1, 2, ...$, and 0 is a density (respectively, an $\mathcal{I}$-density) point of B. Then for an open set $A = X \times B$ and $x \in X$ we have $\langle x, 0 \rangle \in D(A)$ (respectively, $\langle x, 0 \rangle \in D_{\mathcal{I}}(A)$), and $\langle x, 0 \rangle$ is not an interior point of $D(A)$ (respectively, of $D_{\mathcal{I}}(A)$). We can merely state that if $A \subset X^2$ is open, then it is contained in $D(A)$ and in $D_{\mathcal{I}}(A)$.

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