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EVERY ALMOST CONTINUOUS FUNCTION IS POLYGONALLY ALMOST CONTINUOUS

Abstract

We show that every almost continuous function $f\colon \mathbb{I} \to \mathbb{R}$ is also polygonally almost continuous. This solves a problem posed by Agronski, Ceder and Pearson (see [ACP]).

1 Preliminaries

By $\mathbb{R}$ we denote the set of all reals, by $\mathbb{I}$ we denote the interval $[0, 1]$. For every set A, by $\mathrm{cl}(A)$ we will denote closure of A.

We will consider following classes of functions from the interval $\mathbb{I}$ to $\mathbb{R}$:

AC A function $f\colon \mathbb{I} \to \mathbb{R}$ is almost continuous (AC) if whenever $U \subset \mathbb{I} \times \mathbb{R}$ is an open set containing the graph of f, then U contains the graph of a continuous function $g\colon \mathbb{I} \to \mathbb{R}$.

PAC A function $f\colon \mathbb{I} \to \mathbb{R}$ is polygonally almost continuous (PAC) if whenever $U \subset \mathbb{I} \times \mathbb{R}$ is an open set containing the graph of f, then U contains the graph of a polygonal (piecewise linear continuous) function $h\colon \mathbb{I} \to \mathbb{R}$ with all its vertices on f.

D A function $f\colon \mathbb{I} \to \mathbb{R}$ is Darboux (D) if the image of $C \subset \mathbb{I}$ is connected in $\mathbb{R}$ whenever C is connected in $\mathbb{I}$.

For properties of these and other Darboux-like classes see e. g. the survey [GN]. In particular, it is known that AC $\subset$ D. Clearly every PAC function is

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AC. Recently Agronsky, Ceder and Pearson asked whether the opposite implication holds ([ACP]). In this note we give a positive answer this question. (We would like to thank Professor Kenneth Kellum for drawing the author's attention to this problem. In particular, Kellum proved that every extendable function as well as every AC function with dense graph is PAC (private communication)).

2 The Result

Theorem 1. *Every* AC *function* $f\colon \mathbb{I} \to \mathbb{R}$ *is* PAC.

PROOF. Let $f\colon \mathbb{I} \to \mathbb{R}$ be AC. Suppose, f is not PAC. For every $x \in \mathbb{I}$, let

$$H_x = \{h\colon [0,x] \to \mathbb{R} \mid h \text{ is polygonally continuous with all its vertices on } f\}.$$

Let $G \subset \mathbb{I} \times \mathbb{R}$ be an open set such that $f \subset G$ and there does not exist a polygonal function $h_1 \in H_1$, $h_1 \subset G$. Define:

- $E = \{(x,y) \in f \mid (\exists h_x \in H_x)\ h_x \subset G\}$;
- $N = \{(x,y) \in f \mid (\neg\exists h_x \in H_x)\ h_x \subset G\}$.

Clearly $E \cup N = f$, $E \cap N = \emptyset$, $(0, f(0)) \in E$, and by the supposition $(1, f(1)) \in N$.

For $S_{(x,y)}$ being an open square with center (x,y) let $3 \cdot S_{(x,y)}$ denote the open square with the center (x,y) and with the diagonal 3 times that of $S_{(x,y)}$. For every $(x,y) \in f$ let $S_{(x,y)}$ be an open square with the center (x,y) such that:

(S1) $3 \cdot S_{(x,y)} \subset G \cap ((0,1) \times \mathbb{R})$ for $x \in (0,1)$,
$3 \cdot S_{(0,f(0))} \cap [0,+\infty) \times \mathbb{R} \subset G$,
$3 \cdot S_{(1,f(1))} \cap (-\infty, 1] \times \mathbb{R} \subset G$;

(S2) either $S_{(x,y)} \cap ([0,x) \times \mathbb{R}) \cap f \subset E$ or $S_{(x,y)} \cap ([0,x) \times \mathbb{R}) \cap f \subset N$, for $x > 0$;

(S3) either $S_{(x,y)} \cap ((x,1] \times \mathbb{R}) \cap f \subset E$ or $S_{(x,y)} \cap ((x,1] \times \mathbb{R}) \cap f \subset N$, for $x < 1$.

Such a $S_{(x,y)}$ exists for every $(x,y) \in f$. Indeed, suppose for example, there exists $(x,y) \in f$ such that for every $S_{(x,y)} \subset G$ there exist $x_1 \in [0,x)$ and $x_2 \in [0,x))$ such that $(x_1, f(x_1)) \in E \cap S_{(x,y)}$ and $(x_2, f(x_2)) \in N \cap S_{(x,y)}$. Now we can find $(p, f(p)) \in E \cap S_{(x,y)} \cap ([0,x) \times \mathbb{R})$ and $(q, f(q)) \in N \cap S_{(x,y)} \cap ((p,x) \times \mathbb{R})$. Then $p < q$, but $(p, f(p)) \to (q, f(q)) \subset S_{(x,y)} \subset G$, with $\alpha \to \beta$ denoting the

line segment linking α and β for every $\alpha \in \mathbb{I} \times \mathbb{R}$, $\beta \in \mathbb{I} \times \mathbb{R}$, $\alpha < \beta$. So, we can extend the polygonal function $h_p \in H_p$, $h_p \subset G$ to a polygonal function $h_q \in H_q$, $h_q \subset G$, contrary to $q \in N$.

For every $(x, y) \in f$ let $R_{(x,y)} \subset S_{(x,y)}$ be an open rectangular neighborhood of (x, y) fulfilling the following conditions (with x_l denoting $\inf\{a \mid (a, b) \in R_{(x,y)}\}$, x_r denoting $\sup\{a \mid (a, b) \in R_{(x,y)}\}$, y_l denoting $\inf\{b \mid (a, b) \in R_{(x,y)}\}$, y_u denoting $\sup\{b \mid (a, b) \in R_{(x,y)}\}$):

(1) If $x > 0$ and $S_{(x,y)} \cap ([0, x) \times \mathbb{R}) \cap f \subset E$, then $f(x_l) \in (y_l, y_u)$.

(2) If $x < 1$ and $S_{(x,y)} \cap ((x, 1] \times \mathbb{R}) \cap f \subset N$, then $f(x_r) \in (y_l, y_u)$.

Such a $R_{(x,y)}$ always exists, because (x, y) is a left side limit point of f for every $x \in (0, 1]$ and (x, y) is a right side limit point of f for every $x \in [0, 1)$ (f is Darboux; so it satisfies Young's condition, see e. g. [GN]).

Note also that $(x_l, f(x_l))$ is a right side limit point of f; so if $R_{(x,y)} \cap ([0, x) \times \mathbb{R}) \cap f \subset E$, then (from (1)) for every $a \in (x_l, x) \cap \mathbb{I}$ there exists $b < a$ such that $(b, f(b)) \in E \cap R_{(x,y)}$.

Because $(x_r, f(x_r))$ is a left side limit point of f, if $R_{(x,y)} \cap ((x, 1] \times \mathbb{R}) \cap f \subset N$, then (from (2)) for every $a \in (x_l, x_r) \cap \mathbb{I}$ there exists $c > a$ such that $(c, f(c)) \in N \cap R_{(x,y)}$; so $R_{(x,y)} \cap ((a, x_r) \times \mathbb{R}) \cap f \not\subset E$. Now for every $R_{(x,y)}$ we have:

(A) If $R_{(x,y)} \cap ((a, x) \times \mathbb{R}) \cap f \subset E$ for some $a \in (x_l, x) \cap \mathbb{I}$, then $R_{(x,y)} \cap ([0, x) \times \mathbb{R}) \cap f \subset E$, and there exists $b < a$ such that $(b, f(b)) \in E \cap R_{(x,y)}$.

(B) If $R_{(x,y)} \cap ((a, x_r) \times \mathbb{R}) \cap f \subset E$ for some $a \in (x_l, x_r) \cap \mathbb{I}$, then $R_{(x,y)} \cap ((x, 1] \times \mathbb{R}) \cap f \subset E$, and there exists $b < a$ such that $(b, f(b)) \in E \cap R_{(x,y)}$.

Let $H = \bigcup_{(x,y) \in f} R_{(x,y)}$. H is open, $H \subset G$ and $f \subset H$. So, there exists a continuous function $g \colon \mathbb{I} \to \mathbb{R}$, $g \subset H$. Because the graph of g is compact, there exists a finite family of sets $\mathcal{R} \subset \{R_{(x,y)} \mid (x, y) \in f\}$ such that $g \subset \bigcup \mathcal{R}$. $R_{(0,f(0))} \in \mathcal{R}$ and $R_{(1,f(1))} \in \mathcal{R}$. Indeed, for every $0 < x < 1$ we have $3 \cdot S_{(x,y)} \subset G \cap ((0, 1) \times \mathbb{R})$; so only the square $S_{(0,f(0))}$ contains points with abscissa 0, and only the square $S_{(1,f(1))}$ contains points with abscissa 1. Moreover, since $\mathcal{R}$ is finite, $\sup\{x \in \mathbb{I} \mid (x, y) \in \bigcup(\mathcal{R} \setminus \{R_{(1,f(1))}\})\} < 1$. Let

$$C = \{x \in \mathbb{I} \mid (\exists R \in \mathcal{R})\ ((x, g(x)) \in R \text{ and } (\exists x_1 \leq x)\ (x_1, f(x_1)) \in E \cap R)\},$$

let $s = \sup C$. From (S3) $R_{(0,f(0))} \cap f \subset E$; so $(0, g(0)) \in R_{(0,f(0))}$, $C \neq \emptyset$ and s is well defined. Since (from (S2)) $R_{(1,f(1))} \cap f \subset N$, $0 < s < 1$.

$\mathcal{R}$ is finite and g is continuous, so there exists $R_{(a,b)} \in \mathcal{R}$ such that $(s, g(s)) \in \mathrm{cl}(R_{(a,b)})$ and there exists $x_1 \leq s$ such that $(x_1, f(x_1)) \in E \cap R_{(a,b)}$.

There exists also an open set $R_{(c,d)} \in \mathcal{R}$ such that $(s, g(s)) \in R_{(c,d)}$. Then $(s, g(s)) \in \mathrm{cl}(R_{(a,b)}) \cap R_{(c,d)}$; so $R_{(a,b)} \cap R_{(c,d)} \neq \emptyset$. Note that $R_{(a,b)} \cup R_{(c,d)} \subset 3 \cdot S_{(p,q)}$, for $S_{(p,q)}$ being this square from $S_{(a,b)}$ and $S_{(c,d)}$ which has greater diameter. Therefore we can connect every two points α, β of $R_{(a,b)} \cup R_{(c,d)}$ by the line segment $\alpha \to \beta$ whole contained in $3 \cdot S_{(p,q)} \subset G$.

In $R_{(c,d)}$ we can find $x_2 > s$ such that $(x_2, f(x_2)) \in N$. Indeed, suppose

$$R_{(c,d)} \cap ((s, 1] \times \mathbb{R}) \cap f \subset E. \qquad (\star)$$

$R_{(c,d)}$ is open and g is continuous; so there exists $t > s$ such that $(t, g(t)) \in R_{(c,d)}$. We have two cases:

1. If $t \leq c$, then $s < c$, and from $(\star)$ we have $R_{(c,d)} \cap ((s, c) \times \mathbb{R}) \cap f \subset E$. From (A) we have $R_{(c,d)} \cap ([0, c) \times \mathbb{R}) \cap f \subset E$ and $(\exists w \leq t)\ (w, f(w)) \in E \cap R_{(c,d)}$. But now $t \in C$, $\sup C \geq t > s$, a contradiction.

2. If $t > c$, then from $(\star)$ and (B) we have $R_{(c,d)} \cap ((c, 1] \times \mathbb{R}) \cap f \subset E$ and $(\exists w \leq t)\ (w, f(w)) \in E \cap R_{(c,d)}$. Now $t \in C$, $\sup C \geq t > s$, a contradiction.

Now we have $x_1 < x_2$, $(x_1, f(x_1)) \in E \cap R_{(a,b)}$, $(x_2, f(x_2)) \in N \cap R_{(c,d)}$; so we can extend the polygonal function $h_{x_1} \in H_{x_1}$, $h_{x_1} \subset G$ via the line segment $(x_1, f(x_1)) \to (x_2, f(x_2))$ contained in $3 \cdot S_{(p,q)} \subset G$ to a polygonal function $h_{x_2} \in H_{x_2}$, $h_{x_2} \subset G$. Thus we have $(x_2, f(x_2))$ belongs to E rather than N. This is a contradiction. □

The following corollary gives a full answer to the question from [ACP].

Corollary 1. *Given any $\varepsilon > 0$ and any open neighborhood G of an almost continuous function f, there exists a polygonal function h with the length of the longest line segment less than ε, such that $h \subset G$ and all vertices of h belong to f.*

PROOF. It is easy to modify previous proof, such that the length of every line segment of polygonal function $h \subset G$ will be less than ε. □

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