Real Analysis Exchange
Vol. (), , pp. 617–628

Martin Dindoš, Mathematics Department, University of North Carolina Chapel Hill, Phillips Hall CB #3250, Chapel Hill NC 27599. e-mail: `dindos@math.unc.edu`

ON A TYPICAL SERIES WITH ALTERNATING SIGNS

Abstract

This paper presents a study of typical series with alternating signs. Namely, given a sequence of real nonnegative numbers whose sum is infinity we consider all possible ways of placing plus or minus signs in front of each of these numbers. Choosing a convenient metric we ask what is the 'size' (in terms of Baire category and porosity) of the set of those choices of + or − for which the resulting series converges. The author of this paper has studied this problem in his paper [D1] for the Euclidean metric. The main goal of this paper is to extend the results from [D1] for other standard metrics, such as the Frèchet or Baire metrics.

1 Introduction

The problem of relatively convergent series is studied in many monographs and articles. As is mentioned in the abstract we study the problem of convergence of the series

$$\sum_{n=1}^{\infty}(-1)^{a_n}b_n\ , \tag{1}$$

where $(a_n)_{n\in N}$ is a sequence of zeros and ones and $(b_n)_{n\in N}$ is a sequence of nonnegative real numbers.

For fixed sequence $(b_n)_{n\in\mathbb{N}}$ we want to consider these two sets

$$\mathcal{C}=\{(a_n)_{n\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}};\ \text{the series (1) converges}\}$$

$$\mathcal{B}=\{(a_n)_{n\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}};\exists M>0\ \forall k\ |\sum_{n=1}^{k}(-1)^{a_n}b_n|\le M\}.$$

Key Words: relatively convergent series, Baire category, porosity, Frèchet and Baire metric

Mathematical Reviews subject classification: Primary 28A21, secondary 28A05, 40A05, 54E52

Received by the editors March 30, 1999

The set $\mathcal{C}$ contains all sequences $(a_n)_{n\in\mathbb{N}}$ for which series (1) converges and the set $\mathcal{B}$ contains sequences $(a_n)_{n\in\mathbb{N}}$ for which the series (1) has bounded partial sums. Clearly $\mathcal{C} \subset \mathcal{B}$. These two sets are not equal to the full space $\{0,1\}^{\mathbb{N}}$ if and only if the series

$$\sum_{n=1}^{\infty} b_n \text{ diverges,} \tag{2}$$

i.e., the sum (2) is infinity. Therefore (2) will be our standard assumption in the whole paper.

On our space $\{0,1\}^{\mathbb{N}}$ there are several choices of metrics that can be considered. In the paper [D1] a function $\varphi : \{0,1\}^{\mathbb{N}} \to [0,1]$ was defined by

$$\varphi((a_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \text{ for } (a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}. \tag{3}$$

Let $d_E(a,b) = |\varphi(a) - \varphi(b)|$. The function d_E is a pseudometric on the space $\{0,1\}^{\mathbb{N}}$. If we drop all sequences of the form $(a_1, a_2, .., a_n, 0, 1, 1, 1, \dots)$ we get

$$\mathcal{M} = \{0,1\}^{\mathbb{N}} \setminus \{(a_1, a_2, .., a_n, 0, 1, 1, 1, \dots);\ a_i \in \{0,1\}\ i = 1, 2, \dots, n\}\ ,$$

on which d_E is a metric and $(\mathcal{M}, d_E)$ is a complete metric space. We refer to d_E as the Euclidean metric. From the results in [D1], which have been proved in more general setting for $b_n \in H$ where H is a Hilbert space, it follows that both sets $\mathcal{C}$ and $\mathcal{B}$ are of first Baire category in $(\mathcal{M}, d_E)$ provided (2) holds. Moreover we also have a result on the Lebesgue measure of the sets $\varphi(\mathcal{C})$ and $\varphi(\mathcal{B})$. The measure of these sets is either 0 or 1 (depending on the given sequence $(b_n)_{n\in\mathbb{N}}$).

In this paper we study the same problem for the Frèchet (d_F) and Baire (d_B) metric which are defined as follows.

$$d_F(a,b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|a_n - b_n|}{1 + |a_n - b_n|} \qquad \textbf{Frèchet metric,}$$

$$d_B(a,b) = \begin{cases} \frac{1}{\min\{n\in\mathbb{N}; a_n \neq b_n\}} & \text{for } a \neq b \\ 0 & \text{for } a = b \end{cases} \quad \textbf{Baire metric.}$$

In the last part of this paper we will assume some additional conditions on the sequence $(b_n)_{n\in\mathbb{N}}$. Under such additional assumptions we will obtain interesting results about the porosity of the sets $\mathcal{B}$ and $\mathcal{C}$ in all three metrics.

It is interesting to investigate relations between our three metrics. On the space $\{0,1\}^{\mathbb{N}}$ we have the following inequalities $d_B \geq d_F \geq \frac{1}{2} d_E$.

Recall that two metrics d, d' are called equivalent if there are positive real numbers c_1 and c_2 such that $c_1 d' \leq d \leq c_2 d'$. Two equivalent metrics always generate the same topology. Notice that our three metrics are not equivalent in the sense defined above. Nevertheless, the topologies generated by Baire and Frèchet metric are the same.

This observation has one important consequence. Any set of first Baire category $A \subset \{0,1\}^{\mathbb{N}}$ in one metric must be of first Baire category also in the other one. Of course for porosity a similar statement is not true because porosity depends on concrete metric not just topology.

This and my previous work [D1] have been motivated by very interesting results that have appeared in works of V. László and T. Šalát. They studied the series

$$\sum_{n=1}^{\infty} a_n b_n, \tag{4}$$

where $\sum_{n=1}^{\infty} b_n$ is a divergent series and $(a_n)_{n\in\mathbb{N}}$ is series of zeros and ones. According to their work in [L-Š] and [Š] the set of sequences $(a_n)_{n\in\mathbb{N}}$ for which the series (4) converges (i.e., $\mathcal{C}$) is of the first Baire category in $\{0,1\}^{\mathbb{N}}$ with the Euclidean metric.

2 Categorical Size of the Set of Convergent Sequences

In this section we present the discussion of the sets $\mathcal{B}$ and $\mathcal{C}$ defined above in terms of their Baire category in the complete metric space $(\{0,1\}^{\mathbb{N}}, d)$. We closely follow the paper [D1]. First we are going to establish the following auxiliary statement.

Lemma 2.1. *Consider the complete metric space $(\{0,1\}^{\mathbb{N}}, d)$ where d is either the Frèchet metric d_F or the Baire metric d_B. Let k be a positive integer, c a real number and $(\alpha_1, \alpha_2, \dots \alpha_k)$ a finite sequence of real numbers. Let $\mathcal{S}_+$ and $\mathcal{S}_-$ be the sets*

$$\mathcal{S}_+ = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}; \alpha_1(-1)^{a_1} + \alpha_2(-1)^{a_2} + \cdots + \alpha_k(-1)^{a_k} > c\},$$
$$\mathcal{S}_- = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}; \alpha_1(-1)^{a_1} + \alpha_2(-1)^{a_2} + \cdots + \alpha_k(-1)^{a_k} < c\}.$$

Then both sets $\mathcal{S}_+$ and $\mathcal{S}_-$ are open subsets of the metric space $(\{0,1\}^{\mathbb{N}}, d)$.

Proof. We will prove this lemma only for the set S_+ since the proof for the other set is similar. Take any $(a_n)_{n\in\mathbb{N}} \in S_+$. We need to show that there is an $\varepsilon > 0$ such that each sequence $(b_n)_{n\in\mathbb{N}}$ whose distance from $(a_n)_{n\in\mathbb{N}}$ is less than ε is also in S_+. This can be achieved quite easily. For the Baire metric just take $\varepsilon = \frac{1}{k+1}$ and for the Frèchet $\varepsilon = \frac{1}{2^{k+1}}$. If $d(a,b) < \varepsilon$, then $a_i = b_i$ for $i = 1, 2, \ldots, k$ which means that b also belongs to S_+. $\square$

Now we are ready to prove that both $\mathcal{B}$ and $\mathcal{C}$ are sets of first Baire category in the complete metric space $(\{0,1\}^{\mathbb{N}}, d)$. Here d is the Frèchet (Baire) metric, respectively.

Theorem 2.2. *Consider the series $\sum_{n=1}^{\infty}(-1)^{a_n} b_n$, where $(b_n)_{n\in\mathbb{N}}$ is a given sequence of non-negative real numbers. Assume that the condition* (2) *holds, namely $\sum_{n=1}^{\infty} b_n = \infty$. Then*

$$\mathcal{C} = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}};\ \textit{series (1) converges}\},$$

as well as

$$\mathcal{B} = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}; \exists M > 0\ \forall k\ |\sum_{n=1}^{k}(-1)^{a_n} b_n| \le M\},$$

are first Baire category sets in the complete metric space $(\{0,1\}^{\mathbb{N}}, d)$ where d is the Frèchet metric d_F or the Baire metric d_B. Moreover the set $\mathcal{B}$ is of type F_σ.

PROOF. Since it is clear that $\mathcal{C} \subset \mathcal{B}$, proving the theorem for the set $\mathcal{B}$ will suffice. This set can be written as

$$\mathcal{B} = \bigcup_{M=1}^{\infty} \bigcap_{k=1}^{\infty} \Big\{(a_n)_{n\in\mathbb{N}}; \Big|\sum_{n=1}^{k}(-1)^{a_n} b_n\Big| \le M\Big\}.$$

Let

$$F_i = \bigcap_{k=1}^{\infty} \Big\{(a_n)_{n\in\mathbb{N}}; \Big|\sum_{n=1}^{k}(-1)^{a_n} b_n\Big| \le i\Big\}.$$

It is clear that $\mathcal{B} = \cup F_i$ and the sets F_i are closed since they are defined as an intersection of closed sets. The fact that any set of the form

$$\Big\{(a_n)_{n\in\mathbb{N}}; \Big|\sum_{n=1}^{k}(-1)^{a_n} b_n\Big| \le M\Big\}$$

is closed can be seen from the Lemma 2.1, namely the complement of this set can be written as a finite intersection of $\mathcal{S}_+$ and $\mathcal{S}_-$ which are open. From this we also have that $\mathcal{B}$ is an F_σ set.

Now we want to show that each set F_i is nowhere dense in $(\{0,1\}^{\mathbb{N}}, d)$. Take any $(a_n)_{n\in\mathbb{N}} \in F_i$. We want to show that in any ε neighborhood of this sequence there is a sequence $(c_n)_{n\in\mathbb{N}}$ not in F_i. Define $(c_n)_{n\in\mathbb{N}}$ by

$$c_n = \begin{cases} a_n, & \text{for } n = 1, 2, \ldots, k, \\ 0, & \text{otherwise.} \end{cases}$$

Here k is chosen such that the distance between $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ is less than that given ε. Namely, take k a positive integer such that

$$\frac{1}{2^{k+1}} < \varepsilon \text{ for the Frèchet metric,}$$
$$\frac{1}{k} < \varepsilon \text{ for the Baire metric.}$$

Now it is clear that the sequence $\displaystyle\sum_{n=1}^{\infty}(-1)^{c_n} b_n$ does not belong to F_i since according to (2) $\sum_{n=k+1}^{\infty}(-1)^{c_n} b_n = \sum_{n=k+1}^{\infty} b_n = \infty$ Hence the theorem is proved. $\square$

It is also quite interesting to ask what is the Borel type of the set $\mathcal{C}$. Here we cannot show that this set is F_σ, however it is definitely Borel measurable since it can be written as

$$\mathcal{C} = \bigcap_{m=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} \bigcap_{l=1}^{\infty} \Big\{ (a_n)_{n\in\mathbb{N}}; \Big| \sum_{n=k}^{k+l} (-1)^{a_n} b_n \Big| \le \frac{1}{m} \Big\},$$

which means $\mathcal{C}$ is a $F_{\sigma\delta}$ set.

3 Porosity

In this section we briefly discuss the question whether in Theorem 2.2 we could replace first Baire category by porosity (or σ-porosity). Hence we want to know whether the sets $\mathcal{C}$ and $\mathcal{B}$ are porous ($\sigma-$porous) in the complete metric space $(\{0,1\}^{\mathbb{N}}, d)$. (See also [D1] for a partial discussion about the Euclidean metric.)

For completeness we briefly outline the definition of a porous set.

Definition 3.1. Let E be a given set in a metric space. Given the number $c \in (0,1]$ we say that a point $x_0 \in E$ is a c-porosity point of the set E if there exists a sequence of open balls B_k with radius $r_k \to 0$ centered at x_0 such that for each k there is a ball G_k of radius ρ_k such that $G_k \subset B_k \setminus E$ and $\overline{\lim}_{k\to\infty} \dfrac{\rho_k}{r_k} \ge c$.

We say that x_0 is a porosity point of the set E_0 provided it is a c-porosity point of the set E_0 for some $c > 0$.

The set E is said to be porous (c-porous) if all its points are porosity (c-porosity) points of E. The set E is called σ-porous if it can be covered by a countable union of porous sets.

Remark 3.2. If in the definition of a point of c-porosity we require that for any ball B_r of radius $r > 0$ centered at x_0 there is a ball G_r of radius $\rho(r) > 0$ such that $G_r \subset B_r \setminus E$ and $\underline{\lim}_{r\to 0+} \frac{\rho(r)}{r} \geq c$, then such point is said to be *very* porous. Following the previous definition we can define very porous, c-very porous and σ-very porous set, respectively.

Naturally, any σ-porous set E is a set of first Baire category. A result from [D1] implies that the attempt to replace first Baire category by porosity in Theorem 2.2 without any additional requirement on $(b_n)_{n\in\mathbb{N}}$ must fail (at least for the Euclidean metric). It follows from the fact that the Lebesgue measure of a porous set (in $\mathbb{R}$) must be zero which contradicts the observation that the measure of the set $\varphi(\mathcal{C})$ could be one. Naturally, this argument does not work in case of Frèchet or Baire metric.

So to get some results about porosity we have to require more about the sequence $(b_n)_{n\in\mathbb{N}}$. This leads to the following definition.

Definition 3.3. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers. We say that such sequence satisfies the condition of *sufficiently large partial sums* (SLPS) if there exists an integer $k > 0$ such that for any $n \in \mathbb{N}$

$$\sum_{i=n}^{kn} b_i > 1 \text{ (SLPS condition).}$$

Example 3.4. *Consider a sequence $(b_n)_{n\in\mathbb{N}}$ that satisfies the following condition:*

$$\underline{\lim}_{n\to\infty} b_n n > 0. \tag{5}$$

Such a sequence satisfies (SLPS).

PROOF. To see that (SLPS) is true is not difficult. Condition (5) implies that for sufficiently large n there is a $c > 0$ such that $b_n \geq \frac{c}{n}$. Therefore

$$\sum_{i=n}^{kn} b_i \geq \sum_{i=n}^{kn} \frac{c}{i} \geq c\int_n^{kn} \frac{1}{x}dx = c\ln\left(\frac{kn}{n}\right) = c\ln k$$

So for n large, we pick k such that $k > e^{\frac{1}{c}}$. By possibly enlarging k a little bit, one can make (SLPS) work for all n. □

Corollary 3.5. *The sequence* $b_n = \dfrac{1}{n}$ *satisfies (SLPS).*

Proposition 3.6. *If the sequence* $(b_n)_{n\in\mathbb{N}}$ *is nonincreasing, then condition (SLPS) is equivalent to* $\underline{\lim}_{n\to\infty} b_n n > 0$.

Proof. It remains to establish that the under the assumption of monotonicity of $(b_n)_{n\in\mathbb{N}}$ (SLPS) implies (5). But this is easy. We have $knb_n \geq \sum_{j=n}^{kn} b_j > 1$; i.e., $nb_n > \frac{1}{k}$. From this (5) follows. □

Remark 3.7. Condition (SLPS) can be equivalently stated as $\sum_{i=n}^{kn-1} b_i > 1$ for all $n \in \mathbb{N}$, (by enlarging k in Definition 3.3 by one).

The next theorem shows that the condition (SLPS) is closely related to the σ-porosity of sets $\mathcal{B}$, $\mathcal{C}$ in the Baire metric.

Theorem 3.8. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a sequence of nonnegative numbers satisfying (SLPS). The set*

$$\mathcal{C} = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}};\ \textit{series (1) converges}\},$$

as well as the set

$$\mathcal{B} = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}; \exists M > 0\ \forall k\ |\sum_{n=1}^{k}(-1)^{a_n} b_n| \leq M\},$$

are σ*-very porous in the complete metric space* $(\{0,1\}^{\mathbb{N}}, d_B)$*, where* d_B *is the Baire metric.*

Proof. Since $\mathcal{C} \subset \mathcal{B}$ we will prove the theorem just for the set $\mathcal{B}$. Write again $\mathcal{B} = \cup F_i$ where

$$F_i = \bigcap_{k=1}^{\infty} \Big\{(a_n)_{n\in\mathbb{N}}; \Big|\sum_{n=1}^{k}(-1)^{a_n} b_n\Big| \leq i\Big\}.$$

First we want to prove that each set F_i is $\frac{1}{k^i}$-porous where k is an integer from Remark 3.7. Consequently the set $\mathcal{B}$ must be σ-porous. Then we will show that sets F_i are actually $\frac{1}{2k^i}$-very porous; hence the set $\mathcal{B}$ is in fact σ-very porous.

Fix the integer i and take any sequence $(a_n)_{n\in\mathbb{N}} \in F_i$. Take $\varepsilon_m = \frac{1}{m-1} > 0$ for any $m \geq 2$ and consider the ball

$$B_m = \{(c_n)_{n\in\mathbb{N}}; d_B(a,c) < \varepsilon_m = \frac{1}{m-1}\}.$$

Define a sequence $(d_n)_{n\in\mathbb{N}}$ by

$$d_n = \begin{cases} a_n, & \text{for } n < m, \\ 0, & \text{for } n \geq m \text{ if } \sum_{j=1}^{m-1}(-1)^{d_j} b_j \geq 0, \\ 1, & \text{for } n \geq m \text{ if } \sum_{j=1}^{m-1}(-1)^{d_j} b_j < 0. \end{cases}$$

It is obvious that the sequence $(d_n)_{n\in\mathbb{N}}$ does not belong to F_i since its tail is either $0, 0, 0, \ldots$ or $1, 1, 1, \ldots$. Also $d_B(a, d) < \varepsilon_m \implies d \in B_m$. Using the assumption (SLPS) we can see that for any n we have $\sum_{j=n}^{k^i n} b_j > i$. Indeed, the sum above can be estimated by

$$\sum_{j=n}^{k^i n} b_j \geq \sum_{j=n}^{k^i n-1} b_j = \sum_{j=0}^{i-1} \left(\sum_{l=k^j n}^{k^{j+1}n-1} b_l \right) ,$$

and for each j the sum $\displaystyle\sum_{l=k^j n}^{k^{j+1}n-1} b_l$ is bigger than 1 by Remark 3.7. Hence we have either $\sum_{j=1}^{k^i m}(-1)^{d_j} b_j > i$ or $\sum_{j=1}^{k^i m}(-1)^{d_j} b_j < -i$. Take now

$$G_m = \{(c_n)_{n\in\mathbb{N}}; d_B(d, c) < \frac{1}{k^i m}\} .$$

We want to see that $G_m \subset B_m \setminus F_i$. Clearly $G_m \subset B_m$ since this set has smaller radius. Pick any $(c_n)_{n\in\mathbb{N}} \in G_m$. We want to show that it does not belong to F_i. Without loss of generality assume now that the tail of the sequence $(d_n)_{n\in\mathbb{N}}$ is $0, 0, 0, \ldots$ and therefore $\sum_{j=1}^{k^i m}(-1)^{d_j} b_j > i$. The distance between $(d_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ is less than $\dfrac{1}{k^i m}$ and this implies that $c_n = d_n$ for $n = 1, 2, 3, \ldots, k^i m$; i.e., $\sum_{j=1}^{k^i m}(-1)^{c_j} b_j = \sum_{j=1}^{k^i m}(-1)^{d_j} b_j > i$, which gives us that $(c_n)_{n\in\mathbb{N}} \notin F_i$. Finally we compute the limit of the radii of G_m and B_m:

$$\lim_{m\to\infty} \frac{r(G_m)}{r(B_m)} = \lim_{m\to\infty} \frac{\frac{1}{k^i m}}{\frac{1}{m-1}} = \frac{1}{k^i}. \tag{6}$$

This proves that the set F_i is $\frac{1}{k^i}$-porous. The $\frac{1}{2k^i}$-very porosity follows from this simple observation.

For any $r > 0$ small there is $m \in N$ such that $r \geq \varepsilon_m = \dfrac{1}{m-1} > \dfrac{r}{2}$.

Hence if we take $G_r = G_m$ in the definition of very porosity we get the estimate

$$\underline{\lim}_{r\to 0+} \frac{\rho(r)}{r} \geq \underline{\lim}_{m\to\infty} \frac{\rho_m}{2\varepsilon_m} \geq \lim_{m\to\infty} \frac{\frac{1}{k^i m}}{\frac{2}{m-1}} = \frac{1}{2k^i} \; .$$

From this we have that each set F_i is $\dfrac{1}{2k^i}$-very porous and therefore $\mathcal{B}$ (and $\mathcal{C}$) are σ-very porous. □

Corollary 3.9. *For the series $\sum_{n=1}^{\infty}(-1)^{a_n}\frac{1}{n}$ the sets $\mathcal{C}$ and $\mathcal{B}$ are σ-very porous in the metric space $(\{0,1\}^{\mathbb{N}}, d_B)$.*

If we try to prove a similar theorem for the Frèchet or Euclidean metric using the assumption (SLPS), we will run into trouble when estimating the quotient of radii (6). This quotient tends to zero and therefore we do not have porosity of the set F_i. A natural solution is to replace (SLPS) by a stronger condition (VLPS), which will be defined later.

Unfortunately it turns out this new condition (VLPS) implies that the sequence $(b_n)_{n\in\mathbb{N}}$ does not have limit equal to zero. Thus the series (1) is never convergent; hence $\mathcal{C} = \emptyset$. However, since the set $\mathcal{B}$ might still be nonempty, a result on porosity of this set has certain value.

Definition 3.10. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers. We say that such a sequence satisfies condition of *very large partial sums* (VLPS) if there exists an integer $k > 0$ such that for any $n \in \mathbb{N}$

$$\sum_{i=n}^{n+k} b_i > 1 \text{ (VLPS condition).}$$

Clearly (VLPS) $\Longrightarrow$ (SLPS). Moreover (VLPS) $\Longrightarrow \mathcal{C} = \emptyset$. For the set $\mathcal{B}$ we get the following.

Theorem 3.11. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers satisfying (VLPS). Then the set $\mathcal{C}$ is empty and*

$$\mathcal{B} = \{(a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}; \exists M > 0 \; \forall k \left|\sum_{n=1}^{k}(-1)^{a_n} b_n\right| \leq M\}$$

is a σ-very porous set in the complete metric space $(\{0,1\}^{\mathbb{N}}, d)$, for the Frèchet (Euclidean) metric d, respectively.

PROOF. This proof follows the proof of Theorem 3.8. Write again $\mathcal{B} = \cup F_i$ where

$$F_i = \bigcap_{k=1}^{\infty} \Big\{ (a_n)_{n\in\mathbb{N}}; \Big| \sum_{n=1}^{k} (-1)^{a_n} b_n \Big| \le i \Big\}.$$

We want to prove that each set F_i is $\dfrac{1}{2^{2ki+2}}$-porous where k is the integer from Definition 3.10 increased by one.

Fix an integer i and take any sequence $(a_n)_{n\in\mathbb{N}} \in F_i$. Take $\varepsilon_m = \frac{1}{2^{m-2}} > 0$ for any integer $m \ge 2$ and consider the ball

$$B_m = \{(c_n)_{n\in\mathbb{N}}; d(a,c) < \varepsilon_m = \frac{1}{2^{m-2}}\}.$$

Define a sequence $(d_n)_{n\in\mathbb{N}}$ by

$$d_n = \begin{cases} a_n, & \text{for } n < m \ , \\ 0, & \text{for } n \ge m \ . \end{cases}$$

It is obvious that the sequence $(d_n)_{n\in\mathbb{N}}$ does not belong to F_i since its tail is $0, 0, 0, \dots$. Also

$$d(a,d) \le \sum_{n \ge m} \frac{1}{2^n} = \frac{1}{2^{m-1}} < \varepsilon_m \implies d \in B_m \ .$$

Using the assumption (VLPS) we can see that for any n we have $\sum_{j=n}^{n+2ki} b_j > i$. Hence we have

$$\sum_{j=1}^{m+2ki} (-1)^{d_j} b_j = \sum_{j=1}^{m-1} (-1)^{d_j} b_j + \sum_{j=m}^{m+2ki} (-1)^{d_j} b_j > -i + 2i = i \ .$$

Take now

$$G_m = \{(c_n)_{n\in\mathbb{N}}; d(d,c) < \frac{1}{2^{m+2ki+1}}\} \ .$$

We want to see that $G_m \subset B_m \setminus F_i$. Clearly $G_m \subset B_m$ using the triangle inequality. Pick now any $(c_n)_{n\in\mathbb{N}} \in G_m$. We want to show that such $(c_n)_{n\in\mathbb{N}}$ is not in F_i.

The distance between $(d_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ is less than $\dfrac{1}{2^{m+2ki+1}}$. In case of the Frèchet metric this implies that $c_n = d_n$ for $n = 1, 2, 3, \dots, m + 2ki$. Therefore

$$\sum_{j=1}^{m+2ki} (-1)^{c_j} b_j = \sum_{j=1}^{m+2ki} (-1)^{d_j} b_j > i \ , \tag{7}$$

which gives us that $(c_n)_{n\in\mathbb{N}} \notin F_i$.

In the case of the Euclidean metric the situation could be a little more complicated.

If $c_n = d_n$ for $n = 1, 2, 3, \dots, m + 2ki$, we again have (7). If this is not true, then there is $t < m + 2ki$ such that

$$\begin{aligned} &c_j = d_j \text{ ,for } j = 1, 2, \dots t \ , \\ &c_t \neq d_t \ . \end{aligned}$$

Clearly $t < m$ since otherwise we would have

$$d_E(c, d) = \sum_{j=t}^{\infty} \frac{c_j - d_j}{2^j} = \sum_{j=t}^{\infty} \frac{c_j}{2^j} \geq \frac{1}{2^t} > \frac{1}{2^{m+2ki}} \ .$$

So $t < m$ and therefore we have an estimate $\left| \sum_{j=t}^{m-1} \frac{c_j - d_j}{2^j} \right| \geq \frac{1}{2^{m-1}}$. This gives

$$\sum_{j=m}^{\infty} \frac{c_j}{2^j} = \Big| \sum_{j=m}^{\infty} \frac{c_j - d_j}{2^j} \Big| = \Big| \sum_{j=t}^{m-1} \frac{c_j - d_j}{2^j} - d_E(c, d) \Big| \tag{8}$$

$$\geq \Big| \sum_{j=t}^{m-1} \frac{c_j - d_j}{2^j} \Big| - d_E(c, d) > \frac{1}{2^{m-1}} - \frac{1}{2^{m+2ki}} \ . \tag{9}$$

Now if for any $j \in \{m, m+1, \dots, m+2ki\}$ $c_j = 0$ we also have

$$\sum_{j=m}^{\infty} \frac{c_j}{2^j} \leq \frac{1}{2^{m-1}} - \frac{1}{2^j} \leq \frac{1}{2^{m-1}} - \frac{1}{2^{m+2ki}}. \tag{10}$$

Since (8) and (10) cannot hold simultaneously we have that $c_j = 1$ for $j = m, m+1, \dots, m+2ki$, and therefore $\sum_{j=m}^{m+2ki} (-1)^{c_j} b_j < -2i$. From this immediately $c = (c_n)_{n\in\mathbb{N}} \notin F_i$.

Finally, we compute the limit of the radii of G_m and B_m.

$$\lim_{m\to\infty} \frac{r(G_m)}{r(B_m)} = \lim_{m\to\infty} \frac{\frac{1}{2^{m+2ki+1}}}{\frac{1}{2^{m-1}}} = \frac{1}{2^{2ki+2}} > 0 \ .$$

This proves that the set F_i is $\dfrac{1}{2^{2ki+2}}$-porous. Using a similar argument as in Theorem 3.8 we get that this set is also $\dfrac{1}{2^{2ki+3}}$-very porous which finishes our proof. $\square$

Example 3.12. *A typical series for which Theorem 3.11 works is* $\sum_{n=1}^{\infty}(-1)^{a_n}$. *The previous theorems give us that the set of sequences* $(a_n)_{n\in\mathbb{N}}$ *for which the partial sums of this series are bounded* ($\mathcal{B}$) *is* σ*-very porous in all three metrics (Baire, Frèchet and Euclidean).*

References

[AMM] H. D. Rudeman, *Advanced problem 6105*, American Math. Monthly, **83** (1976).

[B] P. Bunschuch, *Konvergenz unendlicher Reihen und Gleeichverteillung mod 1*, Arch. Math. (1977), 518–523.

[C] E. Cesàro, *Sur une distribution de signes*, Rend. Acc. Lincei **4** (1888), 133.

[D1] M. Dindoš, *On series with alternating signs in Euclidean metric*, Real Analysis Exchange, this issue. Preprint (donwnloadable at `http://www.math.unc.edu/Grads/dindos/publ.html`).

[D2] M. Dindoš, *On sign distribution in relatively convergent series*, Preprint (1999).

[P-S] G. Pólya and G. Segö, *Problems and theorems in analysis,* Springer Verlag, Berlin (1978).

[R] J. F. Randolph, *Basic real and abstract analysis,* Academic press Inc., New York and London (1968).

[Š] T. Šalát, *On subseries of divergent series*, Matematický časopis, **18** (1968), 312–338.

[Z] V. Zanelli, *Sign distribution and absolutely divergent series*, Periodica Mathematica Hungarica, **28(1)** (1994), 1–17.