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CONVERGENCE THEOREMS

Abstract

In this query we ask if the following 11 classical theorems on convergence are equivalent: the Lebesgue–Beppo Levi Theorem [2, p. 141], a theorem on the integration of a series with positive terms [2, p. 142], the Fatou Lemma I [1, p. 172], the Fatou Lemma II [2, p. 140], the Fatou Lemma III [2, p. 140], Lebesgue's Dominated Convergence Theorem I [1, p. 172], Lebesgue's Dominated Convergence Theorem II [1, p. 173], Lebesgue's Dominated Convergence Theorem III [2, pp. 149-50], Vitali's Theorem [2, p. 152], Lebesgue's Dominated Convergence Theorem for Bounded Functions I [2, p. 127], Lebesgue's Dominated Convergence Theorem for Bounded Functions II.

Are the following 11 classical assertions equivalent?[1] ?

Query. *Are the following theorems equivalent?*

1. (Lebesgue–Beppo Levi) [2, p. 141].
If $\{f_n\}_n$ is a sequence of positive, increasing, Lebesgue measurable functions on a measurable set E converging to a function f a.e. on E, then

$$\lim_{n\to\infty}(\mathcal{L})\int_E f_n(t)dt = (\mathcal{L})\int_E f(t)\,dt\,.$$

2. (The Integration of a Series with Positive Terms) [2, Theorem 11, p. 142]. *If $\{f_n\}_n$ is a sequence of positive, increasing, Lebesgue measurable functions on a measurable set E and $\sum_{n=1}^{\infty} f_n(t) = f(t)$ for $t \in E$, then*

$$(\mathcal{L})\int_E f(t)dt = \sum_{n=1}^{\infty}(\mathcal{L})\int_E f_n(t)dt\,.$$

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[1]The author was convinced that the answer is yes, but he never finished the proof.

3. (The Fatou Lemma I) [1, p. 172]. *If $\{f_n\}_n$ is a sequence of positive, increasing, Lebesgue measurable functions on a measurable set E, then*

$$(\mathcal{L})\int_E \liminf_{n\to\infty} f_n(t)dt = \liminf_{n\to\infty}(\mathcal{L})\int_E f_n(t)dt\,.$$

4. (The Fatou Lemma II) [2, p. 140]. *If $\{f_n\}_n$ is a sequence of positive, Lebesgue measurable functions on a measurable set E, converging a.e. on E to a function f, then*

$$(\mathcal{L})\int_E f(t)dt \;\le\; \sup_n\left\{(\mathcal{L})\int_E f_n(t)dt\right\}\,.$$

5. (The Fatou Lemma III) [2, footnote, p. 140]. *If $\{f_n\}_n$ is a sequence of positive, Lebesgue measurable functions on a measurable set E, finite a.e. on E and $\{f_n\}_n$ converges in measure to a finite function f, then*

$$(\mathcal{L})\int_E f(t)dt \;\le\; \sup_n\left\{(\mathcal{L})\int_E f_n(t)dt\right\}\,.$$

6. (Lebesgue's Dominated Convergence Theorem I) [1, p. 172].
If $\{f_n\}_n$ is a sequence of Lebesgue measurable functions on a measurable set E, and g is a Lebesgue integrable function such that $f_n(t) \le g(t)$ a.e. on E, then

$$(\mathcal{L})\int_E \liminf_{n\to\infty} f_n(t)dt \;\le\; \liminf_{n\to\infty}(\mathcal{L})\int_E f_n(t)dt \;\le$$

$$\limsup_{n\to\infty}(\mathcal{L})\int_E f_n(t)dt \;\le\; (\mathcal{L})\int_E \limsup_{n\to\infty} f_n(t)dt\,.$$

7. (Lebesgue's Dominated Convergence Theorem II) [1, p. 173].
If $\{f_n\}_n$ is a convergent sequence of Lebesgue measurable functions on a measurable set E, and g is a Lebesgue integrable function such that $f_n(t) \le g(t)$ a.e. on E, then

$$\lim_{n\to\infty}(\mathcal{L})\int_E f_n(t)dt \;=\; (\mathcal{L})\int_E \lim_{n\to\infty} f_n(t)dt\,.$$

8. (Lebesgue's Dominated Convergence Theorem III) [2, pp. 149-50].
If $\{f_n\}_n$ is a sequence of Lebesgue measurable functions on a measurable

set E converging in measure to a function f, and g is a Lebesgue integrable function such that $f_n(t) \leq g(t)$ a.e. on E, then

$$\lim_{n\to\infty} (\mathcal{L}) \int_E f_n(t)dt \;=\; (\mathcal{L}) \int_E f(t)dt\,.$$

9. (Vitali) [2, p. 152]. *Let $\{f_n\}_n$ be a sequence of Lebesgue integrable functions on a measurable set E, converging in measure to a function f. If the functions of the sequence $\{f_n\}_n$ have equi–absolutely continuous integrals then f is Lebesgue integrable and*

$$\lim_{n\to\infty} (\mathcal{L}) \int_E f_n(t)dt \;=\; (\mathcal{L}) \int_E f(t)dt\,.$$

10. (Lebesgue's Dominated Convergence Theorem for Bounded Functions I) [2, p. 127]. *Let $\{f_n\}_n$ be a sequence of bounded Lebesgue measurable functions on a measurable set E, converging in measure to a bounded Lebesgue measurable function f. If there exists a positive constant K such that $|f_n(t)| < K$ for all n, then*

$$\lim_{n\to\infty} (\mathcal{L}) \int_E f_n(t)dt \;=\; (\mathcal{L}) \int_E f(t)dt\,.$$

11. (Lebesgue's Dominated Convergence Theorem for Bounded Functions II). *Let $\{f_n\}_n$ be a sequence of bounded Lebesgue measurable functions on a measurable set E, converging to a bounded Lebesgue measurable function f. If there exists a positive constant K such that $|f_n(t)| < K$ a.e. on E for all n, then*

$$\lim_{n\to\infty} (\mathcal{L}) \int_E f_n(t)dt \;=\; (\mathcal{L}) \int_E f(t)dt\,.$$

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