Real Analysis Exchange
Vol. (), , pp. 753–770

Francis Jordan,* Department of Mathematics, University of Louisville, Louisville, KY 40208. e-mail: `fejord01@athena.louisville.edu`

A NEW CARDINAL INVARIANT RELATED TO ADDING REAL FUNCTIONS

Abstract

Let $F \subseteq \mathbb{R}^{\mathbb{R}}$. The additivity of F, briefly $\mathrm{A}(F)$, is the minimum cardinality of a family $G \subseteq \mathbb{R}^{\mathbb{R}}$ with the property that $h + G \subseteq F$ for no $h \in \mathbb{R}^{\mathbb{R}}$. In this paper we consider the notion of super-additivity which we will denote by A^*. If $F \subseteq \mathbb{R}^{\mathbb{R}}$, then $\mathrm{A}^*(F)$ is the minimum cardinality of a family of functions G with the property that for any $H \subseteq \mathbb{R}^{\mathbb{R}}$ if $|H| < \mathrm{A}(F)$, there is a $g \in G$ such that $g + H \subseteq F$. We calculate the super-additivities of the families of Darboux-like functions and their complements.

1 Preliminaries

In what follows we will use standard terminology and notation as in [2]. In particular, the set of all functions from a set X into a set Y will be denoted by Y^X. Given a set X and $f, g \in X^X$ we denote their composition by $f \circ g$. The characteristic function of a set $A \subseteq \mathbb{R}$ will be denoted by χ_A. The symbol $|X|$ will denote the cardinality of the set X. The successor of a cardinal κ will be denoted by κ^+. We denote by $[X]^{<\kappa}$, $[X]^{\kappa}$, and $[X]^{\leq\kappa}$ the sets of all subsets of X of cardinality less than κ, equal to κ, and less than or equal to κ, respectively. The cardinality of the real numbers $\mathbb{R}$ will be denoted by $\mathfrak{c}$. Given a cardinal number κ we let $\mathrm{cf}(\kappa)$ denote the cofinality of κ. We say a cardinal κ is regular provided that $\mathrm{cf}(\kappa) = \kappa$. For functions $f, g \in \mathbb{R}^{\mathbb{R}}$ let $[f = g]$ denote the set $\{x \in \mathbb{R} \colon f(x) = g(x)\}$. We define $[f < g]$ and $[f \leq g]$ in a similiar way. Functions will be identified with their graphs. For a set $S \subseteq X \times Y$ we let $\mathrm{dom}(S) = \{x \in X \colon (\exists y \in Y)(\langle x, y\rangle \in S)\}$.

We will need some cardinals which have combinatorial descriptions. For a cardinal κ we define

Key Words: cardinal functions, extendable functions, connectivity functions, peripherially continuous functions, almost continuous functions, Darboux functions.

Mathematical Reviews subject classification: Primary 26A15; Secondary 54A25

Received by the editors November 8, 1999

*This paper was written under supervision of K. Ciesielski. The author wishes to thank him for many helpful conversations.

$d_\kappa = \min\{|F|\colon F \subseteq \kappa^\kappa \ \&\ (\forall g \in \kappa^\kappa)(\exists f \in F)(|[f = g]| = \kappa)\}$,

$e_\kappa = \min\{|F|\colon F \subseteq \kappa^\kappa \ \&\ (\forall g \in \kappa^\kappa)(\exists f \in F)(|[f = g]| < \kappa)\}$,

$e^1_\kappa = \min\{|F|\colon F \subseteq \kappa^\kappa \ \&\ \left(\forall G \in [\kappa^\kappa]^{<e_\kappa}\right)(\exists f \in F)(\forall g \in G)(|[f = g]| = \kappa)\}$,

$d^1_\kappa = \min\{|F|\colon F \subseteq \kappa^\kappa \ \&\ \left(\forall G \in [\kappa^\kappa]^{<d_\kappa}\right)(\exists f \in F)(\forall g \in G)(|[f = g]| < \kappa)\}$.

2 Introduction

The cardinal function called additivity was orginally defined by Natkaniec [10] for families $\mathcal{F} \in \mathbb{R}^\mathbb{R}$ to be

$$\mathrm{A}(\mathcal{F}) = \min(\{|F|\colon F \subseteq \mathbb{R}^\mathbb{R} \ \&\ (\forall g \in \mathbb{R}^\mathbb{R})(\exists f \in F)(f + g \notin \mathcal{F})\} \cup \{(2^\mathfrak{c})^+\}).$$

This cardinal function has been studied intensively and has been generalized to include families in $(\mathbb{R}^m)^{\mathbb{R}^n}$ see [8], [10], and [3]. We will restrict the scope of this paper to $\mathbb{R}^\mathbb{R}$ and consider a new cardinal function which is based on the additivity function. Before continuing let us recall some basic facts about additivity.

Proposition 1. [9, Proposition 1] *Let $\mathcal{P}, \mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$. Then,*

(i) *$\mathcal{F} = \emptyset$ if and only if $\mathrm{A}(\mathcal{F}) = 1$,*

(ii) *$\mathcal{F} = \mathbb{R}^\mathbb{R}$ if and only if $\mathrm{A}(\mathcal{F}) = |2^\mathfrak{c}|^+$,*

(iii) *if $\mathcal{F} \subseteq \mathcal{P}$ then $\mathrm{A}(\mathcal{F}) \le \mathrm{A}(\mathcal{P})$, and*

(iv) *if $\mathcal{F} \neq \emptyset$ then $2 = \mathrm{A}(\mathcal{F})$ if and only if $\mathcal{F} - \mathcal{F} \neq \mathbb{R}^\mathbb{R}$.*

Given $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ the definition of additivity implies that $\mathbb{R}^\mathbb{R}$ has the property

$$(*) \qquad \left(\forall G \in \left[\mathbb{R}^\mathbb{R}\right]^{<\mathrm{A}(\mathcal{F})}\right)(\exists f \in \mathbb{R}^\mathbb{R})(f + G \subseteq \mathcal{F}).$$

A natural question that arises is wether or not $\mathbb{R}^\mathbb{R}$ is the only subset of $\mathbb{R}^\mathbb{R}$ to satisfy $(*)$. In particular, one might want to find the smallest cardinality of a family $F \subseteq \mathbb{R}^\mathbb{R}$ that satisfies $(*)$. This consideration leads to the definition of super-additivity. If $F \subseteq \mathbb{R}^\mathbb{R}$ we define the super-additvity of $\mathcal{F}$ to be

$$\mathrm{A}^*(\mathcal{F}) = \min\{|F|\colon F \subseteq \mathbb{R}^\mathbb{R} \ \&\ \left(\forall G \in \left[\mathbb{R}^\mathbb{R}\right]^{<\mathrm{A}(\mathcal{F})}\right)(\exists f \in F)(f + G \subseteq \mathcal{F})\}.$$

We list some basic facts about super-additivity

Proposition 2. *Let $\mathcal{F},\mathcal{E} \subseteq \mathbb{R}^{\mathbb{R}}$. Then,*

(i) $\mathcal{F} \in \{\emptyset, \mathbb{R}^{\mathbb{R}}\}$ *if and only if* $\mathrm{A}^*(\mathcal{F}) = 1$ *and*

(ii) *if* $\mathrm{A}(\mathcal{F}) = \mathrm{A}(\mathcal{E})$ *and* $\mathcal{F} \subseteq \mathcal{E}$ *then* $\mathrm{A}^*(\mathcal{F}) \geq \mathrm{A}^*(\mathcal{E})$.

Proof. We show (i). If $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$, then, by Proposition 1(ii), $\mathrm{A}(\mathcal{F}) = (2^{\mathfrak{c}})^+$. Let $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<(2^{\mathfrak{c}})^+}$. Clearly, $\chi_\emptyset + G \subseteq \mathbb{R}^{\mathbb{R}} = \mathcal{F}$. So, $\mathrm{A}^*(\mathcal{F}) = 1$. If $\mathcal{F} = \emptyset$, then, by Proposition 1(i), $\mathrm{A}(\mathcal{F}) = 1$. Since $\left[\mathbb{R}^{\mathbb{R}}\right]^{<1} = \{\emptyset\}$ and $\chi_\emptyset + \emptyset \subseteq \mathcal{F}$, it follows that $\mathrm{A}^*(\mathcal{F}) = 1$. Suppose now that $\mathrm{A}^*(\mathcal{F}) = 1$. We show that $\mathcal{F} \in \{\emptyset, \mathbb{R}^{\mathbb{R}}\}$. Assume that $\mathcal{F} \neq \emptyset$. By Proposition 1(i), $\mathrm{A}(\mathcal{F}) > 1$. Since $\mathrm{A}^*(\mathcal{F}) = 1$, there is an $h \in \mathbb{R}^{\mathbb{R}}$ such that $h + g \in \mathcal{F}$ for any $g \in \mathbb{R}^{\mathbb{R}}$. So, $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}} = h + \mathbb{R}^{\mathbb{R}} \subseteq \mathcal{F}$. Thus, $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$.

We show (ii). Let $\kappa = \mathrm{A}(\mathcal{F}) = \mathrm{A}(\mathcal{E})$. Suppose $F \subseteq \mathbb{R}^{\mathbb{R}}$ and $|F| < \mathrm{A}^*(\mathcal{E})$. Then, there exists a $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\kappa}$ such that $f + G$ is not contained in $\mathcal{E}$ for every $f \in F$. But $\mathcal{F} \subseteq \mathcal{E}$; so $f + G$ is not contained in $\mathcal{F}$ for every $f \in F$. Thus, $\mathrm{A}^*(\mathcal{F}) \geq \mathrm{A}^*(\mathcal{E})$. □

Next we point out a basic relationship between additivity and super-additivity.

Proposition 3. *If $\mathcal{F} \notin \{\mathbb{R}^{\mathbb{R}}, \emptyset\}$, then*

$$\max\{\mathrm{A}(\mathcal{F}), \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{F})\} \leq \mathrm{A}^*(\mathcal{F}).$$

Proof. We first show that $\mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{F}) \leq \mathrm{A}^*(\mathcal{F})$. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be a witness to the definition of $\mathrm{A}^*(\mathcal{F})$, i.e., $|F| = \mathrm{A}^*(\mathcal{F})$ and

$$(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathcal{F})})(\exists f \in F)(f + G \subseteq \mathcal{F}).$$

Since $\mathrm{A}(\mathcal{F}) \geq 2 > 1$, we see that F also satisfies

$$(\forall g \in \mathbb{R}^{\mathbb{R}})(\exists f \in F)(f + g \in \mathcal{F}).$$

Since $\mathcal{F} = \mathbb{R}^{\mathbb{R}} \setminus (\mathbb{R}^{\mathbb{R}} \setminus \mathcal{F})$ we see that $\mathrm{A}^*(\mathcal{F}) = |F| \geq \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{F})$.

We show that $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}^*(\mathcal{F})$. By way of contradiction assume $\mathrm{A}(\mathcal{F}) > \mathrm{A}^*(\mathcal{F})$. Then there is an $F \subseteq \mathbb{R}$ such that $|F| < \mathrm{A}(\mathcal{F})$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathcal{F})}\right)(\exists f \in F)(f + G \subseteq \mathcal{F}). \tag{1}$$

Since $\mathcal{F} \neq \mathbb{R}^{\mathbb{R}}$, for every $f \in F$ there is a $g_f \in \mathbb{R}^{\mathbb{R}}$ such that $f + g_f \notin \mathcal{F}$. Let $G = \{g_f \colon f \in F\}$. Notice that $|G| \leq |F| < \mathrm{A}(\mathcal{F})$. By (1) there is an $f \in F$ such that $f + G \subseteq \mathcal{F}$. In particular, $f + g_f \in \mathcal{F}$ but this contradicts the choice of g_f. Thus, $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}^*(\mathcal{F})$. □

3 The Results

We will primarily be concerned with calculating the super-additivities of the following families of functions from $\mathbb{R}$ into $\mathbb{R}$ and their complements. Some combinatorial characterizations of these cardinals are also given. We give general descriptions of these families that will work for any function from one space to another where the spaces are assumed to have the appropriate structure.

Dar: $f \in Y^X$ is a Darboux function if and only if $f[C]$ is connected in Y for every connected subset C of X.

Con: $f \in Y^X$ is a connectivity function if and only if the graph of f restricted to C is connected in $X \times Y$ for every connected subset C of X.

AC: $f \in Y^X$ is an almost continuous function if and only if every open set in $X \times Y$ containing f also contains some continuous function $g \in Y^X$.

Ext: $f \in Y^X$ is an extendable function if and only if there is a connectivity function $g\colon X \times [0,1] \to Y$ such that $f(x) = g(x,0)$ for every $x \in X$.

PR: $f \in \mathbb{R}^{\mathbb{R}}$ is a perfect road function if and only if for every $x \in \mathbb{R}$ there is a perfect set $P \subset \mathbb{R}$ such that x is a bilateral limit point of P and $f|_P$ is continuous at x.

PC: $f \in Y^X$ is a peripherally continuous function if and only if for every $x \in X$ and pair of open sets $U \subset X$ and $V \subset Y$ such that $x \in U$ and $f(x) \in V$ there is an open neighborhood W of x with $\mathrm{cl}(W) \subset U$ and $f[\mathrm{bd}(W)] \subset V$, where $\mathrm{cl}(W)$ and $\mathrm{bd}(W)$ denote the boundary and the closure of W, respectively.

SZ: $f \in Y^X$ is a Sierpiński-Zygmund function if and only if $f|_A$ is continuous for no set $A \subseteq X$ of cardinality $\mathfrak{c}$.

The diagrams below describe the relations between the above families in $\mathbb{R}^{\mathbb{R}}$ except SZ. The symbol $\longrightarrow$ denotes containment. All inclusions are proper.

$$\begin{tikzcd}
 & \mathrm{AC} \arrow[r] & \mathrm{Con} \arrow[r] & \mathrm{Dar} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[drr] & & & & \mathrm{PC} \\
 & & \mathrm{PR} \arrow[urr] & &
\end{tikzcd}$$

It is clear from the definition of super-additivity and Proposition 3 that it would useful to know the additivities of these families. Fortunately, there is good bit that we know about these values.

Proposition 4.

(i) (Ciesielski, Recław [7]) $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$ *and* $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$*;*

(ii) (Ciesielski, Miller [5]) $\mathrm{A}(\mathrm{Dar}) = \mathrm{A}(\mathrm{Con}) = \mathrm{A}(\mathrm{AC}) = e_{\mathfrak{c}}$*;*

(iii) (Ciesielski, Natkaniec [6]) $\mathrm{A}(\mathrm{SZ}) = d_{\mathfrak{c}}$*;*

(iv) (Ciesielski [4](see[9])) $\mathrm{A}(\neg\mathrm{PC}) = \omega_1$*;*

(v) (Jordan [9]) $\mathrm{A}(\neg\mathrm{PR}) = \mathrm{A}(\neg\mathrm{Ext}) = 2^{\mathfrak{c}}$*;*

(vi) (Jordan [9]) *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$*, then* $\mathrm{A}(\neg\mathrm{SZ}) = e_{\mathfrak{c}}$*;*

(vii) (Jordan [9]) *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$*, then* $d_{\mathfrak{c}} = \mathrm{A}(\neg\mathrm{Dar}) = \mathrm{A}(\neg\mathrm{Con}) = \mathrm{A}(\neg\mathrm{AC})$. □

We first calculate the super-additivies of the families of functions we are concerned with.

Theorem 5. *If* $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{PR}, \mathrm{PC}\}$*, then* $\mathrm{A}^*(\mathcal{F}) = \mathrm{A}^*(\neg\mathcal{F}) = 2^{\mathfrak{c}}$.

Theorem 6. $\mathrm{A}^*(\mathrm{AC}) = \mathrm{A}^*(\mathrm{Con}) = \mathrm{A}^*(\mathrm{Dar}) = e^1_{\mathfrak{c}}$.

Theorem 7. $\mathrm{A}^*(\mathrm{SZ}) = d^1_{\mathfrak{c}}$.

Theorem 8. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$*, then* $d^1_{\mathfrak{c}} = \mathrm{A}^*(\neg\mathrm{Dar}) = \mathrm{A}^*(\neg\mathrm{Con}) = \mathrm{A}^*(\neg\mathrm{AC})$.

Theorem 9. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$*, then* $e^1_{\mathfrak{c}} = \mathrm{A}^*(\neg\mathrm{SZ})$.

We also have a purely combinatorial result which will allow us to say something about the values $d^1_{\mathfrak{c}}$ and $e^1_{\mathfrak{c}}$.

Theorem 10. *If* $|\mathfrak{c}^{<\mathfrak{c}}| = \mathfrak{c}$ *and* $\mathfrak{c} = \lambda^+$*, then* $d_{\mathfrak{c}} \le e_{\mathfrak{c}} = e^1_{\mathfrak{c}} = d^1_{\mathfrak{c}}$.

Finally, we quote two consistency results.

Proposition 11. (Ciesielski, Natkaniec [6]) *Let* $\lambda \ge \kappa \ge \omega_2$ *be cardinals such that* $\mathrm{cf}(\lambda) > \omega_1$ *and* κ *is regular. Then it is relatively consistent with ZFC+CH that* $2^{\mathfrak{c}} = \lambda$ *and* $\mathrm{A}(\mathrm{Dar}) = \mathrm{A}(\mathrm{SZ}) = \kappa$. □

Proposition 12. (Ciesielski, Natkaniec [6]) *Let* $\lambda > \omega_2$ *be a cardinal such that* $\mathrm{cf}(\lambda) > \omega_1$*. Then it is relatively consistent with ZFC+CH that* $2^{\mathfrak{c}} = \lambda$*, and* $\mathrm{A}(\mathrm{SZ}) = \mathfrak{c}^+ < 2^{\mathfrak{c}} = \mathrm{A}(\mathrm{Dar})$. □

Since, CH implies that $|[\mathfrak{c}^{<\mathfrak{c}}]| = \mathfrak{c}$, Propositions 11, 12 and 4 together with Theorem 10 imply that

Corollary 13. *Let $\lambda \geq \kappa \geq \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$ and for $\mathcal{F} \in \{\mathrm{AC}, \mathrm{Con}, \mathrm{Dar}, \mathrm{SZ}\}$ $\mathrm{A}(\mathcal{F}) = \mathrm{A}^*(\mathcal{F}) = \mathrm{A}(\neg\mathcal{F}) = \mathrm{A}^*(\neg\mathcal{F}) = \kappa$.* □

Corollary 14. *Let $\lambda > \omega_2$ be a cardinal such that $\mathrm{cf}(\lambda) > \omega_1$. Then it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$, and for $\mathcal{F} \in \{\mathrm{Dar}, \mathrm{Con}, \mathrm{AC}\}$*

$$\begin{aligned}\mathrm{A}(\mathrm{SZ}) = \mathrm{A}(\neg\mathcal{F}) = \mathfrak{c}^+ < 2^{\mathfrak{c}} &= \mathrm{A}(\mathcal{F}) = \mathrm{A}^*(\mathcal{F}) \\ &= \mathrm{A}^*(\neg\mathcal{F}) = \mathrm{A}(\neg\mathrm{SZ}) = \mathrm{A}^*(\neg\mathrm{SZ}) = \mathrm{A}^*(\mathrm{SZ}). \quad \square\end{aligned}$$

We prove Theorems 5 and 7 at this time. We will prove the other Theorems in later sections since the proofs are somewhat long.

PROOF OF THEOREM 5. Let $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{PR}, \mathrm{PC}\}$. By (i) and (v) of Proposition 4, we have $\max\{\mathrm{A}(\mathcal{F}), \mathrm{A}(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{F})\} = 2^{\mathfrak{c}}$. Since $\mathcal{F} \notin \{\mathbb{R}^{\mathbb{R}}, \emptyset\}$ Proposition 3 implies that $\mathrm{A}^*(\mathcal{F}) = 2^{\mathfrak{c}} = \mathrm{A}^*(\mathbb{R}^{\mathbb{R}} \setminus \mathcal{F})$. □

To begin the proof of Theorem 7 we quote a theorem about SZ functions which may be found in [11].

Proposition 15. (Sierpiński, Zygumund [11]) *$f \in \mathbb{R}^{\mathbb{R}}$ is in* SZ *if and only if $|[f = h]| < \mathfrak{c}$ for every continuous function h defined on a G_δ-set of cardinality $\mathfrak{c}$.* □

PROOF OF THEOREM 7. We show that $\mathrm{A}^*(\mathrm{SZ}) \leq d_{\mathfrak{c}}^1$. Let H stand for the family of all functions $h \in \mathbb{R}^{\mathbb{R}}$ such that $h|_A$ is continuous on a G_δ-set A of cardinality $\mathfrak{c}$ and equal to zero elsewhere. Note that $|H| = \mathfrak{c}$. Pick $F \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|F| = d_{\mathfrak{c}}^1$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<d_{\mathfrak{c}}}\right)(\exists f \in F)(\forall g \in G)(|[f = g]| < \mathfrak{c}). \tag{2}$$

We claim that F satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{SZ})}\right)(\exists f \in F)(f + G \subseteq \mathrm{SZ}). \tag{3}$$

Let $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{SZ})}$ be arbitrary. By Proposition 4(iii) we have $|G| < d_{\mathfrak{c}}$. It is shown in [6] that $d_{\mathfrak{c}} > \mathfrak{c}$. It follows that $\{h - g\colon g \in G \ \&\ h \in H\}$ is a set of cardinality strictly less than $d_{\mathfrak{c}}$. By (2) there is an $f \in F$ such $|[f = h - g]| < \mathfrak{c}$ for every $g \in G$ and $h \in H$. So, by Proposition 15, $(f + g)|_A$

is continuous for no set A of cardinality $\mathfrak{c}$ for every $g \in G$. Thus, F satisfies (3) and $\mathrm{A}^*(\mathrm{SZ}) \le d^1_{\mathfrak{c}}$.

We show that $d^1_{\mathfrak{c}} \le \mathrm{A}^*(\mathrm{SZ})$. Pick $F \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|F| = \mathrm{A}^*(\mathrm{SZ})$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{SZ})}\right)(\exists f \in F)(\forall g \in G)(f + g \in \mathrm{SZ}). \tag{4}$$

Let $F_1 = \{-f \colon f \in F\}$, notice $|F_1| = |F|$. We show that F_1 satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<d_{\mathfrak{c}}}\right)(\exists f \in F_1)(\forall g \in G)(|[f = g]| < \mathfrak{c}). \tag{5}$$

Let $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<d_{\mathfrak{c}}}$ be arbitrary. By Proposition 4(iii) we have $|G| < \mathrm{A}(\mathrm{SZ})$. By (4) there is an $f \in F$ such that $f + g \in \mathrm{SZ}$ for every $g \in G$. In particular, $|(f+g)^{-1}(\{0\})| < \mathfrak{c}$ for every $g \in G$. It follows that $|[-f = g]| < \mathfrak{c}$ for every $g \in G$. Thus, F_1 satisfies (5) and $d^1_{\mathfrak{c}} \le \mathrm{A}^*(\mathrm{SZ})$. □

4 Proof of Theorem 6

Our first goal is to show that $\mathrm{A}^*(\mathrm{Dar}) = \mathrm{A}^*(\mathrm{Con}) = \mathrm{A}^*(\mathrm{AC})$. To do this we will need to define the following family of functions. Let Dar_1 denote the collection of all $f \in \mathbb{R}^{\mathbb{R}}$ such that $f[(a,b)] = \mathbb{R}$ for every $a < b$. Clearly, $\mathrm{Dar}_1 \subseteq \mathrm{Dar}$.

Lemma 16. $\mathrm{A}^*(\mathrm{Dar}) = \mathrm{A}^*(\mathrm{Dar}_1)$.

Proof. It follows from [5, Theorem 2.4] that $\mathrm{A}(\mathrm{Dar}) = \mathrm{A}(\mathrm{Dar}_1)$. By Proposition 2(ii) we have $\mathrm{A}^*(\mathrm{Dar}) \le \mathrm{A}^*(\mathrm{Dar}_1)$. We show that $\mathrm{A}^*(\mathrm{Dar}_1) \le \mathrm{A}^*(\mathrm{Dar})$. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| = \mathrm{A}^*(\mathrm{Dar})$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar})}\right)(\exists f \in F)(f + G \subseteq \mathrm{Dar}). \tag{6}$$

We show that F also satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar}_1)}\right)(\exists f \in F)(f + G \subseteq \mathrm{Dar}_1) \tag{7}$$

which will complete the proof. Let $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar}_1)}$. Note that $|G| < \mathrm{A}(\mathrm{Dar})$. Put $G_1 = G \cup \{g + r \cdot \chi_{\mathbb{Q}} \colon g \in G \mathbin{\&} r \in \mathbb{R}\}$. Since $\mathrm{A}(\mathrm{Dar}) > \mathfrak{c}$ [5] we have $G_1 < \mathrm{A}(\mathrm{Dar})$. By (6) there is an $f \in F$ such that $f + G_1 \subseteq \mathrm{Dar}$. We claim that $f + G \subseteq \mathrm{Dar}_1$. By way of contradiction assume there is some $g \in G$ and $a < b$ such that $(f+g)[(a,b)] \ne \mathbb{R}$. Since $(f+g)[(a,b)]$ is an interval, we may assume without loss of generality that there is some $M > 0$ which is an upper bound for $(f+g)[(a,b)]$. Let $q \in (a,b)\cap\mathbb{Q}$ and $k = M-(f+g)(q)$. Now $f+g+(k+1)\cdot\chi_{\mathbb{Q}} \notin \mathrm{Dar}$, since $(f+g+(k+1))[(a,b)\setminus\mathbb{Q}]$ is bounded above by M but $(f+g+(k+1)\cdot\chi_{\mathbb{Q}})(q) = M+1$. However, $g+(k+1)\cdot\chi_{\mathbb{Q}} \in G_1$ so by the choice of f we have $f+g+(k+1)\cdot\chi_{\mathbb{Q}} \in \mathrm{Dar}$ giving a contradiction. Thus, F satisfies (7). □

We will need a result of K.Kellum [10, Theorem 1.2].

Proposition 17. *There is a family $\mathcal{B}$ of closed subsets of $\mathbb{R}^2$ such that $|\mathcal{B}| = \mathfrak{c}$, $\mathrm{dom}(B)$ is a non-degenerate interval for every $B \in \mathcal{B}$ and $f \in \mathrm{AC}$ if and only if $f \cap B \neq \emptyset$ for each $B \in \mathcal{B}$.*

Lemma 18. $\mathrm{A}^*(\mathrm{Dar}) = \mathrm{A}^*(\mathrm{Con}) = \mathrm{A}^*(\mathrm{AC})$.

PROOF. By Proposition 4(ii) and Proposition 2(ii) we have

$$\mathrm{A}^*(\mathrm{Dar}) \leq \mathrm{A}^*(\mathrm{Con}) \leq \mathrm{A}^*(\mathrm{AC}).$$

So it is enough for us to show that $\mathrm{A}^*(\mathrm{AC}) \leq \mathrm{A}^*(\mathrm{Dar})$. By Lemma 16 there is an $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| = \mathrm{A}^*(\mathrm{Dar})$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar})}\right)(\exists f \in F)(f+G \subseteq \mathrm{Dar}_1). \tag{8}$$

We claim that F also satisfies $\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{AC})}\right)(\exists f \in F)(f+G \subseteq \mathrm{AC})$ which will complete the proof. Let $\mathcal{B}$ be as in Proposition 17. For each $B \in \mathcal{B}$ let $h_B \in \mathbb{R}^{\mathbb{R}}$ be such that $h_B|_{\mathrm{pr}_x(B)} \subseteq B$ and zero otherwise. Let $G \subseteq \mathbb{R}^{\mathbb{R}}$ and $|G| < \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{Dar}_1)$. Let $G_1 = \{g - h_B \colon g \in G \;\&\; B \in \mathcal{B}\}$. Since $\mathfrak{c} < \mathrm{A}(\mathrm{Dar})$ we have $|G_1| < \mathrm{A}(\mathrm{Dar})$. By (8) there is an $f \in F$ such that $f+G_1 \subseteq \mathrm{Dar}_1$. We claim that $f+G \subseteq \mathrm{AC}$. Fix $g \in G$. Let $B \in \mathcal{B}$ be arbitrary. Since $f+(g-h_B) \in \mathrm{Dar}_1$ there is an $r \in \mathrm{pr}_x(B)$ such that $f+(g-h_B)(r) = 0$. So $\langle r, (f+g)(r)\rangle = \langle r, h_B(r)\rangle \in B$. Thus, $f+g \in \mathrm{AC}$. □

To complete the proof of Theorem 6 it is enough to prove the following lemma.

Lemma 19. $\mathrm{A}^*(\mathrm{Dar}_1) = e^1_{\mathfrak{c}}$.

PROOF. We show that $\mathrm{A}^*(\mathrm{Dar}_1) \leq e^1_{\mathfrak{c}}$. Let $\{P_\alpha\}_{\alpha\in\mathfrak{c}}$ be a partition of $\mathbb{R}$ such that $|P_\alpha| = \mathfrak{c}$ for every $\alpha \in \mathfrak{c}$ and for every non-degenerate open interval U there is an $\alpha \in \mathfrak{c}$ such that $P_\alpha \subseteq U$. For each $\alpha \in \mathfrak{c}$ let $\{p_{\alpha,\beta} \colon \beta \in \mathfrak{c}\}$ be an injective enumeration of P_α. Let $F \subseteq \mathbb{R}^{\mathfrak{c}}$ be such that $|F| = e^1_{\mathfrak{c}}$ and

$$\left(\forall G \in [\mathbb{R}^{\mathfrak{c}}]^{<e_{\mathfrak{c}}}\right)(\exists f \in F)(|[f=g]| = \mathfrak{c}). \tag{9}$$

For each $f \in F$ define $f^* \in \mathbb{R}^{\mathbb{R}}$ by $f^*(p_{\alpha,\beta}) = f(\beta)$. Let $F^* = \{f^* : f \in F\}$. Note that $|F^*| = |F| = e^1_{\mathfrak{c}}$. It is enough to show that F^* satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar}_1)}\right)(\exists f^* \in F^*)(f^* + G \subseteq \mathrm{Dar}_1). \tag{10}$$

Let $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar}_1)}$. By [5, Theorem 2.4] and Proposition 4(ii) $\mathrm{A}(\mathrm{Dar}_1) = \mathrm{A}(\mathrm{Dar}) = e_{\mathfrak{c}}$. So $|G| < e_{\mathfrak{c}}$. For each $\alpha \in \mathfrak{c}$, $g \in G$, and $r \in \mathbb{R}$ let $g_{\alpha,r} \in \mathbb{R}^{\mathfrak{c}}$ be defined by $g_{\alpha,r}(\beta) = r - g(p_{\alpha,\beta})$ for each $\beta \in \mathfrak{c}$. Let

$$G_1 = \{g_{\alpha,r} : r \in \mathbb{R} \;\&\; g \in G\}.$$

Since $e_{\mathfrak{c}} > \mathfrak{c}$,it follows that $|G_1| < e_{\mathfrak{c}}$. By (9) there is an $f \in F$ such that $|[f = g_1]| = \mathfrak{c}$ for every $g_1 \in G_1$. We claim that $f^* + g \in \mathrm{Dar}_1$ for every $g \in G$. Fix $g \in G$, $r \in \mathbb{R}$, and a non-degenerate open interval U. We must show that $r \in (f^* + g)[U]$. By the way we defined our partition there is an $\alpha \in \mathfrak{c}$ such that $P_\alpha \subseteq U$. Let $\beta \in [f = g_{\alpha,r}]$. Then $f^*(p_{\alpha,\beta}) = r - g(p_{\alpha,\beta})$. So, $(f^* + g)(p_{\alpha,\beta}) = r$ and $p_{\alpha,\beta} \in P_\alpha \subseteq U$. Thus, F^* satisfies (10) establishing the inequality.

We now show that $\mathrm{A}^*(\mathrm{Dar}_1) \geq e^1_{\mathfrak{c}}$. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| = \mathrm{A}^*(\mathrm{Dar}_1)$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\mathrm{Dar}_1)}\right)(\exists f \in F)(f + G \subseteq \mathrm{Dar}_1). \tag{11}$$

Let $\Theta \in \mathbb{R}^{\mathbb{R}}$ be an additive function such that $|\Theta^{-1}(r)| = \mathfrak{c}$ for every $r \in \mathbb{R}$. Let $F_1 = \{\Theta \circ f : f \in F\}$, note that $|F_1| \leq |F|$. It is enough for us to show that F_1 satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<e_{\mathfrak{c}}}\right)(\exists f_1 \in F_1)(\forall g \in G)(|[f_1 = g]| = \mathfrak{c}). \tag{12}$$

Let $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<e_{\mathfrak{c}}}$. For each $g \in G$ pick $g_1 \in \mathbb{R}^{\mathbb{R}}$ such that $\Theta \circ g_1 = -g$. Let $G_1 = \{g_1 : g \in G\}$. Notice that $|G_1| \leq |G|$. By (11) there is an $f \in F$ such that for every $g_1 \in G_1$ we have $f + g_1 \in \mathrm{Dar}_1$. Put $f_1 = \Theta \circ f$. By our choice of Θ and the fact that $(f + g_1)[\mathbb{R}] = \mathbb{R}$, we have

$$|(f_1 - g)^{-1}(0)| = |(\Theta \circ f + \Theta \circ g_1)^{-1}(0)| = |\Theta \circ (f + g_1)^{-1}(0)| = \mathfrak{c}.$$

In particular, for each $g \in G$ we have $|f_1 = g| = \mathfrak{c}$. Thus, F_1 satisfies (12). $\square$

5 Proofs of Theorems 8 and 9

Our first goal will be to prove Theorem 8. Towards this end we introduce two more cardinals the first of which appears in [9].

$$d^*_{\mathfrak{c}} = \min\{|F|\colon F \subseteq \mathfrak{c}^{\mathfrak{c}} \;\&\; (\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{\mathfrak{c}})(\exists f \in F)(\forall g \in G)(|[f = g]| = \mathfrak{c})\}.$$

$$\kappa_1 = \min\{|F|\colon F \subseteq [\mathfrak{c}^{\mathfrak{c}}]^{\mathfrak{c}} \left(\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{<d^*_{\mathfrak{c}}}\right)(\exists A \in F)(\forall g \in G)$$
$$(\exists f \in A)(|[f = g]| < \mathfrak{c})\}$$

Lemma 20. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$, *then* $\kappa_1 \le \mathrm{A}^*(\neg\mathrm{AC})$.

PROOF. Let $\mathcal{B}$ be as in Proposition 17. Enumerate $\mathcal{B}$ injectively by $\{B_\alpha\colon \alpha \in \mathfrak{c}\}$. For each $\alpha \in \mathfrak{c}$ let $h_\alpha \in \mathbb{R}^{\mathbb{R}}$ be such that $h_\alpha|_{\mathrm{pr}_x(B_\alpha)} \subseteq B_\alpha$ and zero otherwise. Let $\{P_\alpha\}_{\alpha\in\mathfrak{c}}$ be a partition of $\mathbb{R}$ such that $P_\alpha \subseteq \mathrm{pr}_x(B_\alpha)$ and $|P_\alpha| = \mathfrak{c}$ for every $\alpha \in \mathfrak{c}$. For each $\alpha \in \mathfrak{c}$ let $\{p_{\alpha,\beta}\colon \beta \in \mathfrak{c}\}$ be an injective enumeration of P_α.

Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| = \mathrm{A}^*(\neg\mathrm{AC})$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\neg\mathrm{AC})}\right)(\exists f \in F)(f + G \subseteq \neg\mathrm{AC}). \tag{13}$$

For each $f \in F$ and $\alpha \in \mathfrak{c}$ define $f_\alpha \in \mathbb{R}^{\mathfrak{c}}$ so that $f_\alpha(\beta) = (h_\alpha - f)(p_{\alpha,\beta})$. For each $f \in F$ let $A_f = \{f_\alpha\colon \alpha \in \mathfrak{c}\}$. Put $F^* = \{A_f\colon f \in F\}$. Note that $|F^*| \le |F| = \mathrm{A}^*(\neg\mathrm{AC})$ and $F^* \subseteq [\mathbb{R}^{\mathfrak{c}}]^{\le\mathfrak{c}}$. It is enough for us to show that F^* satisfies

$$\left(\forall G \in [\mathbb{R}^{\mathfrak{c}}]^{<d^*_{\mathfrak{c}}}\right)(\exists A \in F^*)(\forall g \in G)(\exists f \in A)(|[f = g]| < \mathfrak{c}). \tag{14}$$

Let $G \in [\mathbb{R}^{\mathfrak{c}}]^{<d^*_{\mathfrak{c}}}$. Since $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$, we have $d^*_{\mathfrak{c}} = d_{\mathfrak{c}}$ by [9, Corollary 12]. It follows from Proposition 4(vii) that $|G| < \mathrm{A}(\neg\mathrm{AC})$. For every $g \in G$ define $g^* \in \mathbb{R}^{\mathbb{R}}$ so that $g^*(p_{\alpha,\beta}) = g(\beta)$. Since $|\{g^*\colon g \in G\}| \le |G| < \mathrm{A}(\neg\mathrm{AC})$, it follows by (13) that there is an $f \in F$ such that $f + g^* \notin \mathrm{AC}$ for every $g \in G$. We show that $A_f \in F^*$ has the property that

$$(\forall g \in G)(\exists h \in A_f)(|[h = g]| < \mathfrak{c}). \tag{15}$$

Fix $g \in G$. By Proposition 17 there is an $\alpha \in \mathfrak{c}$ such that $(f + g^*) \cap B_\alpha = \emptyset$. It follows that

$$(f + g^*)|_{P_\alpha} \cap h_\alpha|_{P_\alpha} = \emptyset. \tag{16}$$

Pick $f_\alpha \in A_f$. By way of contradiction assume that $f_\alpha(\beta) = g(\beta)$ for some $\beta \in \mathfrak{c}$. Then $(h_\alpha - f)(p_{\alpha,\beta}) = g(\beta) = g^*(p_{\alpha,\beta})$ but this contradicts (16). Thus, $[f_\alpha = g] = \emptyset$ so A_f satisfies (15). Therefore, F^* satisfies (14). $\square$

To continue the proof it will be useful for us to define the following families of functions. Let $\mathrm{Dar}(\mathfrak{c})$ stand for the set of all $f \in \mathbb{R}^{\mathbb{R}}$ with the property that $|f^{-1}(y) \cap (a,b)| = \mathfrak{c}$ for all $a, b, y \in \mathbb{R}$ such that $a < b$. Let Dar^* denote the set of Darboux functions f which are nowhere constant (i.e. if $a < b$, then $|f[(a,b)]| > 1$).

Lemma 21. $\mathrm{A}^*(\neg\mathrm{Dar}) = \mathrm{A}^*(\neg\mathrm{Dar}^*)$.

Proof. By [9, Lemma 25] $\mathrm{A}(\neg\mathrm{Dar}) = \mathrm{A}(\neg\mathrm{Dar}^*)$. It follws by Proposition 2(ii) that $\mathrm{A}^*(\neg\mathrm{Dar}^*) \le \mathrm{A}^*(\neg\mathrm{Dar})$.

We show the other inequality. Let $\mathcal{I}$ be the family of collections of mutually disjoint non-degenerate open intervals. Since there are continuum many open intervals and the cardinality of any disjoint collection of open intervals is at most ω, it follows that $|\mathcal{I}| = \mathfrak{c}$. For each $I \in \mathcal{I}$ pick $h_I \in \mathrm{Dar}(\mathfrak{c})$ such that $h_I(x) = 0$ if x is an endpoint of any $i \in I$. Let k_I be defined by $k_I(x) = \chi_{\cup I}(x) \cdot h_I(x)$ for each $x \in \mathbb{R}$. Let $K = \{k_I \colon I \in \mathcal{I}\}$. Note that $|K| = \mathfrak{c}$. Suppose that $F \subseteq \mathbb{R}^{\mathbb{R}}$ and $|F| < \mathrm{A}^*(\neg\mathrm{Dar})$. Then by definition of $\mathrm{A}^*(\neg\mathrm{Dar})$ there is a $G \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|G| < \mathrm{A}(\neg\mathrm{Dar}) = \mathrm{A}(\neg\mathrm{Dar}^*)$ and

$$(\forall f \in F)(\exists g \in G)(f + g \in \mathrm{Dar}). \tag{17}$$

Let $G_1 = \{g + k \colon g \in G \ \&\ k \in K\} \cup G$. Note that $|G_1| < \mathrm{A}(\neg\mathrm{Dar}^*)$ since $|K| = \mathfrak{c} < \mathrm{A}(\neg\mathrm{Dar}^*)$. It is enough to show that G_1 satisfies

$$(\forall f \in F)(\exists g \in G)(f + g \in \mathrm{Dar}^*). \tag{18}$$

Let $f \in F$. By (17) there is a $g \in G$ such that $f + g \in \mathrm{Dar}$. If $f + g \in \mathrm{Dar}^*$ there is nothing to do so assume $f + g \in \mathrm{Dar} \setminus \mathrm{Dar}^*$. The set of points at which $f + g$ is constant form a countable collection J of mutally disjoint non-degenerate open intervals such that $f + g$ is constant on each $j \in J$ and is nowhere-constant on $\mathbb{R} \setminus \bigcup J$. Since $g + k_J \in G_1$, it is enough to show that $(f + k_J) + g \in \mathrm{Dar}^*$.

We first show that $(f + k_J) + g$ is nowhere-constant. Let $x \in \mathbb{R}$ be arbitrary. If $x \in \mathrm{cl}\,(\bigcup J)$, then any open nieghborhood U about x contains a non-degenerate sub-interval i of some $j \in J$. Thus,

$$((f+g) + k_J)[U] \supseteq ((f+g) + k_J)[i] = \{r\} + k_J[i] = \{r\} + \mathbb{R} = \mathbb{R}, \tag{19}$$

where $\{r\} = (f+g)[j]$. So $(f + k_J) + g$ is not constant at x. If $x \notin \mathrm{cl}\,(\bigcup J)$, then there is a neighborhood $U \subseteq \mathbb{R} \setminus \mathrm{cl}\,(\bigcup J)$ of x such that k_J is equal to 0 on U, and $(f + k_J + g)|_U = (f+g)|_U$ which is non-constant on U. So, $f + g$ is non-constant at x. Thus, $(f + k_J) + g$ is nowhere-constant.

We now must show that $(f+k_J)+g$ is Darboux. Let $i\subseteq\mathbb{R}$ be a non-degenerate open interval. If $i\cap j\neq\emptyset$ for some $j\in J$, then i contains a non-trival sub-interval of j, so, arguing as in (19), $((f+k_J)+g)[i]=\mathbb{R}$. If $i\cap j=\emptyset$ for all $j\in J$, then $((f+k_J)+g)[i]=(f+g)[i]$. In either case $((f+k_J)+g)[i]$ is an interval. Thus, $(f+k_J)+g$ is Darboux. So, $(f+k_J)+g\in\mathrm{Dar}^*$ and G_1 satisfies (18) completing the proof. □

Lemma 22. $\mathrm{A}^*(\neg\mathrm{Dar}(\mathfrak{c}))=\mathrm{A}^*(\neg\mathrm{Dar})$.

PROOF. By [9, Lemma 27] $\mathrm{A}(\neg\mathrm{Dar})=\mathrm{A}(\neg\mathrm{Dar}(\mathfrak{c}))$. It follws by Proposition 2(ii) that $\mathrm{A}^*(\neg\mathrm{Dar}(\mathfrak{c}))\leq\mathrm{A}^*(\neg\mathrm{Dar})$.

We show the other inequality. By [9, Lemma 26] there is an additive function $\Theta\in\mathbb{R}^\mathbb{R}$ such that $\Theta\circ h\in\mathrm{Dar}(\mathfrak{c})$ for every $h\in\mathrm{Dar}^*$. Notice that Θ is a surjection. Let $F\subseteq\mathbb{R}^\mathbb{R}$ and $|F|<\mathrm{A}^*(\neg\mathrm{Dar})$. For each $f\in F$ pick $f_1\in\mathbb{R}^\mathbb{R}$ such that $\Theta\circ f_1=f$. Let $F_1=\{f_1\colon f\in F\}$. Note that $|F_1|\leq|F|<\mathrm{A}^*(\neg\mathrm{Dar})$. By the definition of $\mathrm{A}(\neg\mathrm{Dar}^*)$ and Lemma 21 there is a $G\subseteq\mathbb{R}^\mathbb{R}$ such that $|G|<\mathrm{A}(\neg\mathrm{Dar})$ and

$$(\forall f_1\in F_1)(\exists g\in G)(f_1+g\in\mathrm{Dar}^*).\tag{20}$$

For each $g\in G$ let $g_1=\Theta\circ g$. Put $G_1=\{g_1\colon g\in G\}$. Note that $|G_1|<\mathrm{A}(\neg\mathrm{Dar}(\mathfrak{c}))$. We will be done if we show that G_1 satisfies

$$(\forall f\in F)(\exists g_1\in G_1)(f+g_1\in\mathrm{Dar}(\mathfrak{c})).\tag{21}$$

Let $f\in F$. By (20) there is a $g\in G$ such that $f_1+g\in\mathrm{Dar}^*$. We now have $f+g_1=(\Theta\circ f_1)+(\Theta\circ g)=\Theta(f_1+g)\in\mathrm{Dar}(\mathfrak{c})$. Thus, G_1 satisfies (21). □

Lemma 23. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}|=\mathfrak{c}$, *then* $\mathrm{A}^*(\neg\mathrm{Dar})\leq d^1_\mathfrak{c}$.

PROOF. By Lemma 22 it is enough to prove that $\mathrm{A}^*(\neg\mathrm{Dar}(\mathfrak{c}))\leq d^1_\mathfrak{c}$. Let $F\subseteq\mathbb{R}^\mathbb{R}$ be such that $|F|=d^1_\mathfrak{c}$ and

$$\left(\forall G\in\left[\mathbb{R}^\mathbb{R}\right]^{<d_\mathfrak{c}}\right)(\exists f\in F)(\forall g\in G)(|[f=g]|<\mathfrak{c}).\tag{22}$$

It is enough for us to show that F satisfies

$$\left(\forall G\in\left[\mathbb{R}^\mathbb{R}\right]^{<\mathrm{A}(\neg\mathrm{Dar}(\mathfrak{c}))}\right)(\exists f\in F)(f+G\subseteq\neg\mathrm{Dar}(\mathfrak{c})).\tag{23}$$

Let $G\in\left[\mathbb{R}^\mathbb{R}\right]^{<\mathrm{A}(\neg\mathrm{Dar}(\mathfrak{c}))}$. Since $|[\mathfrak{c}]^{<\mathfrak{c}}|=\mathfrak{c}$ we have, by [9, Lemma 27] and Proposition 4(vii), that $|G|<d_\mathfrak{c}$. By (22) there is an $f\in F$ such that $|[f=-g]|<\mathfrak{c}$ for every $g\in G$. In particular, $|(f+g)^{-1}(\{0\})|<\mathfrak{c}$ for each $g\in G$. So $f+g\notin\mathrm{Dar}(\mathfrak{c})$ for every $g\in G$ and so F satisfies (23). □

Proof of Theorem 8 We have shown that

$$\kappa_1 \leq \mathrm{A}^*(\neg\mathrm{AC}) \leq \mathrm{A}^*(\neg\mathrm{Con}) \leq \mathrm{A}^*(\neg\mathrm{Dar}) \leq d^1_{\mathfrak{c}}.$$

So it is enough to show that $d^1_{\mathfrak{c}} \leq \kappa_1$. Let $W = \bigcup\{\mathfrak{c}^\alpha : \alpha < \mathfrak{c}\}$. Note that $|W| = \mathfrak{c}$ by our assumption that $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$. Let $V = \{\langle\alpha,\xi\rangle : \xi \leq \alpha < \mathfrak{c}\}$. Let $F \subseteq [\mathfrak{c}^V]^{\mathfrak{c}}$ be such that $|F| = \kappa_1$ and

$$\left(\forall G \in \left[\mathfrak{c}^V\right]^{<d^*_{\mathfrak{c}}}\right)(\exists A \in F)(\forall g \in G)(\exists f \in A)(|[f = g]| < \mathfrak{c}). \tag{24}$$

For each $A \in F$ let $A = \{f_\beta : \beta \in \mathfrak{c}\}$. For each $A \in F$ let $f_A \in W^{\mathfrak{c}}$ be such that $f_A(\alpha) \in \mathfrak{c}^\alpha$ and $f_A(\alpha)(\beta) = f_\beta\langle\alpha,\beta\rangle$. Let $F^* = \{f_A : A \in F\}$. Note that $|F^*| \leq |F| = \kappa_1$. It is enough for us to show that F^* satisfies

$$\left(\forall G \in [W^{\mathfrak{c}}]^{<d_{\mathfrak{c}}}\right)(\exists f \in F^*)(\forall g \in G)(|[f = g]| < \mathfrak{c}). \tag{25}$$

Let $G \subseteq W^{\mathfrak{c}}$ and $|G| < d_{\mathfrak{c}}$. For every $g \in G$ let $g_1 \in \mathfrak{c}^V$ be defined by $g_1\langle\alpha,\beta\rangle = g(\alpha)(\beta)$ for all $\beta \in \mathrm{dom}(g)$ and zero otherwise. Let $G_1 = \{g_1 : g \in G\}$ and notice that $|G_1| < d_{\mathfrak{c}} \leq d^*_{\mathfrak{c}}$. By (24) there is an $A \in F$ such that

$$(\forall g_1 \in G_1)(\exists f \in A)(|[f = g_1]| < \mathfrak{c}). \tag{26}$$

We claim that $|[f_A = g]| < \mathfrak{c}$ for every $g \in G$. Fix $g \in G$. There is an $f_\beta \in A$ such that $|[g_1 = f_\beta]| < \mathfrak{c}$. Thus, for all but strickly less than $\mathfrak{c}$-many $\alpha > \beta$ we have $g(\alpha)(\beta) = g_1\langle\alpha,\beta\rangle \neq f_\beta\langle\alpha,\beta\rangle = f_A(\alpha)(\beta)$. It follows that $|[g = f_A]| < \mathfrak{c}$. So, by (26) the claim is proved. Therefore, F^* satisfies (25). □

We now work to prove Theorem 9. Towards this end we define some other cardinals.

$$e^*_{\mathfrak{c}} = \min\{|F| : F \subseteq \mathfrak{c}^{\mathfrak{c}} \ \& \ \left(\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{\mathfrak{c}}\right)(\exists f \in F)(\forall g \in G)(|[g = f]| < \mathfrak{c})\}.$$

$$\kappa_2 = \min\{|F| : F \subseteq [\mathfrak{c}^{\mathfrak{c}}]^{\mathfrak{c}} \ \& \ \left(\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{<e^*_{\mathfrak{c}}}\right)(\exists A \in F)$$
$$(\forall g \in G)(\exists f \in A)(|[f = g]| = \mathfrak{c})\}.$$

Lemma 24. *If $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$, then $\kappa_2 \leq \mathrm{A}^*(\neg\mathrm{SZ}) \leq e^1_{\mathfrak{c}}$.*

Proof. Let H stand for the family of all functions $h \in \mathbb{R}^{\mathbb{R}}$ such that $h|_A$ is continuous for some G_δ set A of cardinality $\mathfrak{c}$ and equal to zero elsewhere. Note that $|H| = \mathfrak{c}$.

We show that $\kappa_2 \leq \mathrm{A}^*(\neg\mathrm{SZ})$. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| = \mathrm{A}^*(\neg\mathrm{SZ})$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\neg\mathrm{SZ})}\right)(\exists f \in F)(f + G \subseteq \neg\mathrm{SZ}). \tag{27}$$

For each $f \in F$ let $A_f \in [\mathbb{R}^{\mathbb{R}}]^{\mathfrak{c}}$ be defined by $\{h - f : h \in H\}$. Let $F^* = \{A_f : f \in F\}$. Notice that $|F^*| \leq |F| = \mathrm{A}^*(\neg\mathrm{SZ})$. It is enough for us to show that F^* satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<e^*_{\mathfrak{c}}}\right)(\exists A \in F^*)(\forall g \in G)(\exists f \in A)(|[f = g]| = \mathfrak{c}). \tag{28}$$

Let $G \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|G| < e^*_{\mathfrak{c}}$. Since $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$, we have by [9, Corollary 13] $e^*_{\mathfrak{c}} = \mathrm{A}(\neg\mathrm{SZ})$. So $|G| < \mathrm{A}(\neg\mathrm{SZ})$. By (27) there is an $f \in F$ such that $f + G \subseteq \neg\mathrm{SZ}$. So for each $g \in G$ there is, by Proposition 15, an $h \in H$ such that $|[f + g = h]| = \mathfrak{c}$. It follows that for every $g \in G$ there is a $k \in A_f$ such that $|[k = g]| = \mathfrak{c}$. Thus, F^* satisfies (28).

We now show that $\mathrm{A}^*(\neg\mathrm{SZ}) \leq e^1_{\mathfrak{c}}$. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| = e^1_{\mathfrak{c}}$ and

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<e_{\mathfrak{c}}}\right)(\exists f \in F)(\forall g \in G)(|[f = g]| = \mathfrak{c}). \tag{29}$$

It is enough for for us to show that F satisfies

$$\left(\forall G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{<\mathrm{A}(\neg\mathrm{SZ})}\right)(\exists f \in F)(\forall g \in G)(f + g \in \neg\mathrm{SZ}). \tag{30}$$

Let $G \subseteq \mathbb{R}^{\mathbb{R}}$ and $|G| < \mathrm{A}(\neg\mathrm{SZ})$. Notice that since $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$ we have, by [9, Corollary 13], $|G| < e_{\mathfrak{c}}$. By (29) there is an $f \in F$ such that $|[f = -g]| = \mathfrak{c}$ for all $g \in G$. So $|(f + g)^{-1}(\{0\})| = \mathfrak{c}$ which implies that $f + g \in \neg\mathrm{SZ}$. Therefore, F satisfies (30). $\square$

To finish the proof of Theorem 9 it is enough, by Lemma 24, to prove the following lemma

Lemma 25. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$, *then* $e^1_{\mathfrak{c}} = \kappa_2$.

PROOF. First notice that Lemma 24 provides the inequality $\kappa_2 \leq e^1_{\mathfrak{c}}$. We show that $e^1_{\mathfrak{c}} \leq \kappa_2$. Let $W = \bigcup\{\mathfrak{c}^{\alpha} : \alpha < \mathfrak{c}\}$. Note that $|W| = \mathfrak{c}$ by our assumption that $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$. Let $V = \{\langle \alpha, \xi \rangle : \xi \leq \alpha < \mathfrak{c}\}$. Let $F \subseteq [W^{\mathfrak{c}}]^{\mathfrak{c}}$ be such that $|F| = \kappa_2$ and

$$\left(\forall G \in [W^{\mathfrak{c}}]^{<e^*_{\mathfrak{c}}}\right)(\exists A \in F)(\forall g \in G)(\exists f \in A)(|[f = g]| = \mathfrak{c}). \tag{31}$$

For each $A \in F$ let $A = \{f_{\alpha} : \alpha \in \mathfrak{c}\}$ and define $f_A \in \mathfrak{c}^V$ by $f_A\langle \alpha, \beta \rangle = f_{\beta}(\alpha)(\beta)$ if $\beta \in \mathrm{dom}(f_{\beta}(\alpha))$ and zero otherwise. Let $F^* = \{f_A : A \in F\}$. Notice that $|F^*| \leq |F| = \kappa_2$. It is enough for us to show that F^* satisfies

$$\left(\forall G \in \left[\mathfrak{c}^V\right]^{<e_{\mathfrak{c}}}\right)(\exists f \in F)(\forall g \in G)(|[f = g]| = \mathfrak{c}). \tag{32}$$

Let $G \subseteq \mathfrak{c}^V$ be such that $|G| < e_\mathfrak{c}$. For each $g \in G$ define $g_1 \in W^\mathfrak{c}$ by $g_1(\alpha)(\beta) = g\langle\alpha,\beta\rangle$ where $\mathrm{dom}(g_1(\alpha)) = \alpha + 1$. Let $G_1 = \{g_1 : g \in G\}$ and notice that $|G_1| \leq |G| < e_\mathfrak{c} \leq e^*_\mathfrak{c}$. By (31) there is an $A \in F$ such that

$$(\forall g \in G_1)(\exists f \in A)(|[f = g]| = \mathfrak{c}). \tag{33}$$

We claim that $f_A \in F^*$ has the property that $|[f_A = g]| = \mathfrak{c}$ for every $g \in G$. Fix $g \in G$. By (33) there is an $\beta \in \mathfrak{c}$ such that $|[g_1 = f_\beta]| = \mathfrak{c}$. In particular,

$$\begin{aligned}
\mathfrak{c} =& |\{\alpha > \beta \colon f_\beta(\alpha) = g_1(\alpha)\}| \\
\leq& |\{\alpha > \beta \colon f_\beta(\alpha)(\xi) = g_1(\alpha)(\xi) \text{ for all } \xi < \alpha\}| \\
\leq& |\{\alpha > \beta \colon f_\beta(\alpha)(\beta) = g_1(\alpha)(\beta)\}| \\
\leq& |\{\alpha > \beta \colon f_A\langle\alpha,\beta\rangle = g\langle\alpha,\beta\rangle\}| \\
\leq& |[f_A = g]|.
\end{aligned}$$

Thus, F^* satisfies (32) which implies that $e^1_\mathfrak{c} \leq \kappa_2$. □

6 Proof of Theorem 10

To prove Theorem 10 we will need a few lemmas and some more cardinals.

$$\begin{aligned}
D_\mathfrak{c} &= \min\{|F| \colon F \subseteq \mathfrak{c}^\mathfrak{c} \ \& \ (\forall g \in \mathfrak{c}^\mathfrak{c})(\exists f \in F)(|[f \leq g]| < \mathfrak{c})\}, \\
b_\mathfrak{c} &= \min\{|F| \colon F \subseteq \mathfrak{c}^\mathfrak{c} \ \& \ (\forall g \in \mathfrak{c}^\mathfrak{c})(\exists f \in F)(|[g \leq f]| = \mathfrak{c})\}.
\end{aligned}$$

The numbers $b_\mathfrak{c}$ and $D_\mathfrak{c}$ are analogs of the bounding number $b = b_\omega$ and the dominating number $D = D_\omega$. We will use the following proposition from [9, Lemma 31] a number of times thoughout this section.

Proposition 26. *If $\mathfrak{c} = \lambda^+$, then the set $\{\langle\alpha,\beta\rangle \in \mathfrak{c}^2 \colon \beta \leq f(\alpha)\}$ is the union of λ-many functions in $\mathfrak{c}^\mathfrak{c}$ for every $f \in \mathfrak{c}^\mathfrak{c}$.*

Our first goal will be to show that under the assumption of $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$ and $\mathfrak{c} = \lambda^+$ we have $e_\mathfrak{c} = e^1_\mathfrak{c} = D_\mathfrak{c}$.

Lemma 27. *If $\mathfrak{c} = \lambda^+$, then $D_\mathfrak{c} = e^*_\mathfrak{c} \geq \kappa_2$.*

PROOF. By [9, Lemma 33] $e^*_\mathfrak{c} = D_\mathfrak{c}$ under the assumption of $\mathfrak{c} = \lambda^+$ so it is enough for us to show that $\kappa_2 \leq D_\mathfrak{c}$. Let $F \subseteq \mathfrak{c}^\mathfrak{c}$ be such that $|F| = D_\mathfrak{c}$ and

$$(\forall g \in \mathfrak{c}^\mathfrak{c})(\exists f \in F)(|[f \leq g]| < \mathfrak{c}). \tag{34}$$

For each $f \in F$ we may find, using Proposition 26, an $A_f \in [\mathfrak{c}^{\mathfrak{c}}]^{\lambda}$ such that $\{\langle\alpha,\beta\rangle \in \mathfrak{c}^2 \colon \beta \leq f(\alpha)\} = \bigcup A_f$. Let $F^* = \{A_f \colon f \in F\}$. Notice that $|F^*| \leq |F| = D_{\mathfrak{c}}$. It is enough for us to show that F^* satisfies

$$\left(\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{<e^*_{\mathfrak{c}}}\right)(\exists A \in F^*)(\forall g \in G)(\exists f \in A)(|[f = g]| = \mathfrak{c}). \tag{35}$$

Let $G \subseteq \mathfrak{c}^{\mathfrak{c}}$ and $|G| < e^*_{\mathfrak{c}} = D_{\mathfrak{c}}$. Since $|G| < D_{\mathfrak{c}}$ there is an $h \in \mathfrak{c}^{\mathfrak{c}}$ such that $|[g \leq h]| = \mathfrak{c}$ for every $g \in G$. By (34) there is an $f \in F$ such that $|[f \leq h]| < \mathfrak{c}$ so $|[g \leq f]| = \mathfrak{c}$ for every $g \in G$. We claim that A_f has the property that

$$(\forall g \in G)(\exists h \in A_f)(|[h = g]| = \mathfrak{c}). \tag{36}$$

Let $g \in G$. By the choice of f we have $|g \cap (\bigcup A_f)| = \mathfrak{c}$. Since $|A_f| = \lambda$ it follows that $|[g = h]| = \mathfrak{c}$ for some $h \in A_f$. Thus, A_f satisfies (36). Therefore, F^* satisfies (35). □

Lemma 28. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$ *and* $\mathfrak{c} = \lambda^+$, *then* $e_{\mathfrak{c}} = e^1_{\mathfrak{c}} = D_{\mathfrak{c}}$.

Proof. By [9, Theorem 10] and [9, Lemma 33] $e_{\mathfrak{c}} = D_{\mathfrak{c}}$. By Lemmas 25 and 27 and Theorem 6 we have $e_{\mathfrak{c}} = \mathrm{A}(\mathrm{Dar}) \leq \mathrm{A}^*(\mathrm{Dar}) = e^1_{\mathfrak{c}} \leq D_{\mathfrak{c}}$. □

Lemma 29. *If* $|[\mathfrak{c}]^{<\mathfrak{c}}| = \mathfrak{c}$ *and* $\mathfrak{c} = \lambda^+$, *then* $e_{\mathfrak{c}} = d^1_{\mathfrak{c}} = D_{\mathfrak{c}}$.

Proof. By Lemma 28 it is enough for us to show that $D_{\mathfrak{c}} = d^1_{\mathfrak{c}}$. Since $\mathfrak{c} \leq d_{\mathfrak{c}}$ it is easy to check that $e^*_{\mathfrak{c}} \leq d^1_{\mathfrak{c}}$. So by Lemma 28 we have $D_{\mathfrak{c}} \leq d^1_{\mathfrak{c}}$. All we must do now is show that $d^1_{\mathfrak{c}} \leq D_{\mathfrak{c}}$.

Let $F \subseteq \mathfrak{c}^{\mathfrak{c}}$ be such that $|F| = D_{\mathfrak{c}}$ and

$$(\forall g \in \mathfrak{c}^{\mathfrak{c}})(\exists f \in F)(|[f \leq g]| < \mathfrak{c}). \tag{37}$$

It is enough for us to show that F satisfies

$$\left(\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{<d_{\mathfrak{c}}}\right)(\exists f \in F)(\forall g \in G)(|[f = g]| < \mathfrak{c}). \tag{38}$$

Let $G \subseteq \mathfrak{c}^{\mathfrak{c}}$ and $|G| < d_{\mathfrak{c}}$. By [9, Lemma 33] $d_{\mathfrak{c}} = b_{\mathfrak{c}}$. Since $|G| < b_{\mathfrak{c}}$ there is an $h \in \mathfrak{c}^{\mathfrak{c}}$ such that $|[h \leq g]| < \mathfrak{c}$ for all $g \in G$. By (37) there is an $f \in F$ such that $|[f \leq h]| < \mathfrak{c}$. It follows that $|[f \leq g]| < \mathfrak{c}$ for every $g \in G$. In particular, we have that $|[f = g]| < \mathfrak{c}$ for every $g \in G$. Thus, F satisfies (38). □

Proof of Theorem 10 Lemmas 28 and 29 yield the equalities $e_{\mathfrak{c}} = e^1_{\mathfrak{c}} = d^1_{\mathfrak{c}}$. The inequality $d_{\mathfrak{c}} \leq d^1_{\mathfrak{c}}$ is a consequence of Proposition 4(iii) and Theorem 7. □

References

[1] J. B. Brown, P. Humke, and M. Laczkowich, *Measurable Darboux functions*, Proc. Amer. Math. Soc. **102** (1988), 603–609.

[2] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts **39**, Cambridge Univ. Press 1997.

[3] K. Ciesielski, *Set Theoretic Real Analysis*, J. Appl. Anal. **3** (1997), 143–190. (Preprint* available[1]

[4] K. Ciesielski, *Private communication.*

[5] K. Ciesielski and A. W. Miller, *Cardinal invariants concerning functions, whose sum is almost continuous*, Real Anal. Exchange **20** (1994/95), 657–672.

[6] K. Ciesielski and T. Natkaniec, *Algebraic properties of the class of Sierpiński-Zygmund functions*, Topology Appl. **79** (1997), 75–99.

[7] K. Ciesielski and I. Recław, *Cardinal invariants concerning extendableand peripherally continuous functions*, Real Anal. Exchange **21** (1995/96), 459–472.

[8] R. G. Gibson and T. Natkaniec, *Darboux-like functions*, Real Anal. Exchange **22**(**2**) (1996-97), 492–533.

[9] F. Jordan, *Cardinal invariants connected with adding real functions*, Real Anal. Exchange **22**(**2**) (1996-97), 696–713.

[10] T. Natkaniec, *Almost Continuity*, Real Anal. Exchange **17**(1991-92), 462–520.

[11] W. Sierpiński and A. Zygmund, *Sur une fonction qui est discontinue sur tout ensemble de puissance du continu*, Fund. Math. **4** (1923), 316–318

[1]Preprints marked by * are available in electronic form. They can be accessed from *Set Theoretic Analysis Web Page.*

http:/www.math.wvu.edu/homepages/kcies/STA/STA.html