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ON A.C. LIMITS AND MONOTONE LIMITS OF SEQUENCES OF JUMP FUNCTIONS

Abstract

The a.c. limits (introduced by Császár and Laczkovich) and the monotone limits of sequences of functions having everywhere finite unilateral limits are investigated.

Let $\mathbb{R}$ be the set of all reals and let $\mathcal{A}$ be a family of functions from $\mathbb{R}$ to $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ belongs to the class $B_1^*(\mathcal{A})$ if there is a sequence of functions $f_n \in \mathcal{A}$ with $f = a.c. \lim_{n\to\infty} f_n$, **i.e.** for each point $x \in \mathbb{R}$ there is a positive integer k such that $f_n(x) = f(x)$ for every $n > k$.

It is evident that $f \in B_1^*(\mathcal{A})$ if and only if there is a sequence of sets A_n, $n = 1, 2, \ldots$, such that for each positive integer n there is a function $g_n \in \mathcal{A}$ such that

$$g_n/A_n = f/A_n, \quad A_1 \subset A_2 \subset \ldots \subset A_n \ldots$$

and

$$\mathbb{R} = \bigcup_{n=1}^{\infty} A_n.$$

In [2, 3] it is proved that in the case of the class $\mathcal{C}$ of all continuous functions the sets A_n, $n = 1, 2, \ldots$, can be closed and that $f \in B_1^*(\mathcal{C})$ if and only if for every nonempty closed set $A \subset \mathbb{R}$ there is an open interval I such that $I \cap A \neq \emptyset$ and the reduced function $f/(A \cap I)$ is continuous.

In this article we will investigate the family $B_1(\mathcal{A})$, where $\mathcal{A}$ is the class $\mathcal{P}$ of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for each point $x \in \mathbb{R}$ there are the both finite unilateral limits $\lim_{t\to x-} f(t)$ and $\lim_{t\to x+} f(t)$. The pointwise limits of sequences of such functions from $\mathcal{P}$ were investigated in [4] . Moreover it is well known ([5], p. 45) that if $f \in \mathcal{P}$ then the set $D(f)$ of all discontinuity points of f is countable.

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1 The A.C. Convergence

Theorem 1. *If $f \in B_1^*(\mathcal{P})$ then f satisfies the following condition*

(T) there is a countable set $A \subset \mathbb{R}$ such that the reduced function $f/(\mathbb{R} \setminus A) \in B_1^*(\mathcal{C}(\mathbb{R} \setminus A))$, where $\mathcal{C}(\mathbb{R} \setminus A)$ denotes the class of all continuous functions $g : \mathbb{R} \setminus A \to \mathbb{R}$.

PROOF. Since $f \in B_1^*(\mathcal{P})$, there is a sequence of functions $f_n \in \mathcal{P}$, $n = 1, 2 \ldots$, such that $f = a.c. \lim_{n\to\infty} f_n$. Put

$$A = \bigcup_{n=1}^{\infty} D(f_n)$$

and observe that the set A is countable. Since the functions $g_n = f_n/(\mathbb{R} \setminus A)$ are continuous for $n = 1, 2, \ldots$ and $f/(\mathbb{R} \setminus A) = a.c. \lim_{n\to\infty} g_n$, the proof is completed. □

The following Example shows that the condition (T) is not sufficient for the relation $f \in B_1^*(\mathcal{P})$.

Example 1. Let

$$g(x) = \sin \frac{1}{x} \text{ for } x \neq 0, \quad g(0) = 1$$

and

$$f(x) = \sum_{n=1}^{\infty} \frac{g(x - w_n)}{2^n},$$

where $(w_n)_n$ is an enumeration of all rationals such that $w_n \neq w_m$ for $n \neq m$, $n, m = 1, 2, \ldots$.

As the sum of an uniformly converging series of functions which are continuous at each irrational point, the function f is continuous at every irrational point. Denote by A the set of all rationals and observe that the reduced function $f/(\mathbb{R} \setminus A)$ is continuous. So the function f satisfies the condition (T).

Now we will prove that f is not in the class $B_1^*(\mathcal{P})$. Assume, by way of contradiction, that there is a sequence of functions $f_n \in \mathcal{P}$, $n = 1, 2, \ldots$, such that $a.c. \lim_{n\to\infty} f_n = f$. Then there is a positive integer k and a set, B, of the second category such that $f_m(x) = f(x)$ for each point $x \in B$ and $m \geq k$. Consequently, the set

$$E = \{x; f_m(x) = f(x) \text{ for } m \geq k\} \supset B$$

is Borelian and of the second category. There is an open interval $I = (a, b)$ such that the set $I \setminus E$ is of the first category. Let $u \in I$ be a rational point such that $u = w_i$ for some positive integer $i > k$. Since all functions

$$x \to g(x - w_n), \quad n \neq i, \quad n = 1, 2, \ldots,$$

are continuous at the point u, the function

$$h(x) = \sum_{i \neq n=1}^{\infty} \frac{g(x - w_n)}{2^n}$$

is also continuous at u. Let $J \subset I$ be an open interval containing u such that

$$|h(t) - h(u)| < \frac{1}{8^i} \quad \text{for} \quad t \in J.$$

There are sequences of points

$$u < u_j, v_j \in J \cap E, \quad j = 1, 2, \ldots,$$

such that

$$u = \lim_{j \to \infty} u_j = \lim_{j \to \infty} v_j,$$

$$|g(u_j - u)| < \frac{1}{8^i} \quad \text{and} \quad |g(v_j - u)| > 1 - \frac{1}{8^i} \quad \text{for} \quad j = 1, 2, \ldots.$$

Observe that

$$f_i(u_j) = f(u_j) = h(u_j) + \frac{g(u_j - w_i)}{2^i} < h(u) + \frac{1}{8^i} + \frac{1}{8^i 2^i} = h(u) + \frac{2^i + 1}{2^i 8^i},$$

and

$$f_i(v_j) = f(v_j) = h(v_j) + \frac{g(v_j - w_i)}{2^i} > h(u) - \frac{1}{8^i} + \frac{1}{2^i} - \frac{1}{2^i 8^i} =$$

$$h(u) + \frac{1}{2^i} - \frac{2^i + 1}{8^i 2^i} = h(u) + \frac{8^i - 2^i - 1}{8^i 2^i} > h(u) + \frac{(2^i + 2)}{8^i 2^i},$$

a contradiction with the assumption that f_i has the finite limit from the right hand side. So f is not in $B_1^*(\mathcal{P})$.

Theorem 2. *Let f be a function such that there is a countable set A and a G_δ-set $B \subset A$ for which the reduced function $f/(\mathbb{R} \setminus A)$ is continuous and for each point $x \in A \setminus B$ there are the finite unilateral limits*

$$\lim_{\mathbb{R} \setminus A \ni t \to x-} f(t) \quad \text{and} \quad \lim_{\mathbb{R} \setminus A \ni t \to x+} f(t).$$

Then the function $f \in B_1^*(\mathcal{P})$.

PROOF. Let $(U_n)_n$ be a sequence of open sets such that

$$B = \bigcap_{n=1}^{\infty} U_n \text{ and } U_1 \supset \ldots \supset U_n \supset \ldots .$$

Since every function f with the finite set of all discontinuity points belongs to $B_1^*(\mathcal{P})$, without loss of the generality we can assume that the set A is infinite. Enumerate all points of the set A in a sequence $(a_n)_n$ such that $a_n \neq a_m$ for $n \neq m$, $n, m = 1, 2, \ldots$.

Next fix a positive integer n and put

$$f_n(x) = f(x) \text{ for } x \in (\mathbb{R} \setminus U_n) \setminus A.$$

For $i \leq n$ we define also

$$f_n(a_i) = f(a_i),$$

and for other $x \in A \setminus U_n$ (**i.e.** $x = a_i$, where $i > n$) let either

$$f_n(x) = \lim_{\mathbb{R}\setminus A \ni t \to x+} f(t)$$

if x is the left endpoint of a component of the set U_n or

$$f_n(x) = \lim_{\mathbb{R}\setminus A \ni t \to x-} f(t)$$

otherwise.

Finishing we define f_n as the linear function on the closures of all components of the set $U_n \setminus \{a_i; i \leq n\}$. Then

$$f_n \in \mathcal{P} \text{ for } n = 1, 2, \ldots \text{ and } f = a.c. \lim_{n\to\infty} f_n,$$

so $f \in B_1^*(\mathcal{P})$ and the proof is completed. □

For the formulation of the generalization of the last theorem we introduce the following notion:

> A set A is said to be an interval set if there is a sequence of nondegenerate intervals I_n, $n = 1, 2, \ldots$, such that
>
> $$A = \bigcup_{n=1}^{\infty} I_n;$$

Theorem 3. *Let f be a function such that there is a countable set A such that the reduced function $f/(\mathbb{R} \setminus A)$ is continuous. Moreover, suppose that there is a sequence of interval sets A_n, $n = 1, 2, \ldots$, such that*

$$B = \bigcap_{n=1}^{\infty} A_n \subset A \text{ and } A_1 \supset \ldots A_n \supset \ldots$$

and every reduced function $f/(\mathbb{R} \setminus A_n)$, $n = 1, 2, \ldots$, has the finite limit at every point x such that x is an endpoint of a component of the set A_n but x does not belong to A_n. Then $f \in B_1^*(\mathcal{P})$.

Proof. The proof is similar as the proof of Theorem 2. □

The following example shows that the assumption of Theorem 3 is essentially more general than that in Theorem 2.

Example 2. Let g be the same function as that from Example 1 and let $C \subset [0, 1]$ be the ternary Cantor set. Put

$$[0, 1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where (a_n, b_n), $n = 1, 2, \ldots$, are components of the open set $[0, 1] \setminus C$. For each positive integer n we find a point

$$c_n \in (a_n, b_n) \text{ such that } g(c_n - a_n) = 0.$$

Let

$$f(x) = \frac{g(x - a_n)}{n} \text{ for } x \in [a_n, c_n], \quad n = 1, 2, \ldots$$

and

$$f(x) = 0 \text{ otherwise on } \mathbb{R}.$$

Observe that for every countable set A the reduced function $f/(\mathbb{R} \setminus A)$ does not have the limit from the right hand side for any point a_n, $n = 1, 2, \ldots$. Evidently, there is not a countable G_δ-set containing the set $B = \{a_n; n = 1, 2, \ldots\}$. But if $A = B$ and for $n = 1, 2, , \ldots$ we define

$$A_n = \bigcup_{k=1}^{\infty} [a_k, \min(a_k + \frac{1}{n}, c_k))$$

then the hypothesis of Theorem 3 is satisfied and the function $f \in B_1^*(\mathcal{P})$.

Next example shows that the assumption of Theorem 3 is not a necessary condition for the relation $f \in B_1^*(\mathcal{P})$.

Example 3. Let C and (a_n, b_n), $n = 1, 2, \ldots$, be the same as those in Example 2. Find a countable set $B \subset C \setminus \{a_n, b_n; n = 1, 2, \ldots\}$ which is dense in C. Enumerate all points of the set B in a sequence $(z_n)_n$ such that $z_n \neq z_m$ for $n \neq m$, $n, m = 1, 2, \ldots$. For every positive integer n find a sequence of closed intervals $I_{n,m} = [c_{n,m}, d_{n,m}] \subset (0, 1)$, $m = 1, 2, \ldots$, such that:

if $(n, m) \neq (k, l)$ then $I_{n,m} \cap I_{k,l} = \emptyset$, $k, l, m, n = 1, 2, \ldots$;

$I_{n,m} \cap C = \emptyset$ for $n, m = 1, 2, \ldots$;

$\lim_{m\to\infty} c_{n,m} = \lim_{m\to\infty} d_{n,m} = z_n$ for $n = 1, 2, \ldots$.

For all positive integers n, m define a continuous function

$$f_{n,m} : I_{n,m} \to [0, \frac{1}{n}]$$

such that

$$f_{n,m}(c_{n,m}) = f_{n,m}(d_{n,m}) = 0 \text{ and } f_{n,m}(I_{n,m}) = [0, \frac{1}{n}].$$

Let

$$f(x) = \begin{cases} f_{n,m}(x) & if & x \in I_{n,m}, \quad n, m = 1, 2, \ldots \\ n^{-1} & if & x = z_n, \quad n = 1, 2, \ldots \\ 0 & otherwise\ on & \mathbb{R}. \end{cases}$$

Observe that for any countable set A there is a point in B where the unilateral limit of the reduced function $f/(\mathbb{R} \setminus A)$ does not exist. Since every interval set containing B is residual in the set C, the function f does not satisfy the assumption of Theorem 3.

But we will prove that $f \in B_1^*(\mathcal{P})$. For this for every positive integer n define

$$f_n(x) = \begin{cases} f_{i,k}(x) & for & x \in I_{i,k}, \quad i, k \leq n \\ i^{-1} & for & x = z_i, \quad i \leq n \\ 0 & otherwise\ on & \mathbb{R} \end{cases}$$

and observe that

$$f_n \in \mathcal{P}, \text{ and } a.c. \lim_{n\to\infty} f_n = f.$$

Theorem 4. *A function f is the a.c. limit of a sequence of functions $f_n \in \mathcal{P}$ if and only if it satisfies the following condition:*

(P) there is a countable set A, a sequence of closed sets A_n and a sequence of functions $g_n \in \mathcal{P}$, $n = 1, 2, \ldots$, such that

$$\mathbb{R} = \bigcup_{n=1}^{\infty} A_n \text{ and } A_n \subset A_{n+1} \text{ for } n = 1, 2, \ldots$$

and

$$g_n/(A_n \setminus A) = f/(A_n \setminus A) \text{ for } n = 1, 2, \ldots .$$

Proof. **Sufficiency.** For the proof of the sufficiency of condition (P) we enumerate the set A in a sequence $(a_n)_n$ and for $n = 1, 2, \ldots$ we define

$$f_n(x) = \begin{cases} f(a_i) & for \quad i \le n \\ g_n(x) & otherwise\ on \quad \mathbb{R}. \end{cases}$$

Then

$$a.c. \lim_{n \to \infty} f_n = f \text{ and } f_n \in \mathcal{P} \text{ for } n = 1, 2, \ldots .$$

Necessity. Let

$$A = \bigcup_{n=1}^{\infty} D(f_n),$$

where $D(f_n)$ denotes the set of all discontinuity points of the function f_n, $n = 1, 2, \ldots$.

Now, we will apply the transfinite induction.

Since $f = a.c. \lim_{n \to \infty} f_n$, there is a positive integer k_0 such that the set

$$B_{k_0} = \{x : \forall_{i \ge k_0} f_i(x) = f(x)\}$$

is of the second category. Consequently, there is an open interval I_0 with rational endpoints such that

$$I_0 \cap B_{k_0} \ne \emptyset \text{ and } f_i(x) = f(x) \text{ for all } x \in I_0 \setminus A \text{ and } i \ge k_0.$$

Fix an ordinal number $\alpha > 0$ and suppose that for every ordinal number $\beta < \alpha$ there are a positive integer k_β and an open interval I_β with rational endpoints such that

$$E_\beta = (I_\beta \setminus A) \setminus \bigcup_{\gamma < \beta} I_\gamma \ne \emptyset,$$

$$f_i(x) = f(x) \text{ for } x \in E_\beta \text{ and } i \ge k_\beta.$$

and

$$D_\alpha = \mathbb{R} \setminus \bigcup_{\beta < \alpha} I_\beta \ne \emptyset.$$

For each point $x \in D_\alpha$ there is a positive integer $k(x)$ such that

$$f_i(x) = f(x) \text{ for } i \geq k(x).$$

By Baire's category theorem there is a positive integer k_α such that the set

$$F_\alpha = \{x \in D_\alpha; k(x) = k_\alpha\}$$

is of the second category in D_α. So, there is an open interval I_α with rational endpoints such that

$$D_\alpha \cap I_\alpha \neq \emptyset$$

and

$$f_i(x) = f(x) \text{ for } x \in E_\alpha \text{ and } i \geq k_\alpha.$$

Let α_0 be the first ordinal number α with $E_\alpha = \emptyset$. Since the family of all intervals with rational endpoints is countable, α_0 is a countable ordinal number. Every set $I_\alpha \cap D_\alpha$, $\alpha < \alpha_0$, is an F_σ set, so

$$I_\alpha \cap D_\alpha = \bigcup_{n=1}^{\infty} F_{n,\alpha},$$

where all sets $F_{n,\alpha}$, $n = 1, 2, \ldots$, $\alpha < \alpha_0$, are closed. Enumerate the set A in a sequence $(a_n)_n$ and all sets $F_{n,\alpha}$, $n = 1, 2, \ldots$, $\alpha < \alpha_0$, in a sequence $(F_{k_i,\alpha_i})_i$. For $n = 1, 2, \ldots$ let

$$A_n = \bigcup_{i=1}^{n} F_{k_i,\alpha_i},$$

and

$$g_n = \begin{cases} f(a_i) & for & i \leq n \\ f_{\max(k_{\alpha_1},\ldots,k_{\alpha_n})} & otherwise\ on & \mathbb{R}. \end{cases}$$

Then

$$g_n/(A_n \setminus A) = f/(A_n \setminus A)$$

and the functions $g_n \in \mathcal{P}$ for $n = 1, 2, \ldots$ and

$$f = a.c. \lim_{n\to\infty} f_n.$$

So the proof is completed. □

2 Monotone Convergence

Remark 1. *It is obvious that a function f is the limit of a pointwise converging sequence of functions $f_n \in \mathcal{P}$ if and only if there is a Baire 1 function g and a countable set A such that*

$$\{x : f(x) \neq g(x)\} \subset A\}.$$

Proof. **Necessity.** If

$$f_n \in \mathcal{P}, \ \ n = 1, 2, \dots \ \text{ and } \ \lim_{n\to\infty} f_n = f$$

then the set

$$A = \bigcup_{n=1}^{\infty} D(f_n),$$

where $D(f_n)$ denotes the set of all discontinuity points of f_n, $n = 1, 2, \dots$, is countable and the reduced function $f/(\mathbb{R} \setminus A)$ is of Baire 1 class. Consequently, there is a Baire 1 function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in \mathbb{R} \setminus A$.

Sufficiency. Since g is of Baire 1 class, there is a sequence of continuous functions g_n, $n = 1, 2, \dots$, such that $g = \lim_{n\to\infty} g_n$. Let $A = \{a_1, \dots, a_n, \dots\}$ and for $n = 1, 2, \dots$ let

$$f_n(x) = \begin{cases} f(a_k) & for \quad k \leq n \\ g_n(x) & otherwise\ on \quad \mathbb{R}. \end{cases}$$

Then

$$f_n \in \mathcal{P} \ \text{ for } \ n = 1, 2, \dots \ \text{ and } \ \lim_{n\to\infty} f_n = f.$$

This completes the proof. □

For the monotone convergence we will prove the following theorem:

Theorem 5. *A function f is the limit of a decreasing sequence of functions $f_n \in \mathcal{P}$ if and only if there are an upper semicontinuous function g and a countable set A such that $f \leq g$ and $f(x) = g(x)$ for all points $x \in \mathbb{R} \setminus A$.*

Proof. **Necessity.** For $n = 1, 2, \dots$ and $x \in \mathbb{R}$ we define

$$g_n(x) = \max(f_n(x), \lim_{t\to x-} f_n(t), \lim_{t\to x+} f_n(t)).$$

From the inequalities $f \leq f_{n+1} \leq f_n$ it follows that $f \leq g_{n+1} \leq g_n$ for $n = 1, 2, \dots$. So there is a function g such that $g_n \searrow g$ with $n \to \infty$. From the definition of g_n and from the inclusion $f_n \in \mathcal{P}$ it follows that every function

g_n, $n = 1, 2, \ldots$, is upper semicontinuous. So, the function g is also upper semicontinuous and $f \leq g$.

Let

$$A = \bigcup_{n=1}^{\infty} D(f_n).$$

The set A is countable. Since all functions f_n, $n = 1, 2, \ldots$, are continuous at all points $x \in \mathbb{R} \setminus A$, we obtain

$$g_n(x) = f_n(x) \text{ for } x \in \mathbb{R} \setminus A, \quad n = 1, 2, \ldots.$$

Consequently,

$$\{x : g(x) \neq f(x)\} \subset A$$

and the proof of the necessity is completed.

Sufficiency. Since g is upper semicontinuous, there is a decreasing sequence of continuous functions g_n, $n = 1, 2, \ldots$, such that

$$g = \lim_{n \to \infty} g_n.$$

Let

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}$$

and for $n = 1, 2, \ldots$ let

$$f_n(x) = \begin{cases} f(x) & for \quad x = a_k, \; k \leq n \\ g_n(x) & otherwise\ on \quad \mathbb{R}. \end{cases}$$

Then the functions $f_n \in \mathcal{P}$ for $n = 1, 2, \ldots$ and

$$f_n \searrow f \text{ with } n \to \infty.$$

This completes the proof. □

Applying the last theorem to the functions f and f_n, $n = 1, 2, \ldots$, we obtain the dual version of Theorem 4.

Theorem 6. *A function f is the limit of an increasing sequence of functions $f_n \in \mathcal{P}$ if and only if there are a lower semicontinuous function g and a countable set A such that $f \geq g$ and*

$$\{x : f(x) \neq g(x)\} \subset A.$$

For the monotone a.c. convergence we have the following theorems:

Theorem 7. *There are functions f and $f_n \in \mathcal{P}$, $n = 1, 2, \ldots$, such that $f = a.c. \lim_{n\to\infty} f_n$ and $f_n \nearrow f$ with $n \to \infty$ and such that for each decreasing sequence of functions $h_n \in \mathcal{P}$, $n = 1, 2, \ldots$, the relation $f = a.c. \lim_{n\to\infty} h_n$ is false.*

Proof. We conserve all notations from Example 3. Let f be the function from Example 3. Then f is the a.c. limit of the sequence of the functions f_n, $n = 1, 2, \ldots$, defined in Example 3 and belonging to $\mathcal{P}$. As an upper semicontinuous function f is the limit of a decreasing sequence of continuous functions (so, belonging to $\mathcal{P}$).

Suppose, to the contrary that there is a decreasing sequence of functions h_n, $n = 1, 2, \ldots$, with $a.c. \lim_{n\to\infty} h_n = f$. Since $h_n \geq f$ for $n = 1, 2, \ldots$, for all $n, m = 1, 2, \ldots$ the inequality $h_n \geq f_m$ is true. There are an open interval K, a countable set E, and a positive integer k such that

$$K \cap C \neq \emptyset \text{ and } h_i(x) = 0 \text{ for } x \in (K \cap C) \setminus E \text{ and } i \geq k.$$

Let $m > k$ be a positive integer with $z_m \in K \cap C$. In every interval $I_{m,j}$, $j = 1, 2, \ldots$, there is a point $u_{m,j} \in I_{m,j}$ at which $f(u_{m,j}) = \frac{1}{m}$. Consequently,

$$h_k(u_{m,j}) \geq \frac{1}{m} \text{ for } j = 1, 2, \ldots.$$

Since

$$z_m = \lim_{j\to\infty} u_{m,j}$$

and z_m is a bilateral accumulation point of the set $K \cap (C \setminus E)$, the function h_k has not at least one unilateral limit at z_m. So it is not in $\mathcal{P}$ and the obtained contradiction proves our theorem. □

Theorem 8. *Let f be a function. Suppose that there are a countable set A, a sequence of closed sets A_n and a sequence of functions $g_n \in \mathcal{P}$ with $g_n \geq f$ ($g_n \leq f$), $n = 1, 2, \ldots$, such that*

$$\mathbb{R} = \bigcup_{n=1}^{\infty} A_n \text{ and } A_n \subset A_{n+1} \text{ for } n = 1, 2, \ldots$$

and

$$g_n/(A_n \setminus A) = f/(A_n \setminus A) \text{ for } n = 1, 2, \ldots.$$

Then there is a decreasing (increasing) sequence of functions $h_n \in \mathcal{P}$, $n = 1, 2, \ldots$, such that $f = a.c. \lim_{n\to\infty} h_n$.

PROOF. We will consider only the first case where $g_n \geq f$, since the case where $g_n \leq f$ is analogous.

Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ and for $n = 1, 2, \ldots$ let

$$h_n(x) = \begin{cases} f(a_i) & \textit{for} \quad i \leq n \\ \min(g_1(x), \ldots, g_n(x)) & \textit{otherwise on} \quad \mathbb{R}. \end{cases}$$

Then the sequence of functions $h_n \in \mathcal{P}$ satisfies all requirements and the proof is completed. □

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