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CONTINUITY IN TERMS OF FUNCTIONAL CONVERGENCE

Abstract

The note presents a new approach to the notion of continuity of real function at a point. It is applied to obtain a characterization of continuity at a point with respect to *-topology (Hashimoto topology), density topology and I-density topology (Wilczyński topology). The latter is closely related to the definition of density point of measurable set formulated by W. Wilczyński in [8].

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. For any sequence $\{t_n\}_{n\in N}$of real numbers decreasing to zero, with $(t_1 < 1)$, we define a sequence of functions $\{f_{t_n}\}_{n\in N}$, $f_{t_n} : [-1, 1] \to \mathbb{R}$ in the following way

$$f_{t_n}(x) = f(t_n \cdot x)$$

The theorem given below presents a new point of view on the notion of continuity. It describes a surprising connection between a continuity of a function at the point and a convergence of an appropriate sequence of functions. We omit the proof since it follows immediately from Cauchy's and Heine's definitions of continuity of a function at a point.

Theorem 1. *The following conditions are equivalent:*

(1) f *is continuous (with respect to the natural topology on the domain and on the range) at* 0,

(2) *for every sequence* $\{t_n\}_{n\in\mathbb{N}}$ *decreasing to zero the sequence of functions* $\{f_{t_n}\}_{n\in\mathbb{N}}$ *converges to* $f(0)$ *uniformly on* $[-1, 1]$,

(3) *for every sequence* $\{t_n\}_{n\in\mathbb{N}}$ *decreasing to zero the sequence of functions* $\{f_{t_n}\}_{n\in\mathbb{N}}$ *converges to* $f(0)$ *on* $[-1, 1]$,

(4) *there exists a decreasing to zero sequence* $\{t_n\}_{n\in\mathbb{N}}$ *such that the sequence of functions* $\{f_{t_n}\}_{n\in\mathbb{N}}$ *converges to* $f(0)$ *uniformly on* $[-1, 1]$,

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(5) *there are two points* $x_1 \in (-1,0)$ *and* $x_2 \in (0,-1)$ *such that for every decreasing to zero sequence* $\{t_n\}_{n\in\mathbb{N}}$ *the sequence of functions* $\{f_{t_n}\}_{n\in\mathbb{N}}$ *converges to* $f(0)$ *at* x_1 *and* x_2.

Now we shall give several applications of this new approach to continuity with respect to the Hashimoto topology, density topology and I-density topology.

Let $\mathcal{S}$ be an algebra and $\mathcal{I}$ a proper ideal of subsets of the real line $\mathbb{R}$. We assume $\mathcal{S}$ and $\mathcal{I}$ to be invariant with respect to linear transformations. If $A \in \mathcal{S}$, we say that A is $\mathcal{S}$-measurable. Similarly, if for a real function f, $f^{-1}(U)$ is $\mathcal{S}-$measurable for every open U, we say that f is $\mathcal{S}$-measurable. If some property holds for points from $A \setminus P$ for some $P \in \mathcal{I}$, we say that it holds $\mathcal{I}$-almost everywhere ($\mathcal{I}$-a.e.) on A. If $B = A \setminus P$ for some $P \in \mathcal{I}$, we say that B is *residual* subset of A or B is residual in A. We use A^c for the complement of A and $A - x$ for $\{y - x : y \in A\}$.

For a given sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers decreasing to zero we shall now consider the convergence of a sequence of functions $\{f_{t_n}\}_{n\in\mathbb{N}}$ on a given residual subset of $[-1,1]$.

Definition 1. We say that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ *converges a.e. uniformly* on a set $A \subset \mathbb{R}$, if there is a residual subset B of A such that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on B.

Theorem 2. *Let* $\mathcal{I}$ *be a* σ*-ideal. For every real function* f*, if there exists a decreasing to zero sequence* $\{t_n\}_{n\in\mathbb{N}}$ *such that the sequence of functions* $\{f_{t_n}\}_{n\in\mathbb{N}}$ *converges to* $f(0)$ *a.e. uniformly on* $[-1,1]$*, then there exists a residual subset* E *of* $\mathbb{R}$ *such that* f *restricted to* $E \cup \{0\}$ *is continuous at* 0.

PROOF. Let $A \in \mathcal{I}$ be the set such that $\{f_{t_n}\}_{n\in\mathbb{N}}$ converges to $f(0)$ uniformly on $[-1,1] - A$. Put $E = \mathbb{R} \setminus \bigcup_n (t_n \cdot A)$ and define a function g such that

$$g(x) = \begin{cases} f(x) & x \in E \\ f(0) & x \in \mathbb{R} \setminus E. \end{cases}$$

Now the sequence $\{g_{t_n}\}_{n\in\mathbb{N}}$ converges uniformly to $f(0)$ on $[-1,1]$. By Theorem 1 (4) the function g is continuous at 0. Hence g restricted to $E \cup \{0\}$ is continuous at 0. As g restricted to $E \cup \{0\}$ equals f restricted to $E \cup \{0\}$, the proof is complete. □

Theorem 3. *Let* $\mathcal{I}$ *be a* $\sigma-$*ideal. For every real function* f*, if there exists a residual subset* E *of* $\mathbb{R}$ *such that* f *restricted to* $E \cup \{0\}$ *is continuous at* 0*, then for every decreasing to zero sequence* $\{t_n\}_{n\in\mathbb{N}}$ *the sequence of functions* $\{f_{t_n}\}_{n\in\mathbb{N}}$ *converges to* $f(0)$ *a.e. uniformly on* $[-1,1]$.

PROOF. Let $A = \mathbb{R} \setminus \{E \cup \{0\}\}$ and $B = \bigcup_n \left(\left(\frac{1}{t_n} \cdot A\right) \cap [-1,1]\right)$. As E is residual, we have $B \in \mathcal{I}$. Define function g such that

$$g(x) = \begin{cases} f(x) & x \in E \cup \{0\} \\ f(0) & x \notin E \cup \{0\}. \end{cases}$$

Clearly, g is continuous at 0, hence the sequence $\{g_{t_n}\}_{n\in\mathbb{N}}$ converges uniformly to $f(0)$ on $[-1,1]$. The sequence $\{f_{t_n}\}_{n\in\mathbb{N}}$ converges to $f(0)$ a.e. uniformly on $[-1,1]$, because f_n restricted to $\mathbb{R}\setminus B$ equals g_n restricted to $\mathbb{R}\setminus B$ for every $n \in \mathbb{N}$, and $B \in \mathcal{I}$. □

Corollary 4. *Let $\mathcal{I}$ be a $\sigma-$ideal. The following conditions are equivalent:*

(1) *there exists a residual subset E of $\mathbb{R}$ such that f restricted to $E \cup \{0\}$ is continuous at* 0*,*

(2) *there exists a decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ such that the sequence of functions $\{f_{t_n}\}_{n\in\mathbb{N}}$ converges to $f(0)$ a.e. uniformly on $[-1,1]$,*

(3) *for every decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ the sequence of functions $\{f_{t_n}\}_{n\in\mathbb{N}}$ converges to $f(0)$ a.e. uniformly on $[-1,1]$.*

Remark 1. The above Corollary gives us a characterization of continuity at a point of a real function with respect to *topology (Hashimoto topology) generated by the basis $\{U \setminus P : U \in \tau, P \in \mathcal{I}\}$, where τ is a natural topology on $\mathbb{R}$ and $\mathcal{I}$ is a σ -ideal (see [3], [5]).

In 1982, W. Wilczyński in his paper [8] formulated the following definition:

Definition 2. We say that the point 0 is an $\mathcal{I}$*-density point* of an $\mathcal{S}$-measurable set A if for any decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that $\chi_{\left\{\frac{1}{t_{n_m}}\cdot A\right\}\cap[-1,1]}$ converges to 1, $\mathcal{I}$-almost everywhere on $[-1,1]$. (Equivalently such that $\liminf_{k\to\infty}\left(\left(\frac{1}{t_{n_m}} \cdot A\right) \cap [-1,1]\right) = [-1,1] \setminus P$ for some $P \in \mathcal{I}$)

We say that the point x is an $\mathcal{I}$-density point of an $\mathcal{S}$-measurable set A if 0 is an $\mathcal{I}$-density point of $A - x$.

If $\mathcal{S}$ is the σ-algebra of a Lebesgue measurable sets and $\mathcal{I}$ is the σ-ideal of sets of measure zero, this defines the usual density point of Lebesgue measurable set. If $\mathcal{S}$ is the σ-algebra of sets having Baire property and $\mathcal{I}$ is the σ-ideal of sets of the first category, this is a definition of an $\mathcal{I}$-density point of a Baire measurable set. The definition and all its consequences were deeply examined in a large number of papers - see [6], [7], and [1]. We adopt here their basic definitions. For $A \in \mathcal{S}$, let $\Phi(A)$ denote the set of all density points of A. The family $\{A \in \mathcal{S},\ A \ \subset \Phi(A)\}$ is a topology on $\mathbb{R}$ called the $\mathcal{I}$-density topology (Wilczyński topology) and is denoted $\tau_{\mathcal{I}}$. We say that $\mathcal{S}$-measurable function f is *topologically $\mathcal{I}$-approximate* continuous at x, if and only if $f^{-1}(f(x) - a, f(x) + a)$ has x as its density point, for every positive a. The continuity of real functions with respect to $\tau_{\mathcal{I}}$-topology was examined mainly for $\mathcal{S}$ the σ-algebra of a Lebesgue measurable sets and $\mathcal{I}$ the σ-ideal of sets of measure zero or $\mathcal{S}$ the σ-algebra $\mathcal{S}$ of Baire measurable sets with $\mathcal{I}$ the σ-ideal of sets of first category.

Remark 2. We say that an $\mathcal{S}$-measurable function f is *restrictively $\mathcal{I}$-approximate* continuous at point x, if and only if there is a $\tau_{\mathcal{I}}$-open neighborhood U of x such that f restricted to U is continuous at x in the natural topology relativised to U. The restrictional continuity of $\mathcal{S}$-measurable real function implies its topological continuity. In the case of $\mathcal{S}$ the σ-algebra of a Lebesgue measurable sets and $\mathcal{I}$ the σ-ideal of sets of measure zero, $\mathcal{I}$ -approximate restrictional continuity and topological $\mathcal{I}$ -approximate continuity coincide (See [4].)

Remark 3. Observe that the definition of an $\mathcal{I}$-density point can be reformulated in the following form: We say that the point 0 is an $\mathcal{I}$-density point of an $\mathcal{S}$-measurable set A if for any sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers decreasing to zero there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that $\chi_A(t_{n_m} \cdot x)$ converges to 1, $\mathcal{I}$-almost everywhere on $[-1, 1]$.

This leads in the natural way to the following definition:

Definition 3. We say that an $\mathcal{S}$-measurable function f *has property* $(*)$ at a point 0, if and only if for any decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that $f(t_{n_m} \cdot x)$ converges to $f(0)$, $\mathcal{I}$ -almost everywhere on $[-1, 1]$.

We say that an $\mathcal{S}$-measurable function f has property $(*)$ at point x, if the function g, $g(y) = f(y - x)$, has property $(*)$ at 0.

We shall show that for an $\mathcal{S}$-measurable function, the property $(*)$ at a point x is equivalent to its $\mathcal{I}$ -approximate topological continuity.

Theorem 5. *If the $\mathcal{S}$-measurable function f has property $(*)$ at a point x, then it is $\mathcal{I}$-approximate topologically continuous at this point.*

PROOF. We shall restrict ourselves to the case $x = 0$ and additionally put $f(0) = 0$. Suppose that f is not $\mathcal{I}$-approximate topologically continuous at 0. Then there exists $a > 0$ such that the set $A = \{x : -a < f(x) < a\}$ has not 0 as its density point. It means that there exists a decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers such that for each subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ the sequence $\chi_{\left\{\frac{1}{t_{n_m}}\cdot A\right\}\cap[-1,1]}$ does not converge to 1, $\mathcal{I}$-almost everywhere on $[-1,1]$. This is equivalent to $\limsup\limits_{k\to\infty}\left(\left(\frac{1}{t_{n_m}}\cdot A^c\right)\cap[-1,1]\right)\notin\mathcal{I}$.

For $x\in A^c$, we have $|f(x)|\geqslant a$, hence for $x\in\limsup\limits_{k\to\infty}\left(\frac{1}{t_{n_m}}\cdot A^c\right)$ the sequence $f(t_{n_m}\cdot x)$ does not converge to $f(0)=0$. It follows that there exists a decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers such that for each subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$, the sequence $f(t_{n_m}\cdot x)$ does not converge to $f(0)=0$, $\mathcal{I}$-a.e. on $[-1,1]$; this means that f does not have property $(*)$ at the point 0. □

Theorem 6. *Let $\mathcal{I}$ be a σ-ideal. If the $\mathcal{S}$-measurable function f is $\mathcal{I}$-approximate topologically continuous at point x then it has property $(*)$ at this point.*

Proof. We restrict ourselves again to the case $x=0$ and additionally we put $f(0)=0$. As f is $\mathcal{I}$-approximate topologically continuous at 0 then $A_\varepsilon = f^{-1}((-\varepsilon,\varepsilon))$ has 0 as its density point, for every $\varepsilon>0$. The latter can be stated equivalently that 0 is a density point of $A_k=f^{-1}\left(\left(-\frac{1}{k},\frac{1}{k}\right)\right)$, for every $k\in N$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers decreasing to zero and put $k=1$. We may choose a subsequence $\left\{t_n^{(1)}\right\}_{n\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ such that $\chi_{\left\{\frac{1}{t_n^{(1)}}\cdot A_1\right\}\cap[-1,1]}$ converges to 1, $\mathcal{I}$-almost everywhere on $[-1,1]$. This means that there exists a set E_1 such that $[-1,1]-E_1\in\mathcal{I}$ and for points from E_1, $\chi_{\left\{\frac{1}{t_n^{(1)}}\cdot A_1\right\}\cap[-1,1]}$ converges to 1. Since $E_1=\liminf\limits_{m\to\infty}\left(\left\{\frac{1}{t_n^{(1)}}\cdot A_1\right\}\cap[-1,1]\right)$ we have $\limsup\limits_{m\to\infty}\left|f\left(t_n^{(1)}\cdot x\right)\right|<1$ for $x\in E_1$. Similarly we can find, for every natural $k>1$, a set $E_k\subset E_{k-1}$ and a subsequence $\left\{t_n^{(k)}\right\}_{n\in\mathbb{N}}$ of $\left\{t_n^{(k-1)}\right\}_{n\in\mathbb{N}}$ such that $[-1,1]\setminus E_k\in\mathcal{I}$ and $\limsup\limits_{m\to\infty}\left|f\left(t_n^{(k)}\cdot x\right)\right|<\frac{1}{k}$ for $x\in E_k$. Let $\left\{t_{n_p}\right\}_{p\in\mathbb{N}}$ be a diagonal subsequence of subsequences $\left\{t_n^{(k)}\right\}_{n\in\mathbb{N}}$, $k\in N$. We have $\limsup\limits_{p\to\infty}\left|f\left(t_{n_p}\cdot x\right)\right|=0$ for x from $\bigcap\limits_{k=1}^{\infty}E_k$. Since $\mathcal{I}$ is a σ-ideal and $[-1,1]\setminus E_k\in\mathcal{I}$ for every $k\in N$ then $[-1,1]\setminus\bigcap\limits_{k=1}^{\infty}E_k\in\mathcal{I}$. So, for any sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $\left\{t_{n_p}\right\}_{p\in\mathbb{N}}$ such that $f\left(t_{n_p}\cdot x\right)$ converges to $f(0)$, $\mathcal{I}$-almost everywhere on $[-1,1]$, i.e., function f has property $(*)$ at point 0. □

If $\mathcal{I}$ is a σ-ideal, Theorems 5 and 6 state that for the $\mathcal{S}$-measurable function f its $\mathcal{I}$-approximate topological continuity and property $(*)$ are equivalent at point x.

Theorem 7. *If the $\mathcal{S}$-measurable function f is $\mathcal{I}$-approximate restrictively continuous at point x then it has property $(*)$ at this point.*

PROOF. Since the restrictional continuity implies the topological continuity, the theorem follows immediately. However, we would like to present the proof with the use of the property $(*)$. We shall restrict our considerations to the case $x = 0$. From the assumption it follows that there exists an $\tau_{\mathcal{I}}$-open neighborhood U of 0 such that f restricted to U is continuous at 0. As 0 is an $\mathcal{I}$-density point of U, for any sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers decreasing to zero there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that $\chi_{\left\{\frac{1}{t_{n_m}}\cdot U\right\}\cap[-1,1]}$ converges to 1, $\mathcal{I}$ -almost everywhere on $[-1,1]$. This means that there exists a set E such that $[-1,1]- E\in\mathcal{I}$ and for points from E, $\chi_{\left\{\frac{1}{t_{n_m}}\cdot U\right\}\cap[-1,1]}$ converges to 1. Put

$$g(x)=\begin{cases} f(x) & x\in U\\ f(0) & x\notin U\end{cases}.$$

By the assumption g is continuous at 0 and by Theorem 1 (3), $g(t_{n_m}\cdot x)$ converges to $f(0)$ everywhere on $[-1,1]$. Therefore, $f(t_{n_m}\cdot x)$ converges to $f(0)$ for points from E, i.e. $\mathcal{I}$ -almost everywhere on $[-1,1]$. □

Now we shall give a definition closely related to Lemma 1 in [7].

Definition 4. We shall say that the pair $(\mathcal{S},\mathcal{I})$ is of type I (is of type II), if for every increasing sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{s_n\}_{n\in\mathbb{N}}$ of real numbers tending to infinity and such that

$$\lim_{n\to\infty}\frac{t_n}{s_n}=1$$

and for every $\mathcal{S}$-measurable set A such that $\liminf\limits_{n\to\infty}(t_n\cdot A)\cap[-1,1]$ is residual in $[-1,1]$ then also $\liminf\limits_{n\to\infty}(s_n\cdot A)\cap[-1,1]$ is residual in $[-1,1]$ (there exists a subsequence $\{n_m\}_{m\in\mathbb{N}}$ such that $\liminf\limits_{m\to\infty}(s_{n_m}\cdot A)\cap[-1,1]$ is residual in $[-1,1]$).

In Lemma 1 in [7] it is proved that the pair $(\mathcal{S},\mathcal{I})$ where $\mathcal{S}$ is the σ-algebra of sets having Baire property and $\mathcal{I}$ the σ-ideal of sets of the first category is of type I. It is not difficult to show that in measure case the pair $(\mathcal{S},\mathcal{I})$, where $\mathcal{S}$ is the σ-algebra of Lebesgue measurable sets and $\mathcal{I}$ the σ-ideal of sets of measure zero, is of type II.

Lemma 8. *Let $\mathcal{I}$ be a $\sigma-$ideal. Suppose a pair $(\mathcal{S},\mathcal{I})$ is of type I (is of type II) and $\{t_n\}_{n\in\mathbb{N}}$,$\{s_n\}_{n\in\mathbb{N}}$ are sequences of real numbers decreasing to zero such that*

$$\lim_{n\to\infty}\frac{t_n}{s_n}=1.$$

If for a real $\mathcal{S}$-measurable function f, $f(t_n\cdot x)$ converges to $f(0)$ $\mathcal{I}$-almost everywhere on $[-1,1]$ then $f(s_n\cdot x)$ converges to $f(0)$ $\mathcal{I}$-almost everywhere on $[-1,1]$. (There is a subsequence $\{s_{n_m}\}_{m\in\mathbb{N}}$ of $\{s_n\}_{n\in\mathbb{N}}$ such that $f(s_{n_m}\cdot x)$ converges to $f(0)$, $\mathcal{I}$-a.e. on $[-1,1]$.)

Proof. Put $A_k=\left\{x\in[-1,1]:|f(x)|<\frac{1}{k}\right\}$. For every decreasing to zero sequence $\{u_n\}_{n\in\mathbb{N}}$ of real numbers we have

$$\begin{aligned}
\bigcup_p\bigcap_{n\geqslant p}\left(\frac{1}{u_n}\cdot A_{k+1}\right)\cap[-1,1] &= \bigcup_p\bigcap_{n\geqslant p}\left\{x\in[-1,1]:|f(u_n\cdot x)|<\frac{1}{k+1}\right\}\\
&=\left\{x\in[-1,1]:\exists_p\forall_{n\geqslant p}\,|f(u_n\cdot x)|<\frac{1}{k+1}\right\}\\
&\subset\left\{x\in[-1,1]:\limsup_n|f(u_n\cdot x)|\leqslant\frac{1}{k+1}\right\}\\
&\subset\left\{x\in[-1,1]:\limsup_n|f(u_n\cdot x)|<\frac{1}{k}\right\}\\
&\subset\left\{x\in[-1,1]:\exists_p\forall_{n\geqslant p}\,|f(u_n\cdot x)|<\frac{1}{k}\right\}\\
&=\bigcup_p\bigcap_{n\geqslant p}\left\{x\in[-1,1]:|f(u_n\cdot x)|<\frac{1}{k}\right\}\\
&=\bigcup_p\bigcap_{n\geqslant p}\left(\frac{1}{u_n}\cdot A_k\right)\cap[-1,1].
\end{aligned}$$

Put $B_k=\bigcup_p\bigcap_{n\geqslant p}\left(\frac{1}{u_n}\cdot A_k\right)\cap[-1,1]$. Now we have

$$\begin{aligned}
B_{k+1}=\bigcup_p\bigcap_{n\geqslant p}\left(\frac{1}{u_n}\cdot A_{k+1}\right)\cap[-1,1]&\subset\left\{x\in[-1,1]:\limsup_n|f(u_n\cdot x)|<\frac{1}{k}\right\}\\
&\subset\bigcup_p\bigcap_{n\geqslant p}\left(\frac{1}{u_n}\cdot A_k\right)\cap[-1,1]=B_k,\ \text{and}
\end{aligned}$$

$$\bigcap_k \left\{x \in [-1,1] : \limsup_n |f(u_n \cdot x)| < \frac{1}{k}\right\} = \bigcap_k \bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{u_n} \cdot A_k\right) \cap [-1,1]$$
$$= \bigcap_k B_k.$$

Hence,

$$\left\{x \in [-1,1] : \lim_n f(u_n \cdot x) = 0\right\} = \left\{x \in [-1,1] : \limsup_n |f(u_n \cdot x)| = 0\right\}$$
$$= \bigcap_k \left\{x \in [-1,1] : \limsup_n |f(u_n \cdot x)| < \frac{1}{k}\right\} = \bigcap_k \bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{u_n} \cdot A_k\right) \cap [-1,1].$$

By the assumption $f(t_n \cdot x)$ converges to $f(0)$, $\mathcal{I}$-almost everywhere on $[-1,1]$, therefore $\left\{x \in [-1,1] : \lim_n f(t_n \cdot x) = 0\right\}$ is residual on $[-1,1]$ and consequently $\bigcap_k \bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{t_n} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$. As $\mathcal{I}$ is a σ-ideal, the countable intersection of sets is residual, if and only if every set is residual and the latter implies that $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{t_n} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$, for every $k \in N$.

Now, if a pair $(\mathcal{S}, \mathcal{I})$ is of type I then also $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{s_n} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$, for every $k \in N$ and $\bigcap_k \bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{s_n} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$. By the last statement $\left\{x \in [-1,1] : \lim_n f(s_n \cdot x) = 0\right\}$ is residual on $[-1,1]$ and $f(s_n \cdot x)$ converges to $f(0)$, $\mathcal{I}$-almost everywhere on $[-1,1]$.

Suppose now that a pair $(\mathcal{S}, \mathcal{I})$ is of type II. Then for $k = 1$ the set $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{t_n} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$ and we may choose a subsequence $\left\{s_n^{(1)}\right\}_{n \in \mathbb{N}}$ of $\{s_n\}_{n \in \mathbb{N}}$ such that $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{s_n^{(1)}} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$. For $k = 2$, the set $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{t_n^{(1)}} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$ and we may choose a subsequence $\left\{s_n^{(2)}\right\}_{n \in \mathbb{N}}$ of $\left\{s_n^{(1)}\right\}_{n \in \mathbb{N}}$ such that $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{s_n^{(2)}} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$. Similarly, we can find for every natural $k > 1$ a subsequence $\left\{s_n^{(k)}\right\}_{n \in \mathbb{N}}$ of $\left\{s_n^{(k-1)}\right\}_{n \in \mathbb{N}}$ such that $\bigcup_p \bigcap_{n \geqslant p} \left(\frac{1}{s_n^{(k)}} \cdot A_k\right) \cap [-1,1]$ is residual on $[-1,1]$. Let $\{s_{n_m}\}_{m \in \mathbb{N}}$ be a diagonal subsequence of subsequences $\left\{s_n^{(k)}\right\}_{n \in \mathbb{N}}$, $k \in \mathbb{N}$. The set $\bigcup_p \bigcap_{m \geqslant p} \left(\frac{1}{s_{n_m}} \cdot A_k\right) \cap$

$[-1,1]$ is residual on $[-1,1]$ for every $k \in \mathbb{N}$ so the set $\bigcap_k \bigcup_p \bigcap_{m \geqslant p} \left(\frac{1}{s_{n_m}} \cdot A_k \right) \cap [-1,1]$ is also residual on $[-1,1]$ and $f(s_{n_m} \cdot x)$ converges to $f(0)$, $\mathcal{I}$ -a.e. on $[-1,1]$. □

Theorem 9. *Let $\mathcal{I}$ be a $\sigma-$ideal. Suppose a pair $(\mathcal{S},\mathcal{I})$ is of type I or II. If for the real $\mathcal{S}$-measurable function f, $f(s_n \cdot x)$ converges to $f(0)$ $\mathcal{I}$-almost everywhere on $[-1,1]$ for some decreasing to zero sequence $\{s_n\}_{n\in\mathbb{N}}$ of real numbers such that $s_{k+1} > r \cdot s_k$, for some $r > 0$ then f has the property $(*)$ at 0.*

PROOF. Let $\{t_n\}_{n\in\mathbb{N}}$ be any sequence of real numbers decreasing to zero. Choose subsequences $\{s_{k_p}\}_{p\in\mathbb{N}}$ and $\{t_{n_p}\}_{p\in\mathbb{N}}$ such that $t_{n_p} \in \left(s_{k_p+1}, s_{k_p}\right]$. We have $0 < r \cdot s_{k_p} < s_{k_p+1} < t_{n_p} \leqslant s_{k_p}$ and $0 < r < \frac{t_{n_p}}{s_{k_p}} \leqslant 1$. We choose again subsequences $\{s_{k_{p_m}}\}_{m\in\mathbb{N}}$ and $\{t_{n_{p_m}}\}_{m\in\mathbb{N}}$ such that the sequence $\left\{\frac{t_{n_{p_m}}}{s_{k_{p_m}}}\right\}_{m\in\mathbb{N}}$ converges to some a, where $0 < r \leqslant a \leqslant 1$. The sequence $\left\{\frac{t_{n_{p_m}}}{a \cdot s_{k_{p_m}}}\right\}_{m\in\mathbb{N}}$ converges to 1 and clearly $f\left(a \cdot s_{k_{p_m}} \cdot x\right)$ converges to $f(0)$, $\mathcal{I}$-almost everywhere on $[-1,1]$. Directly from Lemma 8 there exists a subsequence $\left\{t_{n_{p_{m_v}}}\right\}_{v\in\mathbb{N}}$ of $\{t_{n_{p_m}}\}_{m\in\mathbb{N}}$ such that $f\left(t_{n_{p_{m_v}}} \cdot x\right)$ converges to $f(0)$, $\mathcal{I}$-almost everywhere on $[-1,1]$. As $\left\{t_{n_{p_{m_v}}}\right\}_{v\in\mathbb{N}}$ is a subsequence of $\{t_n\}_{n\in\mathbb{N}}$, f has the property $(*)$ at 0. □

Theorem 10. *Let $\mathcal{S}$ be any algebra containing the Borel sets and $\mathcal{I}$ be an ideal. If $\{s_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers decreasing to zero such that $\liminf_n \frac{s_{n+1}}{s_n} = 0$, then there exists an $\mathcal{S}$-measurable function f such that $f(s_n \cdot x)$ converges to $f(0)$, $\mathcal{I}$ -almost everywhere on $[-1,1]$ and f has not the property $(*)$ at 0.*

PROOF. The example from the necessary part of the Theorem 1 in [9] is good here. □

As the consequence of Theorems 9 and 10, we have the following generalization of Theorem 1 in [9] and Theorem 1 in [2].

Corollary 11. *Let $\mathcal{S}$ be any algebra containing the Borel sets and $\mathcal{I}$ a σ-ideal. Suppose a pair $(\mathcal{S},\mathcal{I})$ is of type I or II. Then for every decreasing to zero sequence $\{s_n\}_{n\in\mathbb{N}}$ the condition that $s_{k+1} > r \cdot s_k$, for some $r > 0$, is for every real function f equivalent to the fact that $\mathcal{I}$-almost everywhere on $[-1,1]$ convergence of $f(s_n \cdot x)$ to $f(0)$ implies that f has the property $(*)$ at 0.*

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