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ON THE EXISTENCE OF VECTOR MEASURES WITH GIVEN MARGINALS

Abstract

A type of Strassen's Theorem for measures taking values in the positive cone of a Banach lattice is proved. We generalize a result of A. Hirshberg and R. M. Shortt and formulate a type of Strassen's Theorem in a topological context via closed sets.

1 Introduction

In joint work, M. März and R. M. Shortt [10, Theorem 3.7] generalize a version of the theorem known in probability theory as "Strassen's Theorem" (see [13], [6], [5, §11.6]) to the context of measures assuming values in a reflexive Banach lattice.

Continuing the line of inquiry of [12], in [8, Theorem 2] A. Hirshberg and R. M. Shortt prove a result of this type for measures taking values in Banach lattices of a certain type: the KB-spaces. Since reflexive $\Rightarrow$ KB $\Rightarrow$ order complete, their result is a generalization of [10, Theorem 3.7].

In this paper we give a formulation of Strassen's Theorem for measures taking values in Banach lattices with order continuous norm [Theorem 3.10]. These spaces occupy a position between the KB-spaces and the order complete Banach lattices (reflexive $\Rightarrow$ KB $\Rightarrow$ order continuous norm $\Rightarrow$ order complete), hence our result is a generalization of [8, Theorem 2].

We also formulate a type of Strassen's Theorem in a topological context via closed sets [Theorem 3.15].

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2 Preliminaries

Let $\mathcal{F}$ be a field of subsets of a set X. Let $\mathbf{B}$ be a Banach space and $\mathbf{B}^*$ its dual.

We remind that a *vector measure* $\mu : \mathcal{F} \to \mathbf{B}$ is an additive set function, i.e., $\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$, for disjoint F_1 and F_2 in $\mathcal{F}$. We denote by $ca(\mathcal{F}, \mathbf{B})$ the vector space of all countably additive vector measures $\mu : \mathcal{F} \to \mathbf{B}$ and by $\|\mu\|$ the semivariation of μ [4].

A class $\mathcal{K}$ of subsets of a set X is *compact* if it has the following property: given a sequence $(K_n)_{n \in N}$ drawn from $\mathcal{K}$ such that $K_1 \cap \ldots \cap K_n \neq \emptyset$ for each $n \in N$, the intersection $\cap_{n=1}^{\infty} K_n$ is non-empty. Let $\mathcal{F}$ be a field of subsets of X and let $\mu : \mathcal{F} \to \mathbf{B}$ be a vector measure taking values in a Banach space $\mathbf{B}$. We say that μ is a *compact measure* if there exists a compact class $\mathcal{K}$ of subsets of X such that, for every $F \in \mathcal{F}$ and $\epsilon > 0$, there are sets $F^{'} \in \mathcal{F}$ and $K \in \mathcal{K}$ with $F^{'} \subseteq K \subseteq F$ and $\|\mu\|(F - F^{'}) < \epsilon$. In this case we say that the class $\mathcal{K}$ *μ-approximates $\mathcal{F}$*.

Now suppose that $\mathcal{F}$ is a σ-field. We say that a vector measure $\mu : \mathcal{F} \to \mathbf{B}$ is *perfect* if the restriction of μ to every countably generated sub-σ-field of $\mathcal{F}$ is compact.

Given a Hausdorff topological space X and its Borel σ-field $\mathcal{B}(X)$, a vector measure $\mu : \mathcal{B}(X) \to \mathbf{B}$ is *tight* if, for each $\epsilon > 0$ and set $B \in \mathcal{B}(X)$, there is some compact set $K \subseteq B$ such that $\|\mu\|(B - K) < \epsilon$. Clearly, every tight measure on the Borel σ-field of a metric space is compact.

A Banach lattice $\mathbf{B}$ is a *KB-space* if every increasing norm bounded sequence of its positive cone $\mathbf{B}^+$ is norm convergent [1, Definition 14.10].

A normed vector lattice $\mathbf{B}$ is said to have *order continuous norm* [11, Definition 5.12] if every order convergent filter in $\mathbf{B}$ norm converges. For information on these spaces, see ([1], [11]). In these sources are to be found the following results.

1. A countably order complete Banach lattice $\mathbf{B}$ has order continuous norm iff no Banach sublattice of $\mathbf{B}$ is vector lattice isomorphic to l^{∞} [11, Theorem 5.14].

2. Every KB-space has order continuous norm [11, page 92]. The converse is not true: the vector lattice c_0 of real null sequences under the supremum norm is an important example of a Banach lattice with order continuous norm, but it is not KB.

3. Every Banach lattice having order continuous norm is order complete [11, page 92]. The converse is not true: l^{∞} is order complete, but it has

not order continuous norm.

3 Strassen's Theorem

Without further notice, in this section, for two sets X_1 and X_2,

1. $\mathcal{P}(X_1 \times X_2)$ is the power set of $X_1 \times X_2$,
2. for every $F \in \mathcal{P}(X_1 \times X_2)$, χ_F is the indicator function of F,
3. $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are the projections of the product space $X_1 \times X_2$,
4. if X_1 and X_2 are Hausdorff topological spaces, by $\mathcal{B}(X_1)$, $\mathcal{B}(X_2)$ and $\mathcal{B}(X_1 \times X_2)$ we denote the Borel σ-fields of X_1, X_2 and $X_1 \times X_2$, respectively.

Definition 3.1. *Let X_1 and X_2 be Hausdorff topological spaces and let μ_1 and μ_2 be tight elements of $ca(\mathcal{B}(X_1), \mathbf{B}^+)$ and of $ca(\mathcal{B}(X_2), \mathbf{B}^+)$, respectively, such that $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of an order complete Banach lattice $\mathbf{B}$. Let $M = \{\mu \in ca(\mathcal{B}(X_1 \times X_2), \mathbf{B}^+) :$ μ is tight and $\mu \circ \pi_1^{-1} = \mu_1$ and $\mu \circ \pi_2^{-1} = \mu_2\}$ (i.e. $M = \{\mu \in ca(\mathcal{B}(X_1 \times X_2), \mathbf{B}^+) : \mu$ is tight and has marginals μ_1 and $\mu_2\}$). For every $F \in \mathcal{P}(X_1 \times X_2)$ we define*

$$S(F) = \begin{cases} 0 & \text{if } M = \emptyset \\ \bigvee\{\mu^*(F) : \mu \in M\} & \text{otherwise,} \end{cases}$$

where 0 is the zero element of $\mathbf{B}$ and $\mu^*(F)$ is the outer measure of F,

and

$$I(F) = \bigwedge\{\sum_{i=1}^{2} \mu_i(B_i) : B_i \in \mathcal{B}(X_i) \text{ and } F \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(B_i)\}.$$

For the properties of I and S see [3] and [9].

Theorem 3.2. *Let $C \in \mathcal{B}(X_1 \times X_2)$ be a closed set. Then*

$$I(C) = \bigwedge\{\sum_{i=1}^{2} \mu_i(C_i) : C_i \text{ closed set in } \mathcal{B}(X_i) \text{ and } C \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(C_i)\}.$$

PROOF. The proof consists of four steps.

1. First we shall prove that, for every $F \in \mathcal{B}(X_1 \times X_2)$,

$$(*) \qquad I(F) = \bigwedge \{\sum_{i=1}^{2} \int f_i d\mu_i : f_i \in \mathcal{L}(X_i) \text{ and } \chi_F \le \sum_{i=1}^{2} f_i \circ \pi_i\},$$

where $\mathcal{L}(X_1)$ is the family of all μ_1-integrable Borel measurable functions defined on X_1, with values in $[0,1]$, and $\mathcal{L}(X_2)$ is the family of all μ_2-integrable Borel measurable functions defined on X_2, with values in $[0,1]$.

Notice that if $B_i \in \mathcal{B}(X_i)$ with $F \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(B_i)$ then $\chi_F \le \sum_{i=1}^{2} \chi_{B_i} \circ \pi_i$, where χ_{B_i} is the indicator function of B_i.

Let f_1 and f_2 be any two elements of $\mathcal{L}(X_1)$ and of $\mathcal{L}(X_2)$, respectively, satisfying $\chi_F \le \sum_{i=1}^{2} f_i \circ \pi_i$. Then this inequality leads to the relation

$$F \subseteq (\{y \in X_1 : f_1(y) \ge s\} \times X_2) \cup (X_1 \times \{z \in X_2 : f_2(z) \ge 1-s\}) \quad \text{for } 0 \le s \le 1.$$

Let x^* be a nonnegative element of $\mathbf{B}^*$. Then, since $x^* \circ \mu_1$ and $x^* \circ \mu_2$ are real-valued measures, as in Proposition 3.3 of [9],

$$\begin{aligned}
&\sum_{i=1}^{2} \int f_i d(x^* \circ \mu_i) = \int_0^1 (x^* \circ \mu_1)(\{y \in X_1 : f_1(y) \ge s\}) ds + \\
&\int_0^1 (x^* \circ \mu_2)(\{z \in X_2 : f_2(z) \ge 1-s\}) ds \\
&\ge \inf_{0 \le s \le 1} x^*(\mu_1(\{y \in X_1 : f_1(y) \ge s\}) + \mu_2(\{z \in X_2 : f_2(z) \ge 1-s\})).
\end{aligned}$$

Therefore, for every nonnegative element x^* of $\mathbf{B}^*$,

$$\begin{aligned}
&x^*(\sum_{i=1}^{2} \int f_i d\mu_i) = \sum_{i=1}^{2} \int f_i d(x^* \circ \mu_i) \\
&\ge \inf_{0 \le s \le 1} x^*(\mu_1(\{y \in X_1 : f_1(y) \ge s\}) + \mu_2(\{z \in X_2 : f_2(z) \ge 1-s\})) \\
&\ge x^*(\bigwedge_{0 \le s \le 1} (\mu_1(\{y \in X_1 : f_1(y) \ge s\}) + \mu_2(\{z \in X_2 : f_2(z) \ge 1-s\})).
\end{aligned}$$

Hence, for every $f_i \in \mathcal{L}(X_i)$ such that $\chi_F \le \sum_{i=1}^{2} f_i \circ \pi_i$,

$$\sum_{i=1}^{2} \int f_i d\mu_i \ge \bigwedge_{0 \le s \le 1} (\mu_1(\{y \in X_1 : f_1(y) \ge s\}) + \mu_2(\{z \in X_2 : f_2(z) \ge 1-s\})).$$

Taking into account that

$$I(F) = \bigwedge\{\sum_{i=1}^{2} \int \chi_{B_i} d\mu_i : B_i \in \mathcal{B}(X_i) \text{ and } \chi_F \leq \sum_{i=1}^{2} \chi_{B_i} \circ \pi_i\},$$

since for every $f_i \in \mathcal{L}(X_i)$, for every $t \in [0,1]$, the set $\{x \in X_i : f_i(x) \geq t\}$ is a Borel subset of X_i, it follows that

$$\begin{aligned} I(F) &\geq \bigwedge\{\sum_{i=1}^{2} \int f_i d\mu_i : f_i \in \mathcal{L}(X_i) \text{ and } \chi_F \leq \sum_{i=1}^{2} f_i \circ \pi_i\} \\ &\geq \bigwedge\{ \bigwedge_{0\leq s\leq 1} (\mu_1(\{x \in X_1 : f_1(x) \geq s\}) + \mu_2(\{z \in X_2 : f_2(z) \geq 1-s\})) : \\ &\qquad f_i \in \mathcal{L}(X_i) \text{ and } \chi_F \leq \sum_{i=1}^{2} f_i \circ \pi_i\} \\ &\geq I(F). \end{aligned}$$

So (*) holds.

2. In this step we prove that, for every closed set $C \in \mathcal{B}(X_1 \times X_2)$,

$$\begin{aligned} (**) \quad & \bigwedge\{\sum_{i=1}^{2} \int f_i d\mu_i : f_i \in \mathcal{L}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} f_i \circ \pi_i\} \\ &= \bigwedge\{\sum_{i=1}^{2} \int h_i d\mu_i : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\}, \end{aligned}$$

where $\mathcal{U}(X_i)$ is the family of all upper semicontinuous elements of $\mathcal{L}(X_i)$. Let x^* be a nonnegative element of $\mathbf{B}^*$. By Proposition 1.31 of [9], for the real valued tight measures $x^* \circ \mu_1$ and $x^* \circ \mu_2$, the following equality holds

$$\begin{aligned} &\inf\{\sum_{i=1}^{2} \int f_i d(x^* \circ \mu_i) : f_i \in \mathcal{L}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} f_i \circ \pi_i\} \\ =&\inf\{\sum_{i=1}^{2} \int h_i d(x^* \circ \mu_i) : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\}. \end{aligned}$$

Hence, for every nonnegative element x^* of $\mathbf{B}^*$ and for every $f_i \in \mathcal{L}(X_i)$ such that $\chi_C \leq \sum_{i=1}^2 f_i \circ \pi_i$,

$$\begin{aligned}
&x^*(\sum_{i=1}^{2} \int f_i d\mu_i) = \sum_{i=1}^{2} \int f_i d(x^* \circ \mu_i) \\
&\geq \inf\{\sum_{i=1}^{2} \int h_i d(x^* \circ \mu_i) : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\} \\
&= \inf\{x^*(\sum_{i=1}^{2} \int h_i d\mu_i) : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\} \\
&\geq x^*(\bigwedge\{\sum_{i=1}^{2} \int h_i d\mu_i : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\}).
\end{aligned}$$

Therefore, for every $f_i \in \mathcal{L}(X_i)$ such that $\chi_C \leq \sum_{i=1}^2 f_i \circ \pi_i$,

$$\sum_{i=1}^{2} \int f_i d\mu_i \geq \bigwedge\{\sum_{i=1}^{2} \int h_i d\mu_i : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\}.$$

So we obtain that

$$\begin{aligned}
&\bigwedge\{\sum_{i=1}^{2} \int f_i d\mu_i : f_i \in \mathcal{L}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} f_i \circ \pi_i\} \\
&\geq \bigwedge\{\sum_{i=1}^{2} \int h_i d\mu_i : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\} \\
&\geq \bigwedge\{\sum_{i=1}^{2} \int f_i d\mu_i : f_i \in \mathcal{L}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} f_i \circ \pi_i\}.
\end{aligned}$$

3. Now we show that, for every closed set $C \in \mathcal{B}(X_1 \times X_2)$,

$$\begin{aligned}
(***) \qquad &\bigwedge\{\sum_{i=1}^{2} \mu_i(C_i) : C_i \text{ closed set in } \mathcal{B}(X_i) \text{ and } C \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(C_i)\} \\
&= \bigwedge\{\sum_{i=1}^{2} \int h_i d\mu_i : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\}.
\end{aligned}$$

We only have to notice that, for every $h_i \in \mathcal{U}(X_i)$, for every $t \in [0,1]$, the set $\{x \in X_i : h_i(x) \geq t\}$ is a closed subset of X_i, and to argue as in step 1.

4. Conclusion. By the previous steps,

$$\begin{aligned} I(C) &= \bigwedge\{\sum_{i=1}^{2} \int f_i d\mu_i : f_i \in \mathcal{L}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} f_i \circ \pi_i\} \\ &= \bigwedge\{\sum_{i=1}^{2} \int h_i d\mu_i : h_i \in \mathcal{U}(X_i) \text{ and } \chi_C \leq \sum_{i=1}^{2} h_i \circ \pi_i\} \\ &= \bigwedge\{\sum_{i=1}^{2} \mu_i(C_i) : C_i \text{ closed set in } \mathcal{B}(X_i) \text{ and } C \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(C_i)\}. \end{aligned}$$

So the theorem is proved. □

Proposition 3.3. *For each $F \in \mathcal{P}(X_1 \times X_2)$, $S(F) \leq I(F)$.*

Proof. For any $F \in \mathcal{P}(X_1 \times X_2)$, $0 \leq I(F)$, therefore the case $M = \emptyset$ is obvious. Otherwise it is enough to notice that, for every μ with marginals μ_1 and μ_2 and for every $(B_1, B_2) \in \mathcal{B}(X_1) \times \mathcal{B}(X_2)$ with $F \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(B_i)$,

$$\begin{aligned} \mu^*(F) &\leq \mu^*(\cup_{i=1}^{2} {\pi_i}^{-1}(B_i)) = \mu(\cup_{i=1}^{2} {\pi_i}^{-1}(B_i)) \\ &\leq \sum_{i=1}^{2} \mu({\pi_i}^{-1}(B_i)) = \sum_{i=1}^{2} \mu_i(B_i). \end{aligned}$$

□

Definition 3.4. [4, Definition 14, page 7] *Let $\mathcal{F}$ be a field of subsets of a set X and let $\mu : \mathcal{F} \to \mathbf{B}$ be a vector measure with values in a Banach space $\mathbf{B}$. The vector measure μ is said to be strongly additive whenever given a sequence $(F_n)_{n \in N}$ of pairwise disjoint members of $\mathcal{F}$, the series $\sum_{n=1}^{\infty} \mu(F_n)$ converges in norm.*

Theorem 3.5. *Let $\mathcal{F}$ be a field of subsets of a set X. Then every vector measure $\mu : \mathcal{F} \to \mathbf{B}^+$ taking values in the positive cone of a Banach lattice with order continuous norm is strongly additive.*

Proof. If $(F_n)_{n \in N}$ is a sequence of pairwise disjoint sets in $\mathcal{F}$, then the sequence of partial sums $s_n = \sum_{i=1}^{n} \mu(F_i)$ is increasing and it is bounded above by $\mu(X)$, hence it is directed $(\leq)$ and majorized. Thus, by [11, Theorem 5.10], the infinite series $\sum_{n=1}^{\infty} \mu(F_n)$ converges weakly. Therefore, by [11, Corollary, page 89], it norm converges. □

Definition 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of a set X_1 and a set X_2, respectively. Henceforward by $\mathcal{A} \times \mathcal{B}$ we denote the field on $X_1 \times X_2$ generated by all rectangles $A \times B$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. By $\mathcal{A} \otimes \mathcal{B}$ we denote the σ-field on $X_1 \times X_2$ generated by $\mathcal{A} \times \mathcal{B}$.*

Given a non-empty set X and a subset F of X by F^c we denote the complement $X - F$ of F.

Theorem 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of a set X_1 and a set X_2, respectively, and let $\mu_1 : \mathcal{A} \to \mathbf{B}^+$ and $\mu_2 : \mathcal{B} \to \mathbf{B}^+$ be vector measures taking values in the positive cone of an order complete Banach lattice $\mathbf{B}$. We assume that $\mu_1(X_1) = \mu_2(X_2) = \alpha$, for some $\alpha \in \mathbf{B}^+$. Let F be an arbitrary set in $\mathcal{A} \times \mathcal{B}$ and let $\mathcal{C}$ be the field on $X_1 \times X_2$ generated by F and the sets in $\mathcal{A} \times \mathcal{B}$. For an element $v \in \mathbf{B}$, with $0 \leq v \leq \alpha$, we consider the following conditions:*

- *(i) There is a vector measure $\mu : \mathcal{C} \to \mathbf{B}^+$ such that $\mu(A \times X_2) = \mu_1(A)$ and $\mu(X_1 \times B) = \mu_2(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (i.e. μ has marginals μ_1 and μ_2) and such that $\mu(F) = v$.*
- *(ii) Whenever $A \times B \subseteq F$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\mu_1(A) + \mu_2(B) \leq \alpha + v$.*
- *(iii) Whenever $A \times B \subseteq F^c$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\mu_1(A) + \mu_2(B) \leq 2\alpha - v$.*

Then (i) is equivalent to the conjunction of (ii) and (iii).

PROOF. This follows from Theorem 2.1 in [7]. □

Theorem 3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-fields of subsets of a set X_1 and a set X_2, respectively, and let $\mu : \mathcal{A} \times \mathcal{B} :\to \mathbf{B}^+$ be a vector measure, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$. Define $\mu_1 : \mathcal{A} \to \mathbf{B}^+$ and $\mu_2 : \mathcal{B} \to \mathbf{B}^+$ by the rule $\mu_1(A) = \mu(A \times X_2)$ and $\mu_2(B) = \mu(X_1 \times B)$.*

If μ_1 is compact and μ_2 is countably additive, then μ is countably additive on $\mathcal{A} \times \mathcal{B}$.

PROOF. This result is proved in the first part of the proof of Theorem 3.1 of [12]. □

Theorem 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-fields of subsets of a set X_1 and a set X_2, respectively, and let $\mu_1 \in ca(\mathcal{A}, \mathbf{B}^+)$ and $\mu_2 \in ca(\mathcal{B}, \mathbf{B}^+)$, with $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$ with order continuous norm. Suppose that μ_1 is compact and that $F \in \mathcal{A} \otimes \mathcal{B}$ is a countable intersection of sets in $\mathcal{A} \times \mathcal{B}$. For any $v \in \mathbf{B}^+$, with $v \leq \alpha$, the following are equivalent:*

(i) *There is a vector measure* $\mu \in ca(\mathcal{A} \otimes \mathcal{B}, \mathbf{B}^+)$ *with* $\mu \circ {\pi_1}^{-1} = \mu_1$ *and* $\mu \circ {\pi_2}^{-1} = \mu_2$ *such that* $\mu(F) \geq v$.

(ii) *For all* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$, *we have* $\mu_1(A) + \mu_2(B) \leq 2\alpha - v$, *whenever* $A \times B \subseteq F^c$.

PROOF. $(i) \Rightarrow (ii)$ Since

$$\begin{aligned} & A \times B \subseteq F^c \Rightarrow F \subseteq (A^c \times X_2) \cup (X_1 \times B^c) \\ \Rightarrow \ & (A \times X_2) \cap F \subseteq (A \times X_2) \cap ((A^c \times X_2) \cup (X_1 \times B^c)) \\ \Rightarrow \ & (A \times X_2) \cap F \subseteq A \times B^c, \end{aligned}$$

we calculate

$$\begin{aligned} & \mu_1(A) = \mu(A \times X_2) = \mu((A \times X_2) \cap F) + \mu((A \times X_2) \cap F^c) \\ \leq \ & \mu(X_1 \times B^c) + \mu((X_1 \times X_2) - F) = \mu_2(B^c) + \mu(X_1 \times X_2) - \mu(F) \\ \leq \ & 2\alpha - \mu_2(B) - v. \end{aligned}$$

$(ii) \Rightarrow (i)$ As in the proof of Theorem 2 in [8], define $I = \bigwedge\{2\alpha - \mu_1(A) - \mu_2(B) : A \times B \subseteq F^c\}$ and $\Sigma = \bigvee\{\mu_1(A) + \mu_2(B) - \alpha : A \times B \subseteq F\}$. It is straightforward that $\Sigma \leq I$: suppose that $A \times B \subseteq F$ and $A_0 \times B_0 \subseteq F^c$. Note that either $A \cap A_0 = \emptyset$ or $B \cap B_0 = \emptyset$. Therefore, $\mu_1(A) + \mu_2(B) + \mu_1(A_0) + \mu_2(B_0) \leq 3\alpha$, and hence $\mu_1(A) + \mu_2(B) - \alpha \leq 2\alpha - \mu_1(A_0) - \mu_2(B_0)$, as desired.

Let $v_0 = v \vee \Sigma$. Since, by (ii), whenever $A \times B \subseteq F^c$, we have $v \leq 2\alpha - \mu_1(A) - \mu_2(B)$, it is clear that $\Sigma \leq v_0 \leq I$. Hence, (ii) and (iii) of Theorem 3.7 hold with v_0 in place of v. Let $\mathcal{C}$ be the field generated by $\mathcal{A} \times \mathcal{B}$ and the set F. By Theorem 3.7, there exists a vector measure $\mu_0 : \mathcal{C} \to \mathbf{B}^+$ with marginals μ_1 and μ_2 and such that $\mu_0(F) = v_0$. By Theorem 3.8 μ_0 is countably additive on $\mathcal{A} \times \mathcal{B}$, because it has countably additive marginals, one of which is compact. Using Theorem 3.5 and Kluvanek's Theorem [4, page 27], we find a countably additive vector measure $\mu : \mathcal{A} \otimes \mathcal{B} \to \mathbf{B}^+$ such that $\mu = \mu_0$ on $\mathcal{A} \times \mathcal{B}$. Choose a decreasing sequence of sets $(F_n)_{n \in N}$ in $\mathcal{A} \times \mathcal{B}$ such that $\cap_{n \in N} F_n = F$. Then

$$\mu(F) = \lim_n \mu(F_n) = \lim_n \mu_0(F_n) \geq \mu_0(F) = v_0 \geq v,$$

establishing the theorem. $\square$

Theorem 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-fields of subsets of a set X_1 and a set X_2, respectively, and let $\mu_1 \in ca(\mathcal{A}, \mathbf{B}^+)$ and $\mu_2 \in ca(\mathcal{B}, \mathbf{B}^+)$, with $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$ with order continuous norm. Suppose that μ_1 is perfect and that $F \in \mathcal{A} \otimes \mathcal{B}$ is a countable intersection of sets in $\mathcal{A} \times \mathcal{B}$. For any $v \in \mathbf{B}^+$, with $v \leq \alpha$, the following are equivalent:*

(i) There is a vector measure $\mu \in ca(\mathcal{A} \otimes \mathcal{B}, \mathbf{B}^+)$ with $\mu \circ {\pi_1}^{-1} = \mu_1$ and $\mu \circ {\pi_2}^{-1} = \mu_2$ such that $\mu(F) \geq v$.

(ii) For all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have $\mu_1(A) + \mu_2(B) \leq 2\alpha - v$, whenever $A \times B \subseteq F^c$.

PROOF. The proof is the same as the one of Theorem 3.9. The only difference is the fact that instead of applying Theorem 3.8 we apply Theorem 3.1 of [12]. □

Proposition 3.11. [2, Proposition 8.1.5] *Let $X_1 \dots X_n \dots$ be a (finite or infinite) sequence of separable metrizable spaces. Then $\mathcal{B}(\Pi_i X_i) = \otimes_i \mathcal{B}(X_i)$.*

Theorem 3.12. *Let X_1 and X_2 be Polish spaces and let μ_1 and μ_2 be elements of $ca(\mathcal{B}(X_1), \mathbf{B}^+)$ and of $ca(\mathcal{B}(X_2), \mathbf{B}^+)$, respectively, with $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$ with order continuous norm. Let $F \in \mathcal{B}(X_1 \times X_2)$ be a countable intersection of sets in $\mathcal{B}(X_1) \times \mathcal{B}(X_2)$. For any $v \in \mathbf{B}^+$, $v \leq \alpha$, the following are equivalent:*

(1) There is a vector measure μ in $ca(\mathcal{B}(X_1 \times X_2), \mathbf{B}^+)$ with $\mu \circ {\pi_1}^{-1} = \mu_1$ and $\mu \circ {\pi_2}^{-1} = \mu_2$ such that $\mu(F) \geq v$.

(2) $I(F) \geq v$.

PROOF. It is enough to observe that μ_1 and μ_2 are tight [10, Theorem 3.2] and that hypothesis (2) is equivalent to (ii) of Theorem 3.9. □

Corollary 3.13. *Let X_1 and X_2 be Polish spaces and let μ_1 and μ_2 be elements of $ca(\mathcal{B}(X_1), \mathbf{B}^+)$ and of $ca(\mathcal{B}(X_2), \mathbf{B}^+)$, respectively, with $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$ with order continuous norm. Then there exists μ in $ca(\mathcal{B}(X_1 \times X_2), \mathbf{B}^+)$ with $\mu \circ {\pi_1}^{-1} = \mu_1$ and $\mu \circ {\pi_2}^{-1} = \mu_2$ (i.e., under these hypotheses, the set M of Definition 3.1 is non-empty).*

PROOF. We only have to notice that $I(X_1 \times X_2) = \alpha$ and to apply Theorem 3.12 with $X_1 \times X_2$ in place of F and with $v = \alpha$. □

Theorem 3.14. Duality Theorem. *Let X_1 and X_2 be Polish spaces and let $\mu_1 \in ca(\mathcal{B}(X_1), \mathbf{B}^+)$ and $\mu_2 \in ca(\mathcal{B}(X_2), \mathbf{B}^+)$, with $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$ with order continuous norm. Let $F \in \mathcal{B}(X_1 \times X_2)$ be a countable intersection of sets in $\mathcal{B}(X_1) \times \mathcal{B}(X_2)$. Then*

$$I(F) = S(F).$$

PROOF. Let $v = I(F)$. By Theorem 3.12, there exists μ in $\mathrm{ca}(\mathcal{B}(X_1 \times X_2), \mathbf{B}^+)$ with $\mu \circ {\pi_1}^{-1} = \mu_1$, $\mu \circ {\pi_2}^{-1} = \mu_2$ and $\mu(F) \geq v$. Thus, $S(F) \geq I(F)$. This inequality and Proposition 3.3 lead to the desired equality. □

The next result extends Proposition 3.8 of [9] to the vector case.

Theorem 3.15. *Let X_1 and X_2 be Polish spaces and let μ_1 and μ_2 be elements of $ca(\mathcal{B}(X_1), \mathbf{B}^+)$ and of $ca(\mathcal{B}(X_2), \mathbf{B}^+)$, respectively, with $\mu_1(X_1) = \mu_2(X_2) = \alpha$, where $\mathbf{B}^+$ is the positive cone of a Banach lattice $\mathbf{B}$ with order continuous norm. Let C be a closed subset of $X_1 \times X_2$. Then, for any $v \in \mathbf{B}^+$, $v \leq \alpha$, there exists a vector measure $\mu \in ca(\mathcal{B}(X_1 \times X_2), \mathbf{B}^+)$ such that $\mu \circ {\pi_1}^{-1} = \mu_1$ and $\mu \circ {\pi_2}^{-1} = \mu_2$, with $\mu(C) \geq v$ if and only if*

$$\sum_{i=1}^{2} \mu_i(C_i) \geq v, \ \textit{for all } C_i \textit{ closed sets in } \mathcal{B}(X_i) \textit{ with } C \subseteq \cup_{i=1}^{2} {\pi_i}^{-1}(C_i).$$

PROOF. Since $C = \cap_{j \in J}(A_{1,j} \times A_{2,j})^c$, with $A_{i,j}$ open subset of X_i, J finite or countable, $i = 1, 2$, this follows at once from Theorem 3.2 and Theorem 3.12. □

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