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$C^{k,1}$ FUNCTIONS AND RIEMANN DERIVATIVES

Abstract

In this work we provide a characterization of $C^{k,1}$ functions of one real variable (that is, k times differentiable with locally Lipschitz k-th derivative) by means of $(k+1)$-th divided differences and Riemann derivatives. In particular we prove that the class of $C^{k,1}$ functions is equivalent to the class of functions with bounded $(k+1)$-th divided difference. From this result we deduce a Taylor's formula for this class of functions and a characterization through Riemann derivatives.

1 Introduction

In this paper we give necessary and sufficient conditions for a real function of one real variable to be of class $C^{k,1}$; that is, k times differentiable with locally Lipschitz k-th derivative. The conditions are on the boundedness of the $(k+1)$-th divided differences and of the $(k+1)$-th Riemann derivatives.

The study of the class of $C^{k,1}$ functions has been renewed since the work of Hiriart-Urruty, Strodiot and Hien Nguyen [7] who introduced the concept of generalized Hessian matrix for $C^{1,1}$ functions proving also second order optimality conditions for nonlinear constrained problems. Later, Luc [10], considering the class of $C^{k,1}$ functions, extended Taylor's formula, proved higher order optimality conditions when derivatives of order greater than k do not exist and provided characterizations of generalized convex functions.

In this section we recall some concepts which are fundamental for understanding the proofs of the results.

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1.1 Peano and Riemann Derivatives

In the following we will consider a function $f : (a, b) \to \mathbb{R}$. For such a function we let

$$\Delta_k f(x;h) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih - \frac{1}{2}kh).$$

Definition 1.1. The k-th Riemann derivative of f at a point $x \in (a, b)$ is defined as $D_k f(x) = \lim_{h\to 0} \Delta_k f(x;h)/h^k$, if this limit exists.

Similarly we can define differences

$$\delta_k f(x;h) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih).$$

The corresponding k-th Riemann-type derivative is denoted by $d_k f(x)$ and is defined as $d_k f(x) = \lim_{h\to 0} \delta_k f(x;h)/h^k$.

We will also consider differences $\tilde{\Delta}_k f(x;h)$ defined recursively by

$$\tilde{\Delta}_1 f(x;h) = f(x+h) - f(x),\ \tilde{\Delta}_k f(x;h) = \tilde{\Delta}_{k-1} f(x;2h) - 2^{k-1}\tilde{\Delta}_{k-1} f(x;h).$$

As observed in [11], we have

$$\tilde{\Delta}_k f(x;h) = a_k f(x + 2^{k-1}h) + a_{k-1} f(x + 2^{k-2}h) + \cdots + a_1 f(x+h) + a_0 f(x),$$

where, for any fixed k, a_j depends only on j $(j = 0, 1, \ldots k-1)$ and $a_k = 1$.

Lemma 1.1. *[11] There are constants $C_0, C_1, \ldots, C_{2^{k-1}-k}$ such that*

$$\tilde{\Delta}_k f(x;h) = \sum_{i=0}^{2^{k-1}-k} C_i \Delta_k f(x + \frac{1}{2}kh + ih; h).$$

The proof of the following lemma is straightforward from the previous result.

Lemma 1.2. *If there exist neighborhoods U of the point x_0 and V of the origin such that $\frac{\Delta_k f(x;h)}{h^k}$ is bounded on $U \times V\backslash\{0\}$, then there exist neighborhoods U' of x_0 and V' of the origin such that $\frac{\tilde{\Delta}_k f(x;h)}{h^k}$ is bounded on $U' \times V'\backslash\{0\}$.*

The proof of the following lemma is similar to that of Lemma 6 in [11].

Lemma 1.3. *Assume that f is bounded in a neighborhood of the point x_0. If there exist neighborhoods U of the point x_0 and V of the origin such that $\dfrac{\tilde{\Delta}_k f(x;h)}{h^k}$ is bounded on $U \times V\backslash\{0\}$, then also $\dfrac{\tilde{\Delta}_{k-1} f(x;h)}{h^{k-1}}$ is bounded on $U \times V\backslash\{0\}$.*

PROOF. From the hypotheses we obtain that there exists a number $\delta > 0$ such that $\forall x \in U$ and $\forall h$ with $|h| \le \delta, h \ne 0$, the following inequalities hold.

$$\left|\tilde{\Delta}_{k-1} f(x;h) - 2^{k-1}\tilde{\Delta}_{k-1} f(x;h/2)\right| \le M|h/2|^k,$$
$$\left|\tilde{\Delta}_{k-1} f(x;h/2) - 2^{k-1}\tilde{\Delta}_{k-1} f(x;h/4)\right| \le M|h/4|^k, \dots$$
$$\left|\tilde{\Delta}_{k-1} f(x;h/2^{n-1}) - 2^{k-1}\tilde{\Delta}_{k-1} f(x;h/2^n)\right| \le M|h/2^n|^k.$$

Multiplying these inequalities by $1, 2^{k-1}, 2^{2(k-1)}, \dots, 2^{(n-1)(k-1)}$ respectively, we obtain by addition

$$\left|\tilde{\Delta}_{k-1} f(x;h) - 2^{n(k-1)}\tilde{\Delta}_{k-1} f(x;h/2^n)\right| \le 2M|h/2|^k,$$

and hence

$$\left|\frac{2^{n(k-1)}\tilde{\Delta}_{k-1} f(x;h/2^n)}{h^{k-1}}\right| \le M'$$

for $\frac{1}{2}\delta \le |h| \le \delta$, by using the boundedness of f. Hence, writing $\xi = h/2^n$, we have

$$\left|\frac{\tilde{\Delta}_{k-1} f(x;\xi)}{\xi^{k-1}}\right| \le M' \quad \text{for } \delta/2^{n+1} \le |\xi| \le \delta/2^n, \ n = 0, 1, \dots,$$

and the lemma is established, since n can be chosen arbitrarily. □

Definition 1.2. If there exist numbers $f_1(x), \dots, f_k(x)$ such that

$$f(x+h) = f(x) + f_1(x)h + \frac{1}{2} f_2(x)h^2 + \cdots + \frac{1}{k!} f_k(x)h^k + o(h^k),$$

where $o(h^k)/h^k \to 0$ as $h \to 0$, then f is said to have a k-th Peano derivative at x. The number $f_k(x)$ is called the k-th Peano derivative of f at x.

We say that f admits k-th Peano derivative on an interval when it admits k-th Peano derivative at any point of this interval.

It is well known that the existence of the ordinary k-th derivative of f at x, $f^{(k)}(x)$, implies the existence of $f_k(x)$ and this in turn implies the existence of $D_k f(x)$.

Lemma 1.4. *[11] If $f_k(x)$ exists, then so does $\lim_{h\to 0} \dfrac{\tilde{\Delta}_k f(x;h)}{h^k}$ and there exists a number λ_k, depending only on k, such that $\lambda_k \lim_{h\to 0} \dfrac{\tilde{\Delta}_k f(x;h)}{h^k} = f_k(x)$.*

For a survey on Riemann and Peano derivatives one can see for instance [2], [6] and [12]. Further properties of Peano and Riemann derivatives are given in [3], [4] and [5]. In this paper we will need the following result.

Theorem 1.1. *[12] If f_k is bounded (upper or lower) on an interval, then $f^{(k)}$ exists on this interval and $f^{(k)} = f_k$.*

1.2 Standard Mollifiers

The function ϕ, defined by

$$\phi(x) = \begin{cases} C \exp(\frac{1}{x^2-1}), & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

is $C^\infty(\mathbb{R})$ and we can choose the constant $C \in \mathbb{R}$ such that $\int_{\mathbb{R}} \phi(x)\, dx = 1$.

Definition 1.3. Let $\varepsilon > 0$. The functions $\phi_\varepsilon(x) = \dfrac{\phi(\frac{x}{\varepsilon})}{\varepsilon}$ are called standard mollifiers.

Definition 1.4. Let $f : (a,b) \to \mathbb{R}$. We say that $f \in C_0^k((a,b))$ if $f \in C^k((a,b))$ and

$$\mathrm{spt}_f = \overline{\{x \in (a,b) : f(x) \neq 0\}} \subset (a,b).$$

Theorem 1.2. *[1] The functions ϕ_ε are $C^\infty(\mathbb{R})$ and satisfy*

i) $\int_{\mathbb{R}} \phi_\varepsilon(x)dx = 1$

ii) $\mathrm{spt}_{\phi_\varepsilon} \subset B(0,\varepsilon)$.

For a bounded function $f : (a,b) \to \mathbb{R}$, and $\varepsilon > 0$ we define functions f_ε by the formula $f_\varepsilon(x) = \int_a^b \phi_\varepsilon(y-x) f(y)\, dy$. Observe that $f_\varepsilon(x) = 0$ if $x \in \mathbb{R} \backslash [a-\varepsilon, b+\varepsilon]$ and that $f_\varepsilon \in C^\infty(\mathbb{R})$.

Theorem 1.3. *[1] Suppose that $f \in L^1_{loc}(a,b)$. Then for a.e. $x \in (a,b)$ we have $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$. If $f \in C((a,b))$, then the convergence is uniform on compact subsets of (a,b).*

Theorem 1.4. *[9] Let $[c,d] \subset (a,b)$. Then $\exists\, \varepsilon_0 > 0$ such that $\forall \varepsilon \leq \varepsilon_0$ and $\forall x \in [c,d]$ the function $y \to \phi_\varepsilon(x-y)$ is $C_0^\infty((a,b))$.*

2 The Main Results

Definition 2.1. A function $f : (a,b) \to \mathbb{R}$ is locally Lipschitz at x_0 when there exists a constant K and a neighborhood U of x_0 such that $|f(x) - f(y)| \leq K|x-y|$, whenever $x, y \in U$.

Definition 2.2. A function $f : (a,b) \to \mathbb{R}$ is of class $C^{k,1}$ at x_0 when $f^{(k)}$ exists in a neighborhood of x_0 and $f^{(k)}$ is locally Lipschitz at x_0.

Theorem 2.1. *Assume that the function $f : (a,b) \to \mathbb{R}$ is bounded on a neighborhood of the point $x_0 \in (a,b)$. Then f is of class $C^{k,1}$ at x_0 if and only if there exist neighborhoods U of x_0 and V of 0 such that $\dfrac{\Delta_{k+1} f(x;h)}{h^{k+1}}$ is bounded on $U \times V \backslash \{0\}$.*

PROOF. i) Sufficiency: From Lemmas 1.2 and 1.3, we have that the boundedness of $\dfrac{\Delta_{k+1} f(x;h)}{h^{k+1}}$ on $U \times V \backslash \{0\}$ implies the existence of neighborhoods U' of x_0 and V' of 0 such that $\dfrac{\tilde{\Delta}_j f(x;h)}{h^j}$ are bounded on $U' \times V' \backslash \{0\}, \forall j = 1, \ldots, k$. Observe that the boundedness of $\dfrac{\tilde{\Delta}_1 f(x;h)}{h}$ means that f is locally Lipschitz at the point x_0 and hence continuous in a neighborhood of x_0. For every x in a neighborhood of x_0 and for ε "sufficiently small", recalling Lemma 1.4 and Theorem 1.4, and using the Lebesgue convergence theorem, for $1 \leq j \leq k$ we have

$$\begin{aligned}
f_\varepsilon^{(j)}(x) &= (-1)^j \int_a^b \phi_\varepsilon^{(j)}(y-x) f(y)\, dy \\
&= (-1)^j \lambda_j \int_a^b \lim_{h\to 0} \frac{\tilde{\Delta}_j \phi_\varepsilon(y-x;h)}{h^j} f(y)\, dy \\
&= (-1)^j \lambda_j \int_a^b \lim_{h\to 0} \frac{\sum_{i=1}^j a_i \phi_\varepsilon(y-x+2^{i-1}h) + a_0 \phi_\varepsilon(y-x)}{h^j} f(y)\, dy \\
&= (-1)^j \lambda_j \lim_{h\to 0} \int_a^b \frac{\sum_{i=1}^j a_i \phi_\varepsilon(y-x+2^{i-1}h) + a_0 \phi_\varepsilon(y-x)}{h^j} f(y)\, dy.
\end{aligned}$$

Now, putting $z = y + 2^{i-1}h$, we obtain

$$\int_a^b \frac{a_i \phi_\varepsilon(y-x+2^{i-1}h)}{h^j} f(y)\, dy = \int_{a+2^{i-1}h}^{b+2^{i-1}h} \frac{a_i f(z-2^{i-1}h)\phi_\varepsilon(z-x)}{h^j}\, dz.$$

Thus

$$(-1)^j \lambda_j \int_a^b \frac{\sum_{i=1}^j a_i \phi_\varepsilon(y-x+2^{i-1}h) + a_0 \phi_\varepsilon(y-x)}{h^j} f(y)\, dy$$

$$=(-1)^j\lambda_j\sum_{i=1}^{j}\int_{a+2^{i-1}h}^{b+2^{i-1}h}\frac{a_i f(z-2^{i-1}h)\phi_\varepsilon(z-x)}{h^j}\,dz$$
$$+(-1)^j\lambda_j\int_a^b\frac{a_0 f(z)\phi_\varepsilon(z-x)}{h^j}\,dz.$$

For $|h|<\dfrac{b-a}{2^{k-1}}$, from Theorem 1.4 for all "sufficiently small" ε, the previous equation is equal to

$$(-1)^j\lambda_j\sum_{i=1}^{j}\int_a^b\frac{a_i f(z-2^{i-1}h)\phi_\varepsilon(z-x)}{h^j}\,dz$$
$$+(-1)^j\lambda_j\int_a^b\frac{a_0 f(z)\phi_\varepsilon(z-x)}{h^j}\,dz$$
$$=(-1)^j\lambda_j\int_a^b\frac{\tilde{\Delta}_j f(z,-h)}{h^j}\phi_\varepsilon(z-x)\,dz=\lambda_j\int_a^b\frac{\tilde{\Delta}_j f(z,-h)}{(-h)^j}\phi_\varepsilon(z-x)\,dz.$$

Hence we get $f_\varepsilon^{(j)}(x)=\lambda_j\lim_{h\to 0}\int_a^b\frac{\tilde{\Delta}_j f(z,h)}{h^j}\phi_\varepsilon(z-x)\,dz$. From the boundedness of $\dfrac{\tilde{\Delta}_j f(x,h)}{h^j}$ we get the existence of a constant M such that $\left|f_\varepsilon^{(j)}(x)\right|\le M$, for every ε "sufficiently small" and for every x in a neighborhood of x_0. In this way we established that $f_\varepsilon^{(j)}(x)$ is bounded (uniformly with respect to ε) on a neighborhood of x_0, $\forall j=1,\ldots,k$. Similarly one can prove that $f_\varepsilon^{(k+1)}(x)$ is bounded on a neighborhood of x_0. Hence, there exists a neighborhood $\tilde{U}$ of x_0 such that for $x\in\tilde{U}$ there is a sequence ε_n converging to 0 such that for all $j=1,...,k$, the sequence $f_{\varepsilon_n}^{(j)}(x)$ converges to a limit which we denote by $\alpha_j(x)$. Notice that the functions $\alpha_j(x)$, $j=1,\ldots,k$ are bounded on $\tilde{U}$.
The functions $f_{\varepsilon_n}(x)$ are of class C^∞ and hence $\forall x,y\in\tilde{U}$

$$f_{\varepsilon_n}(y)=f_{\varepsilon_n}(x)+\sum_{i=1}^{k}\frac{f_{\varepsilon_n}^{(i)}(x)}{i!}(y-x)^i+\frac{f_{\varepsilon_n}^{(k+1)}(\xi_n)}{(k+1)!}(y-x)^{k+1},$$

where $\xi_n\in(x,y)$. Recalling Theorem 1.3, taking the limit for $n\to+\infty$ it follows that $f_{\varepsilon_n}^{(k+1)}(\xi_n)$ converges to a limit which we denote by $\beta(x,y)$. Moreover

$$f(y)=f(x)+\sum_{i=1}^{k}\frac{\alpha_i(x)}{i!}(y-x)^i+\frac{1}{(k+1)!}\beta(x,y)(y-x)^{k+1}.$$

Observing that $\beta(x,y)$ is bounded for $x,y\in\tilde{U}$, we have that $\forall x\in\tilde{U},\alpha_k(x)$ is the k-th Peano derivative of f at x. From Theorem 1.1 it follows that

$\alpha_k(x) = f^{(k)}(x), \forall x \in \tilde{U}$. Furthermore the functions $f_\varepsilon^{(k+1)}$ are bounded uniformly with respect to ε, for ε "sufficiently small" and thus the functions $f_{\varepsilon_n}^{(k)}$ satisfy the following uniform Lipschitz condition

$$\left| f_{\varepsilon_n}^{(k)}(y) - f_{\varepsilon_n}^{(k)}(x) \right| \le B \left| y - x \right|, \forall x, y \in \tilde{U}.$$

Since $f_{\varepsilon_n}^{(k)}(x)$ and $f_{\varepsilon_n}^{(k)}(y)$ converge to $f^{(k)}(x)$ and $f^{(k)}(y)$ respectively, we see that $f^{(k)}$ is Lipschitzian on $\tilde{U}$.
ii) Necessity: Assume that f is of class $C^{k,1}$ at x_0. Set

$$\overline{\Delta}_1 f(x; s_1) = f(x + s_1) - f(x),$$

and recursively define

$$\overline{\Delta}_{k+1} f(x; s_1, \dots, s_{k+1}) = \overline{\Delta}_k f(x + s_{k+1}; s_1, \dots, s_k) - \overline{\Delta}_k f(x; s_1, \dots, s_k),$$

where $x \in (a, b), s_i \in \mathbb{R}, i = 1, \dots, k+1$ and $|s_i|$ is "sufficiently small". Applying the mean value theorem k times we get

$$\frac{\overline{\Delta}_{k+1} f(x; s_1, \dots, s_{k+1})}{s_{k+1} s_k \cdots s_1} = \frac{(\overline{\Delta}_k f)'(x + \theta_{k+1} s_{k+1}; s_1, \dots, s_k)}{s_k \cdots s_1}$$
$$\cdots = \frac{\overline{\Delta}_1 f^{(k)}(x + \theta_{k+1} s_{k+1} + \cdots + \theta_2 s_2; s_1)}{s_1},$$

where $\theta_i \in (0, 1), i = 2, \dots, k+1$. Since f is of class $C^{k,1}$ at x_0, there exist a constant M, a neighborhood $\tilde{U}$ of x_0 and a number $\delta > 0$ such that

$$\left| \frac{\overline{\Delta}_{k+1} f(x; s_1, \dots, s_{k+1})}{s_{k+1} s_k \cdots s_1} \right| \le M, \forall x \in \tilde{U},\ |s_i| < \delta,\ s_i \ne 0,\ i = 1, \dots, k+1.$$

Now the assertion follows easily observing that if $s_1 = s_2 = \cdots = s_{k+1} = h$, then $\overline{\Delta}_{k+1} f(x; s_1, \dots, s_{k+1}) = \delta_{k+1} f(x; h) = \Delta_{k+1} f(x + \frac{k+1}{2} h; h)$. □

Corollary 2.1. *Assume that the function f is bounded on a neighborhood of x_0. Then f is of class $C^{k,1}$ at x_0 if and only if there exist neighborhoods U of x_0 and V of 0 such that $\frac{\delta_{k+1} f(x; h)}{h^{k+1}}$ is bounded on $U \times V \backslash \{0\}$.*

PROOF. The proof is straightforward remembering that

$$\delta_{k+1} f(x; h) = \Delta_{k+1} f(x + \frac{k+1}{2} h; h).$$

□

Corollary 2.2. *(Taylor's formula). If f is of class $C^{k,1}$ at x_0, there exist sequences ε_n converging to 0 and $\xi_n \in (x_0, x_0+h)$ such that $f_{\varepsilon_n}^{(k+1)}(\xi_n)$ converges to a limit $\beta(x_0, x_0+h)$ and*

$$f(x_0+h) = f(x_0) + \sum_{i=1}^{k} \frac{f^{(i)}(x_0)}{i!} h^i + \frac{\beta(x_0, x_0+h)}{(k+1)!} h^{k+1}.$$

PROOF. It is included in the proof of the previous theorem. □

Theorem 2.2. *Assume that f is continuous and $D_{k+1}f(x)$ exists on a neighborhood of the point x_0. Then f is of class $C^{k,1}$ at x_0 if and only if $D_{k+1}f(x)$ is bounded on a neighborhood U of x_0 and there exists a function $g \in L^1(U)$ such that $\left| \frac{\Delta_{k+1} f(x;h)}{h^{k+1}} \right| \leq g(x)$, for every $x \in U$ and h in a neighborhood of 0 $(h \neq 0)$.*

PROOF. i) Sufficiency. Arguing in a fashion similar to that of the previous theorem and using Lebesgue's theorem, we obtain for ε "sufficiently small" and for every x in a neighborhood of x_0

$$\begin{aligned} f_\varepsilon^{(k+1)}(x) &= \lambda_{k+1} \lim_{h\to 0} \int_a^b \frac{\Delta_{k+1} f(z;h)}{h^{k+1}} \phi_\varepsilon(z-x)\, dz \\ &= \lambda_{k+1} \int_a^b \lim_{h\to 0} \frac{\Delta_{k+1} f(z;h)}{h^{k+1}} \phi_\varepsilon(z-x)\, dz \\ &= \lambda_{k+1} \int_a^b D_{k+1} f(z) \phi_\varepsilon(z-x)\, dz. \end{aligned}$$

It follows that $f_\varepsilon^{(k+1)}(x)$ is bounded by a constant M on a neighborhood x_0 (uniformly with respect to ε). Using the integral representation of divided differences (see for instance [8], Ch. 6, Theorem 2), we have

$$\begin{aligned} &\frac{\Delta_{k+1} f_\varepsilon(x;h)}{h^{k+1}} \\ &\qquad = \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_k} f_\varepsilon^{(k+1)}(t_{k+1}h + \cdots + t_1 h + x - \frac{k}{2}h)\, dt_{k+1}. \end{aligned}$$

For x and h in suitable neighborhoods respectively of x_0 and of 0, the left member in the previous inequality is bounded by a constant M. Sending ε to 0 and recalling Theorem 1.3, we get the existence of neighborhoods U of x_0 and V of 0 such that $\frac{\Delta_{k+1} f(x;h)}{h^{k+1}}$ is bounded on $U \times V \backslash \{0\}$. The assertion

now follows recalling Theorem 2.1.
ii) Necessity. The proof is similar to that of the necessary condition in Theorem 2.1. □

Remark 2.1. Theorems 2.1 and 2.2 extend the elementary condition which relates the Lipschitz condition on $f^{(k)}$ and the boundedness of $f^{(k+1)}$. We generalize this relation without requiring any differentiability hypothesis and linking the existence and the Lipschitz behavior of $f^{(k)}$ to the boundedness of $\dfrac{\Delta_{k+1} f(x,h)}{h^{k+1}}$ or of the Riemann derivatives.

Remark 2.2. It is well known [11] that if for every $x \in (a,b)$, $\Delta_{k+1} f(x,h) = O(h^{k+1})$, then f_{k+1} exists a.e. $x \in (a,b)$. In Theorem 2.1, under the stronger hypothesis that $\dfrac{\Delta_{k+1} f(x,h)}{h^{k+1}}$ is bounded on a rectangle as a function of x and h, we prove that $f^{(k)}$ exists on an interval and furthermore is Lipschitz.

Remark 2.3. Conditions similar to those of Theorem 2.2, expressed in terms of $d_{k+1} f(x)$ can be proved analogously.

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