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ON SMALL SUBSETS OF THE SPACE OF DARBOUX FUNCTIONS

Abstract

We prove that the set $\mathcal{F}$ of all bounded functionally connected functions is boundary in the space of all bounded Darboux functions (with the metric of uniform convergence). Next we prove that the set of bounded upper (lower)semi-continuous Darboux functions and the set of all bounded quasi-continuous functionally connected functions is porous at each point of the space $\mathcal{F}$.

Since 1875, when Darboux functions were defined, many papers have appeared about their properties. Proofs of many properties of real Darboux functions of real variables are very important, because the family of Darboux functions includes many important classes of functions; for example, continuous functions and functionally connected functions. (See the next page for the definition.) Mutual inclusions among several families of Darboux-like functions inspired questions concerning the size of particular sets in spaces of functions. One of the questions is how strong the inclusions are. Similar issues have already been considered in a great number of papers such as [4], [5] and [6]. In [3] Jędrzejewski noticed that each continuous function is functionally connected and each functionally connected function is a Darboux function. It is not difficult to show that there exists a discontinuous, functionally connected function and there exists a Darboux function which is not functionally connected. Moreover, it turns out that in the space of bounded Darboux functions (with the metric of uniform convergence) the set of bounded functionally connected functions is boundary (its complement is a dense set) and in the space of bounded functionally connected functions the set of bounded upper (lower)

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semi-continuous functions is porous at each point of this space. We shall also show that bounded quasi-continuous functionally connected functions form a porous set in the space of bounded functionally connected functions.

We use the following symbols and notion.

If (X,d) is a metric space, then $B(x,r)$ denotes the open ball with center at x and radius $r>0$. Let $M \subset X$, $x \in X$, $R>0$. Then $\gamma(x,R,M)$ denotes the supremum of the set of all $r>0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$. The set M is called porous at x if $\limsup_{R\to 0^+} \frac{\gamma(x,R,M)}{R} > 0$.

The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. Let $\Gamma(f)$ denote the graph of a function $f:\mathbb{R}\to\mathbb{R}$.

We say that a function $f:\mathbb{R}\to\mathbb{R}$ is a Darboux function if an image of each interval is a connected set. By $\mathcal{D}$ we denote a set of all bounded Darboux functions $f:\mathbb{R}\to\mathbb{R}$.

By $\mathcal{C}_u^*$ ($\mathcal{C}_l^*$) we mean the set of all bounded upper (lower) semi-continuous Darboux functions $f:\mathbb{R}\to\mathbb{R}$. Let $\mathcal{C}$ denote the set of all bounded continuous functions $f:\mathbb{R}\to\mathbb{R}$ and let $\mathcal{DB}_1$ denote the set of all bounded Darboux Baire 1 functions $f:\mathbb{R}\to\mathbb{R}$.

We say that a function $f:\mathbb{R}\to\mathbb{R}$ is functionally connected if for each $a,b\in\mathbb{R}, a<b$ and for each continuous function $g:[a,b]\to\mathbb{R}$ such that $(f(a)-g(a))(f(b)-g(b))<0$ there exists $x\in(a,b)$ such that $f(x)=g(x)$. By $\mathcal{F}$ we mean the set of all bounded functionally connected functions $f:\mathbb{R}\to\mathbb{R}$.

The following characterization of functionally connected functions will be useful in the further consideration.

Theorem. ([4][Theorem II.2]) *A function $f:\mathbb{R}\to\mathbb{R}$ is a functionally connected iff for an arbitrary continuous function $g:\mathbb{R}\to\mathbb{R}$ function $f+g$ is a Darboux one.*

Then (from this Theorem and Theorem II.3.2 from [1])

$$\mathcal{C}_u^* \subset \mathcal{DB}_1 \subset \mathcal{F} \text{ and} \mathcal{C}_l^* \subset \mathcal{DB}_1 \subset \mathcal{F}.$$

A function $f:\mathbb{R}\to\mathbb{R}$ is quasi-continuous at x_0 if for each $\varepsilon>0$ and $\delta>0$ there exists an open set $U\subset(x_0-\delta,x_0+\delta)$ such that $f(U)\subset (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$. If the function f is quasi-continuous at each point, we say that f is quasi-continuous. By $\mathcal{Q}$ we denote a set of all bounded quasi-continuous functions $f:\mathbb{R}\to\mathbb{R}$. Let $\mathcal{Q}^*=\mathcal{Q}\cap\mathcal{F}$.

By ρ we denote the metric of uniform convergence.

Theorem 1. *The set $\mathcal{F}$ is boundary in the space $\mathcal{D}$.*

PROOF. Let $f \in \mathcal{D}$ and let $\varepsilon > 0$. We shall show that there exists $g \in B(f, \varepsilon) \setminus \mathcal{F}$. First, let us suppose that f is not a constant function . There are points a and b such that $f(a) \neq f(b)$. For example assume $f(a) < f(b)$. Let $N \geq 3$ be such that $\frac{f(b)-f(a)}{N} < \frac{\varepsilon}{4}$. Let $f(a) = p_0 < p_1 < p_2 < ... < p_{N-1} < p_N = f(b)$ be points such that
$p_i - p_{i-1} = \frac{f(b)-f(a)}{N}$ for $i = 1, 2, ..., N$ and

$$[f(a), f(b)] = [p_0, p_1] \cup [p_1, p_2] \cup ... \cup [p_{N-1}, p_N].$$

Obviously, for all $i \in \{1, 2, ...N\}$

$$f^{-1}_{|[a,b]}\ (p_{i-1}, p_i) \text{ is a c-dense-in-itself set.} \tag{1}$$

So, there exist sets $F^{(i)}_\alpha$, $\alpha < \mathfrak{c}$, such that $f^{-1}_{|[a,b]}(p_{i-1}, p_i) = \bigcup_{\alpha<\mathfrak{c}} F^{(i)}_\alpha$, where sets $F^{(i)}_\alpha$, $\alpha < \mathfrak{c}$, are pairwise disjoint and dense in $f^{-1}_{|[a,b]}(p_{i-1}, p_i)$. Let $\mathcal{F}_i = \{F^{(i)}_\alpha : \alpha < \mathfrak{c}\}$ for $i = 1, 2, ...N$.

Let

$$\begin{aligned} &\xi_1 : \mathcal{F}_1 \to [p_0, p_2], \\ &\xi_i : \mathcal{F}_i \to [p_{i-2}, p_{i+1}] \ \ (i \in \{2, ..., N-1\}), \\ &\xi_N : \mathcal{F}_N \to [p_{N-2}, p_N] \end{aligned}$$

be bijections. Let $g^* : \mathbb{R} \to \mathbb{R}$ be defined by

$$g^*(x) = \begin{cases} f(x) & \text{if } x \notin (a, b); \\ f(x) & \text{if } x \in (a, b) \text{ and} \\ & f(x) \in (-\infty, f(a)) \cup (f(b), +\infty) \cup \{p_i : i = 0, 1, ..., N\}; \\ \xi_i(K) & \text{if } x \in K \in \mathcal{F}_i, i = 1, 2, ..., N. \end{cases}$$

We define a function $g : \mathbb{R} \to \mathbb{R}$ as follows.

$$g(x) = \begin{cases} g^*(x) & \text{if } x \notin (a, b) \text{ or if } x \in (a, b) \text{ and } g^*(x) \neq l(x); \\ p_i & \text{if } x \in (a, b),\ g^*(x) = l(x) \\ & \ \text{and } g^*(x) \in (p_{i-1}, p_i), i = 1, 2, ..., N; \\ p_{i-1} & \text{if } x \in (a, b),\ g^*(x) = l(x) = p_i \\ & \ \text{and } f(x) < p_i \ (i = 1, 2, ..., N); \\ p_{i+1} & \text{if } x \in (a, b),\ g^*(x) = l(x) = p_i \\ & \ \text{and } f(x) \geq p_i \ (i = 0, 1, ..., N-1). \end{cases}$$

where l is a linear function such that $l(a) = f(a)$ and $l(b) = f(b)$.

First note that for an arbitrary $i \in \{1, 2, ..., N\}$ and for an arbitrary nondegenerate interval $I \subset \mathbb{R}$

$$I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i) \neq \emptyset \;\Rightarrow\; \forall_{\alpha<\mathfrak{c}}\; \exists_{x_1 \neq x_2}\; x_1, x_2 \in I \cap F^{(i)}_\alpha. \tag{2}$$

((2) follows from (1) and the fact that $F^{(i)}_\alpha$ is dense in $f^{-1}_{|[a,b]}(p_{i-1}, p_i)$ for each $\alpha < \mathfrak{c}$.) We will show that for an arbitrary nondegenerate interval $I \subset \mathbb{R}$ and for an arbitrary $i \in \{2, 3, ..., N-1\}$

$$I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i) \neq \emptyset \Rightarrow g(I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i)) \supset [p_{i-2}, p_{i+1}]. \tag{3}$$

Let $i \in \{2, 3, ..., N-1\}$ be fixed and let I be a nondegenerate interval such that $I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i) \neq \emptyset$. Let $y_0 \in [p_{i-2}, p_{i+1}]$. Then $y_0 \in \xi_i(\mathcal{F}_i)$. So, there exists $\alpha < \mathfrak{c}$ such that $y_0 = \xi_i(F^{(i)}_\alpha)$. By (2) there exists $x_1, x_2 \in I \cap F^{(i)}_\alpha$ such that $x_1 \neq x_2$. Then $g^*(x_1) = \xi_i(F^{(i)}_\alpha) = g^*(x_2)$. Hence $g^*(x_1) \neq l(x_1)$ or $g^*(x_2) \neq l(x_2)$. For example, assume $g^*(x_1) \neq l(x_1)$. Obviously $x_1 \in (a, b)$. Hence $g(x_1) = g^*(x_1) = \xi_i(F^{(i)}_\alpha) = y_0$. So $y_0 \in g(I)$ finishing the proof of (3) .

In the analogous way we can show that for an arbitrary nondegenerate interval $I \subset \mathbb{R}$

$$I \cap f^{-1}_{|[a,b]}(p_0, p_1) \neq \emptyset \Rightarrow g(I \cap f^{-1}_{|[a,b]}(p_0, p_1)) \supset [p_0, p_2] \tag{4}$$

and

$$I \cap f^{-1}_{|[a,b]}(p_{N-1}, p_N) \neq \emptyset \Rightarrow g(I \cap f^{-1}_{|[a,b]}(p_{N-1}, p_N)) \supset [p_{N-2}, p_N]. \tag{5}$$

Now, we will show that

$$g_{|[a,b]} \quad \text{is a Darboux function .} \tag{6}$$

Let $I \subset [a, b]$ be a nondegenerate interval. Then there are two possible cases.

1. $I \subset \{x \in [a, b] : f(x) < f(a) \text{ or } f(x) > f(b)\}$. Then $I \subset (a, b)$ and $g(I) = f(I)$ is a connected set since f is a Darboux function.
2. $I \cap \{x \in [a, b] : f(x) \in [f(a), f(b)]\} \neq \emptyset$.

Let $i_1 = \min\{i \in \{1, 2, ..., N\} : I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i) \neq \emptyset\}$ and $i_2 = \max\{i \in \{1, 2, ..., N\} : I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i) \neq \emptyset\}$. Obviously $i_1 \leq i_2$. Moreover for all $i \in \{i_1, ..., i_2\}$

$$I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i) \neq \emptyset. \tag{7}$$

Consider the four following subcases.
2a. $I \subset \{x \in [a,b] : f(x) \in [f(a), f(b)]\}$.

Then from the fact that f is a Darboux function and from the definition of i_1, i_2 we can infer that $I \subset \bigcup_{i=i_1}^{i_2} f^{-1}_{|[a,b]}\,[p_{i-1}, p_i]$. Hence

$$g^*(I) \subset \bigcup_{i=i_1}^{i_2} g^*(f^{-1}_{|[a,b]}\,(p_{i-1}, p_i)) \cup \bigcup_{i=i_1-1}^{i_2} g^*(f^{-1}_{|[a,b]}\,\{p_i\}) =$$

$$\bigcup_{i=i_1}^{i_2} g^*(\bigcup_{\alpha<\mathfrak{c}} F_\alpha^{(i)}) \cup \bigcup_{i=i_1-1}^{i_2} \{p_i\} = \bigcup_{i=i_1}^{i_2} \xi_i(\mathcal{F}_i) \cup \bigcup_{i=i_1-1}^{i_2} \{p_i\} = \bigcup_{i=i_1}^{i_2} \xi_i(\mathcal{F}_i),$$

so

$$g^*(I) \subset \bigcup_{i=i_1}^{i_2} \xi_i(\mathcal{F}_i). \tag{8}$$

And again, consider the following subcases.
1. $i_1 \geq 2,\;\; i_2 \leq N-1$.

In that case from (8) and the definition of ξ_i, $i \in \{2, ..., N-1\}$, it follows that

$$g^*(I) \subset [p_{i_1-2}, p_{i_2+1}]. \tag{9}$$

Now, we will show that

$$g(I) = [p_{i_1-2}, p_{i_2+1}]. \tag{10}$$

Indeed, by (7), (3) and the definition of i_1 and i_2, we have

$$g(I) \supset \bigcup_{i=i_1}^{i_2} [p_{i-2}, p_{i+1}] = [p_{i_1-2}, p_{i_2+1}].$$

We will prove that $g(I) \subset [p_{i_1-2}, p_{i_2+1}]$. Let $y_0 \in g(I)$. Then there exists $x_0 \in I$ such that $y_0 = g(x_0)$.
The following cases are possible.

- If $g^*(x_0) \neq l(x_0)$, then $g(x_0) = g^*(x_0) \in [p_{i_1-2}, p_{i_2+1}]$ (by (9)).

- If $g^*(x_0) = l(x_0)$ and there exists $i_0 \in \{i_1-1, ..., i_2+1\}$ such that $g^*(x_0) \in (p_{i_0-1}, p_{i_0})$ (by (9)), then $g(x_0) = p_{i_0} \in [p_{i_1-1}, p_{i_2+1}] \subset [p_{i_1-2}, p_{i_2+1}]$.

- If $g^*(x_0) = l(x_0)$ and there exists $i_0 \in \{i_1-2, ..., i_2+1\}$ such that $g^*(x_0) = p_{i_0}$ (by (9)) and $f(x_0) < p_{i_0}$, then $i_0 > i_1 - 2$. (If $i_0 = i_1 - 2$, $f(x_0) < p_{i_0} = p_{i_1-2} < p_{i_1-1}$, which contradicts the definition of i_1 and the fact that f is a Darboux function.) Hence $g(x_0) = p_{i_0-1} \in [p_{i_1-2}, p_{i_2}] \subset [p_{i_1-2}, p_{i_2+1}]$.

- If $g^*(x_0) = l(x_0)$ and there exists $i_0 \in \{i_1 - 2, ..., i_2 + 1\}$ such that $g^*(x_0) = p_{i_0}$ (by (9)) and $f(x_0) \geq p_{i_0}$, then $i_0 < i_2 + 1$. (If $i_0 = i_2 + 1$, $f(x_0) \geq p_{i_0} = p_{i_2+1} > p_{i_2}$, which contradicts the definition of i_2 and the fact that f is a Darboux function.) Hence $g(x_0) = p_{i_0+1} \in [p_{i_1-1}, p_{i_2+1}] \subset [p_{i_1-2}, p_{i_2+1}]$.

The proof of (10) is now complete. So $g(I)$ is a connected set.

2. $i_1 = 1$, $i_2 \leq N - 1$.

Note that from (9) and the definition of ξ_i, $i \in \{1, ..., N-1\}$, it follows that

$$g^*(I) \subset [p_0, p_{i_2+1}]. \tag{11}$$

Now we will show that

$$g(I) = [p_0, p_{i_2+1}]. \tag{12}$$

Indeed, by (7), (3) and (4) and the definition of i_1 and i_2, we have

$$g(I) \supset [p_0, p_2] \cup \bigcup_{i=2}^{i_2} [p_{i-2}, p_{i+1}] = [p_0, p_{i_2+1}].$$

We will prove that $g(I) \subset [p_0, p_{i_2+1}]$. Let $y_0 \in g(I)$. Then there exists $x_0 \in I$ such that $y_0 = g(x_0)$.

There are possible the following cases.

- If $g^*(x_0) \neq l(x_0)$, then $g(x_0) = g^*(x_0) \in [p_0, p_{i_2+1}]$.

- If $g^*(x_0) = l(x_0)$ and there exists $i_0 \in \{1, ..., i_2 + 1\}$ such that $g^*(x_0) \in (p_{i_0-1}, p_{i_0})$ (by (11)), then $g(x_0) = p_{i_0} \in [p_1, p_{i_2+1}] \subset [p_0, p_{i_2+1}]$.

- If $g^*(x_0) = l(x_0)$ and there exists $i_0 \in \{0, .., i_2+1\}$ such that $g^*(x_0) = p_{i_0}$ (by (11)) and $f(x_0) < p_{i_0}$, then $i_0 > 0$. (If $i_0 = 0, f(x_0) < p_{i_0} = p_0 = f(a)$, which contradicts the assumption **2a.** and the fact that f is a Darboux function.) Hence $g(x_0) = p_{i_0-1} \in [p_{i_1-1}, p_{i_2}] \subset [p_0, p_{i_2+1}]$.

- If $g^*(x_0) = l(x_0)$ and there exists $i_0 \in \{0, ..., i_2+1\}$ such that $g^*(x_0) = p_{i_0}$ (by (11)) and $f(x_0) \geq p_{i_0}$, then $i_0 < i_2+1$. (If $i_0 = i_2+1$, $f(x_0) \geq p_{i_0} = p_{i_2+1}$, which contradicts the definition of i_0 and the fact that f is Darboux function.) Hence $g(x_0) = p_{i_0+1} \in [p_{i_1-1}, p_{i_2+1}] = [p_0, p_{i_2+1}]$.

The proof of (12) is now complete. So $g(I)$ is a connected set.

3. In the way similar to the previous cases we can show that for $i_1 \geq 1$, $i_2 = N$ we have $g(I) = [p_{i_1-2}, p_N]$ and for $i_1 = 1$, $i_2 = N$ we have $g(I) = [p_0, p_N]$, so $g(I)$ is a connected set in these cases.

2b. $I \subset \{x \in [a,b] : \ f(x) \leq f(b)\}$ and $I \cap \{x \in [a,b] : f(x) < f(a)\} \neq \emptyset$.

From the fact f is a Darboux function it follows that $i_1 = 1$. Then, by (7), (3) and (4), we have

$$g(I \cap f^{-1}[f(a), f(b)]) \supset \bigcup_{i=1}^{i_2} g(I \cap f^{-1}_{|[a,b]}(p_{i-1}, p_i)) \supset [p_0, p_{i_2+1}].$$

In a way analogous to case (2) we can show that

$$g(I \cap f^{-1}[f(a), f(b)]) \subset [p_0, p_{i_2+1}],$$

so

$$g(I \cap f^{-1}[f(a), f(b)]) = [p_0, p_{i_2+1}].$$

Let $m = \inf\{f(x) : \ x \in I\}$. Then, by our assumption, $m < f(a)$. Moreover (by the fact that f is a Darboux function, the definition of m and the assumption of 2).

$$f(I \cap \{x : \ f(x) < f(a)\}) = \begin{cases} [m, f(a)) & \text{if} \quad m \in f(I), \\ (m, f(a)) & \text{if} \quad m \notin f(I). \end{cases}$$

So,

$$\begin{aligned} g(I) &= g(I \cap \{x : \ f(x) \in [f(a), f(b)]\}) \cup g(I \cap \{x : \ f(x) < f(a)\}) \\ &= [f(a), p_{i_2+1}] \cup f(I \cap \{x : \ f(x) < f(a)\}) \\ &= \begin{cases} [f(a), p_{i_2+1}] \cup [m, f(a)) = [m, p_{i_2+1}] & \text{if } m \in f(I), \\ [f(a), p_{i_2+1}] \cup (m, f(a)) = (m, p_{i_2+1}] & \text{if } m \notin f(I). \end{cases} \end{aligned}$$

Hence $g(I)$ is an interval, so it is a connected set.

2c. $I \subset \{x \in [a,b] : \ f(x) \geq f(a)\}$ and $I \cap \{x \in [a,b] : \ f(x) > f(b)\} \neq \emptyset$

2d. $I \cap \{x \in [a,b] : \ f(x) < f(a)\} \neq \emptyset$ and $I \cap \{x \in [a,b] : \ f(x) > f(b)\} \neq \emptyset$.

These cases are analogous to the previous ones. The proof of (6) is thus complete.

From condition (6) and the fact that $g_{|(-\infty,a]} = f_{|(-\infty,a]}$ and $g_{|[b,+\infty)} = f_{|[b,+\infty)}$ are Darboux functions, it follows that g is a Darboux function ([7], Lemma 1). We will show that

$$g \notin \mathcal{F}. \tag{13}$$

Suppose that $g \in \mathcal{F}$. Then $g - l \in \mathcal{D}$ (Theorem II.2 from [4]). Note that

$$0 \notin (g - l)(a, b). \tag{14}$$

Indeed, if $0 \in (g - l)(a, b)$, there exists $c \in (a, b)$ such that $(g - l)(c) = 0$. So $g(c) = l(c)$. Hence

$$(c, g(c)) \in \Gamma(g_{|(a,b)}) \cap \Gamma(l). \tag{15}$$

But, on the other hand, by the definition of the function g, we can easily show that $\Gamma(g_{|(a,b)}) \cap \Gamma(l) = \emptyset$. This condition contradicts (15) completing the proof of (14).

Obviously $(a, b) \cap f^{-1}_{|[a,b]}(p_0, p_1) \neq \emptyset$ and $(a, b) \cap f^{-1}_{|[a,b]}(p_{N-1}, p_N) \neq \emptyset$. So, by (4), $g(a, b) \supset [p_0, p_2]$ and by (5), $g(a, b) \supset [p_{N-2}, p_N]$. Hence, in particular, there exists $x_1 \in (a, b)$ such that $g(x_1) = p_0$ and there exists $x_2 \in (a, b)$ such that $g(x_2) = p_N$.
Thus we have

$$(g - l)(x_1) = p_0 - l(x_1) = f(a) - l(x_1) < f(a) - f(a) = 0$$

and

$$(g - l)(x_2) = p_N - l(x_2) = f(b) - l(x_2) > f(b) - f(b) = 0.$$

Hence the above conditions and (14) contradicts the fact that $g - l \in D$.
the proof of (13) is now complete.

Now we shall show that

$$g \in B(f, \varepsilon). \tag{16}$$

Let $x \in \mathbb{R}$. Consider the following cases.
1. $x \notin (a, b)$. Then $|f(x) - g(x)| = |f(x) - f(x)| < \frac{\varepsilon}{2}$.
2. $x \in (a, b)$ and $g^*(x) \neq l(x)$.

Then the following subcases are possible.

2a. $f(x) \in (-\infty, f(a)) \cup (f(b), +\infty) \cup \{p_i : i = 0, 1, ...N\}$. Then

$$|f(x) - g(x)| = |f(x) - f(x)| < \frac{\varepsilon}{2}.$$

2b. $x \in F^{(i)}_\alpha \in \mathcal{F}_i$, $i = 1, 2, ..., N$, $\alpha < \mathfrak{c}$. Since $F^{(i)}_\alpha \subset f^{-1}_{|[a,b]}\ (p_{i-1}, p_i)$,

$f(x) \in (p_{i-1}, p_i)$. So

$$f(x) - g(x) = f(x) - g^*(x) = f(x) - \xi_i(F_\alpha^{(i)}) < p_i - \xi_i(F_\alpha^{(i)})$$
$$< \begin{cases} p_1 - p_0, & \text{if } i = 1, \\ p_i - p_{i-2}, & \text{if } i = 2, 3, ..., N-1, \\ p_N - p_{N-2}, & \text{if } i = N, \end{cases}$$
$$= \begin{cases} \frac{f(b)-f(a)}{N}, & \text{if } i = 1, \\ \frac{f(b)-f(a)}{N} + \frac{f(b)-f(a)}{N}, & \text{if } i = 2, 3, ..., N-1, \\ \frac{f(b)-f(a)}{N} + \frac{f(b)-f(a)}{N}, & \text{if } i = N, \end{cases}$$
$$< \begin{cases} \frac{\varepsilon}{4}, & \text{if } i = 1, \\ \frac{\varepsilon}{4} + \frac{\varepsilon}{4}, & \text{if } i = 2, 3, ..., N-1 \\ \frac{\varepsilon}{4} + \frac{\varepsilon}{4}, & \text{if } i = N, \end{cases} \leq \frac{\varepsilon}{2}$$

and

$$f(x) - g(x) = f(x) - g^*(x) = f(x) - \xi_i(F_\alpha^{(i)}) > p_{i-1} - \xi_i(F_\alpha^{(i)})$$
$$> \begin{cases} p_0 - p_2, & \text{if } i = 1, \\ p_{i-1} - p_{i+1}, & \text{if } i = 2, 3, ..., N-1, \\ p_{N-1} - p_N, & \text{if } i = N, \end{cases}$$
$$= \begin{cases} -\frac{f(b)-f(a)}{N} - \frac{f(b)-f(a)}{N}, & \text{if } i = 1, \\ -\frac{f(b)-f(a)}{N} - \frac{f(b)-f(a)}{N}, & \text{if } i = 2, 3, ..., N-1, \\ -\frac{f(b)-f(a)}{N}, & \text{if } i = N, \end{cases}$$
$$> \begin{cases} -\frac{\varepsilon}{4} - \frac{\varepsilon}{4}, & \text{if } i = 1, \\ -\frac{\varepsilon}{4} - \frac{\varepsilon}{4}, & \text{if } i = 2, 3, ..., N-1 \\ -\frac{\varepsilon}{4}, & \text{if } i = N. \end{cases} \geq -\frac{\varepsilon}{2}.$$

So $|f(x) - g(x)| < \frac{\varepsilon}{2}$.

3. $x \in (a, b)$, $g^*(x) = l(x)$, $g^*(x) \in (p_{i_0-1}, p_{i_0})$, $i_0 = 1, 2, ..., N$. Then

$$f(x) \in (p_{i_0-2}, p_{i_0+1}). \tag{17}$$

Indeed, in the other case there are three possible cases.

- $f(x) \in (-\infty, f(a)] \cup [f(b), +\infty)$.

Then $g^*(x) = f(x) \notin (p_{i_0-1}, p_{i_0})$, which contradicts our assumption.

-there exists $j \in \{1, 2, ..., i_0 - 2, i_0 + 2, ..., N\}$ such that $f(x) \in (p_{j-1}, p_j)$. Then $g^*(x) \in [p_{j-2}, p_{j+1}]$. But $[p_{j-2}, p_{j+1}] \cap (p_{i_0-1}, p_{i_0}) = \emptyset$. So $g^*(x) \notin (p_{i_0-1}, p_{i_0})$, which contradicts our assumption.

-there exists $j \in \{0, 1, ..., N\}$ such that $f(x) = p_j$.
Then $g^*(x) = f(x) = p_j \notin (p_{i_0-1}, p_{i_0})$, which contradicts our assumption.

The proof of (17) is thus completed..

Hence $f(x) - g(x) < p_{i_0+1} - p_{i_0} = \frac{f(b)-f(a)}{N} < \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$ and $f(x) - g(x) > p_{i_0-2} - p_{i_0} = -[\frac{f(b)-f(a)}{N} + \frac{f(b)-f(a)}{N}] > -(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}) = -\frac{\varepsilon}{2}$.
So $|f(x) - g(x)| < \frac{\varepsilon}{2}$.

4. $x \in (a, b)$, $g^*(x) = l(x)$, $g^*(x) = p_{i_0}$, $i_0 = 1, 2, ..., N$, $f(x) < p_{i_0}$.
Assume that $i_0 \geq 2$ (for $i_0 = 1$ the proof is analogous). Note that

$$f(x) \geq p_{i_0-2}. \tag{18}$$

Indeed, suppose that $f(x) < p_{i_0-2}$. If $i_0 = 2$, then $f(x) < p_0 = f(a)$ and $g^*(x) = f(x) < f(a) = p_0 < p_2$, which contradicts our assumption. Therefore, assume that $i_0 \geq 3$. Then the following cases are possible.

-If $f(x) < f(a)$, then $g^*(x) = f(x) < f(a) = p_0 < p_{i_0}$, which contradicts our assumption.
-If $f(x) = p_i$, $i = 0, 1, ..., i_0 - 3$, then $g^*(x) = f(x) = p_i \leq p_{i_0-3} < p_{i_0}$, which contradicts our assumption.
-If $f(x) \in (p_{i-1}, p_i)$, $i = 1, 2, ..., i_0 - 2$, then (by the definition of ξ_i) $g^*(x) \leq p_{i+1} \leq p_{i_0-1} < p_{i_0}$, which contradicts our assumption completing the proof of (18).

Thus, we have

$$f(x) - g(x) = f(x) - p_{i_0-1} > -\frac{f(b) - f(a)}{N} > \frac{-\varepsilon}{4} > -\frac{\varepsilon}{2}$$

and

$$f(x) - g(x) = f(x) - p_{i_0-1} < \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.$$

Hence $|f(x) - g(x)| < \frac{\varepsilon}{2}$.

5. $x \in (a, b)$, $g^*(x) = l(x)$, $g^*(x) = p_{i_0}$, $i_0 = 0, 1, ..., N - 1$, $f(x) \geq p_{i_0}$.
Assume that $i_0 \leq N - 2$ (for $i_0 = N - 1$ the proof is analogous). Like the procedure in 4. we can show that $f(x) \leq p_{i_0+2}$. So we have

$$f(x) - g(x) \leq p_{i_0+2} - p_{i_0+1} = \frac{f(b) - f(a)}{N} < \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$$

$$f(x) - g(x) \geq p_{i_0} - p_{i_0+1} = -(p_{i_0+1} - p_{i_0}) = -\frac{f(b) - f(a)}{N} > -\frac{\varepsilon}{4} > -\frac{\varepsilon}{2}.$$

Hence $|f(x)-g(x)| < \frac{\varepsilon}{2}$. Thus we have proved that for all $x \in \mathbb{R}$ $|f(x)-g(x)| < \frac{\varepsilon}{2}$ completing the proof of (16).

The proof of the theorem is finished in case f is not a constant function. If f is a constant function, then there exists $\hat{f} \in \mathcal{D}$ such that $\rho(f, \hat{f}) < \frac{\varepsilon}{2}$ and $\hat{f}$ is not a constant function (for example $\hat{f}(x) = f(x) + \frac{\varepsilon}{2\pi} \arctan x$). Then from the first part of the proof we infer that there exists $g \in B(\hat{f}, \frac{\varepsilon}{2}) \setminus \mathcal{F}$. Hence $g \in B(f, \varepsilon)$ and $g \notin \mathcal{F}$. $\square$

Theorem 2. *The set $\mathcal{C}_u^*$ is porous at each point of the set $\mathcal{F}$.*

PROOF. Let $f \in \mathcal{F}$ and let $R > 0$. Consider the following cases.

1. Assume that there exists a point x_0 of continuity of f. Let $\delta > 0$ be such that

$$f([x_0 - \delta, x_0 + \delta]) \subset (f(x_0) - \frac{R}{8}, f(x_0) + \frac{R}{8}).$$

We will show that there exists $g \in \mathcal{F}$ such that

$$B(g, \frac{R}{16}) \subset B(f, R) \backslash \mathcal{C}_u^*. \tag{19}$$

Define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin (x_0 - \delta, x_0 + \delta), \\ l_1(x) & \text{if } x \in [x_0 - \delta, x_0], \\ \frac{R}{8} \sin \frac{1}{x - x_0} + f(x_0) & \text{if} x \in (x_0, x_0 + \frac{\delta}{2}], \\ l_2(x) & \text{if } x \in [x_0 + \frac{\delta}{2}, x_0 + \delta]; \end{cases}$$

where l_1 is a linear function such that $l_1(x_0 - \delta) = f(x_0 - \delta)$, and $l_1(x_0) = f(x_0)$; l_2 is a linear function such that $l_2(x_0 + \frac{\delta}{2}) = \frac{R}{8} \sin \frac{2}{\delta} + f(x_0)$ and $l_2(x_0 + \delta) = f(x_0 + \delta)$.
By Lemma 1.4 of [7], $g_{|[x_0 - \delta, x_0 + \delta]}$ is a bounded Darboux Baire 1 function, so $g_{|[x_0 - \delta, x_0 + \delta]}$ is functionally connected. By Lemma 5.2 from [3], we infer that g is also functionally connected. Note that $\rho(g, f) \leq \frac{R}{2}$. Then

$$B(g, \frac{R}{16}) \subset B(f, R). \tag{20}$$

Now we will show that

$$B(g, \frac{R}{16}) \cap \mathcal{C}_u^* = \emptyset. \tag{21}$$

If $h \in B(g, \frac{R}{16})$, then $h(x_0) < f(x_0) + \frac{R}{16}$. On the other hand, consider a sequence $x_k = x_0 + \frac{2}{\pi + 4k\pi}$, $k \in \mathbb{N}$.
Then $x_k \to x_0$ and $x_k \in (x_0, x_0 + \frac{\delta}{2}]$ for k large enough. Thus, we have

$$\limsup_{x\to x_0} h(x) \geq \limsup_{k\to\infty}[g(x_k) - \frac{R}{16}] = f(x_0) + \frac{R}{16} > h(x_0),$$

so the function h is not upper semi-continuous at the point x_0, which finishes the proof of (21).

According to (20) and (21) we may infer that $\gamma(f, R, \mathcal{C}_u^*) \geq \frac{R}{16}$. Therefore we deduce that

$$\limsup_{R\to 0^+} \frac{\gamma(f, R, \mathcal{C}_u^*)}{R} \geq \frac{1}{16} > 0,$$

so $\mathcal{C}_u^*$ is porous at f and we have proved the theorem in case 1.

2. Assume f has no points of continuity. Then there exists x_0, such that the function f is not upper semi-continuous at x_0. Indeed, if f is upper semi-continuous (at every point), then it is in Baire class 1. Thus, it has a point of continuity which contradicts our assumption.
We define a function $g : \mathbb{R} \to \mathbb{R}$ $g(x) = f(x)$. We will show that for R small enough

$$B(g, R) \cap \mathcal{C}_u^* = \emptyset. \tag{22}$$

Indeed, there exists $x_n \to x_0, x_n \neq x_0$ such that $f(x_n) \to \alpha > f(x_0)$. Consider $R > 0$ such that $R < \frac{\alpha - f(x_0)}{3}$. If $\eta \in B(g, R)$, then

$$\eta(x_0) < f(x_0) + R < \frac{\alpha + 2f(x_0)}{3} < \frac{2\alpha + f(x_0)}{3}$$

and

$$\limsup_{x\to x_0} \eta(x) \geq \limsup_{n\to\infty}[f(x_n) - R] > \frac{2\alpha + f(x_0)}{3}.$$

Hence $\limsup_{x\to x_0} \eta(x) > \eta(x_0)$. So, the function η is not upper semi-continuous at x_0. Thus $\eta \notin \mathcal{C}_u^*$ completing the proof of(22).

Thus, for R small enough $\gamma(f, R, \mathcal{C}_u^*) = R$. Therefore we deduce that

$$\limsup_{R\to 0^+} \frac{\gamma(f, R, \mathcal{C}_u^*)}{R} = \limsup_{R\to 0^+} \frac{R}{R} = 1,$$

so $\mathcal{C}_u^*$ is porous at f and we have proved this theorem in the case 2. $\square$.

In an analogous way we can establish the following result.

Theorem 3. *The set $\mathcal{C}_l^*$ is porous at each point of the set $\mathcal{F}$.*

The above theorem implies the following assertion.

Theorem 4. *The set $\mathcal{C}$ is porous at each point of the set $\mathcal{F}$.*

Theorem 5. *The set $\mathcal{Q}^*$ is porous at each point of the set $\mathcal{F}$.*

Proof. Let $f \in \mathcal{F}$ and $R > 0$. Consider the following cases.

1. Assume that the function f is quasi-continuous at every point. Let x_0 be an arbitrary point and fix $\delta > 0$. There exists an open interval $(a, b) \neq \emptyset$ such that $[a, b] \subset (x_0 - \delta, x_0 + \delta)$ and $f([a, b]) \subset (f(x_0) - \frac{R}{6}, f(x_0) + \frac{R}{6})$. We will show that there exists $g \in \mathcal{F}$ such that

$$B(g, \frac{R}{24}) \subset B(f, R) \backslash \mathcal{Q}^*. \tag{23}$$

Let $a_1 = a + \frac{b-a}{4}$ and $b_1 = b - \frac{b-a}{4}$. Let $E \subset [a_1, b_1]$ be a bilaterally c-dense in itself F_σ set of null measure such that $a_1, b_1 \notin E$. From [1], Theorem II.2.4 we can deduce that there exists a Darboux Baire 1 function $h : [a_1, b_1] \to [0, \frac{R}{6}]$ such that $\frac{R}{6} \in h((a_1, b_1))$, $h(x) = 0$ for $x \notin E$ and $0 < h(x) \leq \frac{R}{6}$ for $x \in E$. We define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin (a, b), \\ l_1(x) & \text{if } x \in (a, a_1], \\ h(x) + f(x_0) & \text{if } x \in (a_1, b_1), \\ l_2(x) & \text{if } x \in [b_1, b), \end{cases}$$

where l_1 is a linear function such that $l_1(a) = f(a)$ and $l_1(a_1) = f(x_0)$; l_2 is a linear function such that $l_2(b_1) = f(x_0)$ and $l_2(b) = f(b)$. Note that $g \in \mathcal{F}$. Indeed, obviously $g_{|[a,a_1]}$, $g_{|[a_1,b_1]}$, $g_{|[b_1,b]}$ are bounded Darboux Baire 1 functions. So (Theorem II.2 from [4] and Theorem II.3.2 from [1]) they are functionally connected. Thus from Lemma 5.2 from [3] and the fact that $g_{|(-\infty,a]} = f_{|(-\infty,a]}$, $g_{|[b,+\infty)} = f_{|[b,+\infty)}$ are functionally connected, we infer that $g \in \mathcal{F}$. Now, let us notice that $\rho(g, f) \leq \frac{1}{3}R$. Then

$$B(g, \frac{R}{24}) \subset B(f, R). \tag{24}$$

We will show that

$$B(g, \frac{R}{24}) \cap \mathcal{Q}^* = \emptyset. \tag{25}$$

Let $\eta \in B(g, \frac{R}{24})$. Let $y_0 \in (a_1, b_1)$ be such that $g(y_0) = f(x_0) + \frac{R}{6}$. Let $\delta_0 > 0$ be a real number such that $(y_0 - \delta_0, y_0 + \delta_0) \subset (a_1, b_1)$ and let $\varepsilon_0 = \frac{R}{24}$.

Let $G \neq \emptyset$ be an arbitrary open interval such that $G \subset (y_0 - \delta_0, y_0 + \delta_0)$. There exists $z_0 \in G$ be such that $g(z_0) = f(x_0)$ (since $(a_1, b_1) \setminus E$ is dense in (a_1, b_1)). Then

$$\eta(z_0) < g(z_0) + \frac{R}{24} = f(x_0) + \frac{R}{24} = g(y_0) - \frac{3}{24}R < \eta(y_0) - \frac{R}{12},$$

so $\eta(z_0) \notin (\eta(y_0) - \frac{R}{12}, \eta(y_0) + \frac{R}{12})$. Then $\eta(G) \not\subset (\eta(y_0) - \frac{R}{12}, \eta(y_0) + \frac{R}{12})$. Hence we showed that there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that for each nonempty open set $G \subset (y_0 - \delta_0, y_0 + \delta_0)$, $\eta(G) \not\subset (\eta(y_0) - \varepsilon_0, \eta(y_0) + \varepsilon_0)$. Hence the function η is not quasi-continuous at the point y_0, which finishes the proof of (25). From (24) and (25) we may conclude that the condition (23) holds.

According to (23) we may infer that $\gamma(f, R, \mathcal{Q}^*) \geq \frac{R}{24}$. Therefore we deduce that $\limsup_{R \to 0^+} \frac{\gamma(f,R,\mathcal{Q}^*)}{R} \geq \limsup_{R \to 0^+} \frac{\frac{R}{24}}{R} = \frac{1}{24} > 0$, so, $\mathcal{Q}^*$ is porous at f and we have proved the theorem in the case 1.

2. Assume that there exists point $x_0 \in \mathbb{R}$ such that the function f is not quasi-continuous. Then there exists $R_0 > 0$ and $\delta_0 > 0$ such that for each nonempty open set $G \subset (x_0 - \delta_0, x_0 + \delta_0)$ we have

$$f(G) \not\subset (f(x_0) - R_0, f(x_0) + R_0).$$

We will show that

$$B(f, \frac{R_0}{4}) \cap \mathcal{Q}^* = \emptyset. \tag{26}$$

Let $\eta \in K(f, \frac{R_0}{4})$. Let $R_\eta = \frac{R_0}{4}$, $\delta_\eta = \delta_0$. Let $G \subset (x_0 - \delta_\eta, x_0 + \delta_\eta)$ be an arbitrary nonempty open set. Then (by assumption) there exists $y_0 \in G$ such that $f(y_0) \notin (f(x_0) - R_0, f(x_0) + R_0)$. Hence $\eta(y_0) < f(y_0) + \frac{R_0}{4} < f(x_0) - \frac{3}{4}R_0$ or $\eta(y_0) > f(y_0) - \frac{R_0}{4} > f(x_0) + \frac{3}{4}R_0$. Thus $\eta(y_0) \notin (f(x_0) - \frac{3}{4}R_0, f(x_0) + \frac{3}{4}R_0)$. Moreover $f(x_0) - \frac{R_0}{4} < \eta(x_0) < f(x_0) + \frac{R_0}{4}$, so

$$(\eta(x_0) - \frac{R_0}{4}, \eta(x_0) + \frac{R_0}{4}) \subset (f(x_0) - \frac{3}{4}R_0, f(x_0) + \frac{3}{4}R_0).$$

Hence $\eta(y_0) \notin (\eta(x_0) - \frac{R_0}{4}, \eta(x_0) + \frac{R_0}{4})$. Consequently, $\eta(G) \not\subset (\eta(x_0) - \frac{R_0}{4}, \eta(x_0) + \frac{R_0}{4})$, so we have shown that there exists $\delta_\eta > 0$ and there exists $R_\eta > 0$ such that for each nonempty open set $G \subset (x_0 - \delta_\eta, x_0 + \delta_\eta)$, $\eta(G) \not\subset (\eta(x_0) - R_\eta, \eta(x_0) + R_\eta)$. Thus the function η is not quasi-continuous at the point x_0, which finishes the proof of (26).

According to (26) for all $R \leq \frac{R_0}{4}$ $B(f,R) \cap \mathcal{Q}^* = \emptyset$. Thus for R small enough $\gamma(f,R,\mathcal{Q}^*) = R$. Therefore we deduce that

$$\limsup_{R\to 0^+} \frac{\gamma(f,R,\mathcal{Q}^*)}{R} = \limsup_{R\to 0^+} \frac{R}{R} = 1$$

so Q^* is porous at f and we have proved this theorem in the case 2. □

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