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S. I. Othman* and V. Anandam*, Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh - 11451, Saudi Arabia.
e-mail: `sadoon@ksu.edu.sa` and `vanandam@ksu.edu.sa`

SINGULARITIES OF BOUNDED HARMONIC FUNCTIONS

Abstract

In a harmonic space, the property that k is a compact set of removable singularities for bounded harmonic functions defined in a neighborhood of k is independent of the neighborhood chosen.

1 Introduction

Let k be compact and w be open such that $k \subset w \subset \mathbb{R}^n$ $(n \geq 2)$. Suppose that any bounded harmonic function in $w \setminus k$ extends as a harmonic function in w. Then, it can be shown (Theorem 9.9, Axler et al [4]) that for any open set $w_0 \supset k$, if u is bounded harmonic in $w_0 \setminus k$, u extends harmonically in w_0. Using a slightly different method, it is proved here that this result is true in Riemannian manifolds and in Riemann surfaces also; actually we work in the context of the Brelot axiomatic potential theory in harmonic spaces and mention some of its consequences.

2 Preliminaries

Let Ω be a connected and locally compact space which is not compact, provided with a sheaf of harmonic functions satisfying the 3 axioms of Brelot [5]. We assume that the constants are harmonic in Ω. There may or may not be any potential > 0 in Ω. In case there is no potential > 0 in Ω, we fix an unbounded harmonic function $H \geq 0$ outside a compact set (Théorème 1.17 [1]); this function is the axiomatic analogue of $\log |x|$ in $\mathbb{R}^2$ and the Evans potential in a parabolic Riemann surface or in a parabolic Riemannian manifold (Nakai

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[7]). Then Lemma l [2] and Theorem 2.2 [3] together allow us to state the following useful assertion.

Lemma. *In a harmonic space Ω with or without potentials > 0, let k be a compact set and w be an open set such that $k \subset w$. Let u be a harmonic function in $w \setminus k$. Then there exist a harmonic function s in $\Omega \setminus k$ and a harmonic function t in w such that $u = s - t$ in $w \setminus k$. Moreover, s can be assumed bounded near infinity if there are potentials > 0 in Ω; otherwise, for a suitable α, $s - \alpha H$ is bounded near infinity.*

3 Nonremovable Singularities

Recall (p.142 [1]) that a set e in a harmonic space is locally polar (resp. polar) in an open set $w \supset e$, if and only if there exists a superharmonic function (resp. a potential) s in w such that $s(x) = \infty$ on e; and a locally polar set e in w is polar if there exist potentials > 0 in w (p.47, Brelot [5]).

Theorem. *In a harmonic space Ω with potentials > 0, let k be compact and w be open such that $k \subset w$. Suppose there exists a bounded harmonic function in $w \setminus k$ which does not extend harmonically in w. Then for any open set $w_0 \supset k$, there exists a bounded harmonic function in $w_0 \setminus k$ which does not extend harmonically in w_0. The same is true in a harmonic space without positive potentials also, provided every point in Ω is locally polar.*

Proof. Clearly it is enough to prove the theorem assuming $w_0 = \Omega$. Note that k is not locally polar in w since there exists a bounded harmonic function u in $w \setminus k$ which is not extendable as a harmonic function in w.

1) Assume that there are potentials > 0 in Ω. In this case k is not polar in Ω (p.47, Brelot [5]). Let $h = \hat{R}_1^k$ in Ω. Recall that

$$R_1^k = \inf\{v : v \text{ superharmonic} \geq 0 \text{ in } \Omega \text{ and } v \geq 1 \text{ on } k\}$$

and $\hat{R}_1^k$ is its lower semicontinuous regularization so that h is a bounded harmonic function in $\Omega \setminus k$ which does not extend harmonically to Ω.

2) Suppose now that there are no potentials > 0 in Ω. Since every point in Ω is locally polar in this case by the assumption, k should contain at least two points x_i $(i = 1, 2)$ such that u does not extend harmonically to any neighborhood of x_i. Consequently we can find two compact sets k_i which are not locally polar and two open sets w_i such that $k \supset k_1 \cup k_2$, $k_i \subset w_i \subset \overline{w}_i \subset w$ and $w_1 \cap w_2 = \emptyset$. Let $u_i = (\hat{R}_1^{k_i})_w$ in w where the suffix w indicates that the infimum is with respect to the functions defined in w, so that u_i is a positive

superharmonic function in w, harmonic in $w \setminus k_i \supset w \setminus k$ but not harmonic in the whole w.

Then using the lemma above, write $u_i = s_i - t_i$ in $w \setminus k$ where s_i is harmonic in $\Omega \setminus k$, t_i is harmonic in w, and $(s_i - \alpha_i H)$ is bounded near infinity. Here $\alpha_i \neq 0$. For otherwise, define

$$v_i = \begin{cases} s_i & \text{in } \Omega \setminus k \\ u_i + t_i & \text{in } w. \end{cases}$$

Then v_i is a nonharmonic bounded superharmonic function in Ω, a contradiction.

Let $s = \alpha_2 s_1 - \alpha_1 s_2$ in $\Omega \setminus k$; s is harmonic and bounded near infinity. In $w \setminus k$, $s = (\alpha_2 u_1 - \alpha_1 u_2) + (\alpha_2 t_1 - \alpha_1 t_2)$ and hence it is bounded and harmonic in $w \setminus k$. Thus, s is bounded harmonic in $\Omega \setminus k$, but s cannot be extended harmonically in Ω. For suppose s extends harmonically in w; this in particular would imply that s extends harmonically in w_1 and consequently (since u_2, t_1 and t_2 are all harmonic in w_1) $u_1 = (\hat{R}_1^{k_1})_w$ is harmonic in w_1 and hence in w, a contradiction. □

Remark. On the necessity of requiring in the above theorem that every point in Ω should be locally polar if Ω has no potentials > 0: Suppose Ω is a harmonic space without positive potentials and k is a compact set as in the theorem. Then it may happen that k reduces to a single point which is not locally polar. This necessitates the consideration of two different possibilities.

1) $k = \{x_0\}$ and $\Omega \setminus k$ is not connected. In this case there is no problem proving the above theorem; for, define a harmonic function h in $\Omega \setminus k$, equal to 1 in one component and equal to 2 in the other components of $\Omega \setminus k$. Then h is bounded and harmonic to $\Omega \setminus k$ which does not extend harmonically to Ω. For an example of such a possibility, let $\Omega = \mathbb{R}$ with the affine functions as harmonic and take $k = \{0\}$.

2) On the other hand, if $k = \{x_0\}$ and $\Omega \setminus k$ is connected, the above theorem may fail; for, it may happen that there exists a bounded harmonic function in $w \setminus k$ which does not extend harmonically to w whereas every bounded harmonic function in $\Omega \setminus k$ extends harmonically to Ω. For an example of such a possibility, let $\Omega = [0, \infty)$ and define h harmonic in (a, b), $a > 0$, if it is affine and in $[0, c)$ if it is constant. Let $k = \{0\}$. Then, given any open set $w = [0, c)$, $c < \infty$ we can find bounded harmonic functions in $(0, c)$ which do not extend harmonically in $w = [0, c)$; but any bounded harmonic function in $(0, \infty)$ being constant, every bounded harmonic function in $\Omega \setminus k$ extends harmonically to Ω.

Converse. *Let e be a closed set in a harmonic space Ω. We know that if e is locally polar in Ω, then $e^0 = \phi$ and for any open set w, if h is bounded harmonic in $w \setminus e$, h extends harmonically to w. Let us propose its converse in the form: Let e be a closed set contained in an open set w, $e^0 = \phi$; if every bounded harmonic function in $w \setminus e$ extends harmonically to w, then e is locally polar in Ω.*

In this form the converse is true if there are potentials > 0 in Ω (see 6.2.16, p.149 Constantinescu and Cornea [6]). For, if e is not locally polar, we can find a compact set $k \subset e$ which is not locally polar, since Ω is σ-compact. Then, $(\hat{R}_1^k)$ is a bounded harmonic function in $w \setminus e \subset w \setminus k$ which does not extend harmonically to w, a contradiction. (Note that the condition $e^0 = \phi$ is necessary. For, consider the example of $\Omega = (0, \infty)$ with harmonic functions as locally affine functions; take $e = (0, 1]$. Then any bounded harmonic function in $\Omega \setminus e$ being a constant extends harmonically to Ω. But e is not locally polar; however, if e is compact, the condition $e^0 = \phi$ is redundant.)

But, as the above remark shows, this converse need not be true if Ω does not have potentials > 0 in Ω. However, if we assume that each point is locally polar when there is no potential > 0 in Ω (as in parabolic Riemann surfaces and in parabolic Riemannian manifolds), the above theorem shows that this converse is valid. For, if e is not locally polar, let us choose a compact set $k \subset e$ which is not locally polar. Let w_0 be a relatively compact domain in Ω containing k; then $(\hat{R}_1^k)_{w_0}$ is a potential > 0 in w_0. Now $(\hat{R}_1^k)_{w_0}$ is bounded and harmonic in $w_0 \setminus k$ which does not extend harmonically to w_0; hence by the above theorem there exists a bounded harmonic function (in $w \setminus k$ and hence) in $w \setminus e$ which does not extend harmonically to w, a contradiction.

In particular, if R is a Riemann surface or a Riemannian manifold of dimension ≥ 2, hyperbolic or parabolic, and if e is a closed set in R, $e^0 = \phi$, then e is locally polar in R if and only if every bounded harmonic function in $R \setminus e$ extends harmonically in R.

Proposition. *Let Ω be a harmonic space without potentials > 0 in Ω where each point is locally polar. Let e be a closed set in Ω, $e^0 = \phi$. There exists a nonconstant positive harmonic function in $\Omega \setminus e$ if and only if there exists a nonconstant bounded harmonic function in $\Omega \setminus e$.*

Proof. Let $s > 0$ be a nonconstant harmonic function in $\Omega \setminus e$. Clearly e is not locally polar since there are no potentials > 0 in Ω. Then as shown above, there exists a bounded harmonic function u in $\Omega \setminus e$ which does not extend harmonically in Ω. Clearly u is a bounded nonconstant harmonic function in $\Omega \setminus e$. □

Remark. The assertion in the above proposition may not be valid in a harmonic space having potentials > 0 (for example, $\Omega = \mathbb{R}^3$ and $e = \{0\}$); nor in a harmonic space where points are not necessarily locally polar.

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