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FUBINI PROPERTIES OF IDEALS

Abstract

Let I and J be σ-ideals on Polish spaces X and Y, respectively. We say that the pair $\langle I, J\rangle$ has the Fubini Property (FP) if for every Borel subset B of $X \times Y$, if all its sections $B_x = \{y : \langle x, y\rangle \in B\}$ are in J, then its sections $B^y = \{x : \langle x, y\rangle \in B\}$ are in I, for every y outside a set from J. We study the question, which pairs of σ-ideals have the Fubini Property. We show, in particular, that:

– $\langle \mathrm{MGR}(X), J\rangle$ satisfies FP, for every J generated by any family of closed subsets of Y ($\mathrm{MGR}(X)$ is the σ-ideal of all meager subsets of X),

– $\langle \mathrm{NULL}_\mu, J\rangle$ satisfies FP, whenever J is generated by any of the following families of closed subsets of Y (NULL_μ is the σ-ideal of all subsets of X, having outer measure zero with respect to a Borel σ-finite continuous measure μ on X):

(i) all closed sets of cardinality ≤ 1,

(ii) all compact sets,

(iii) all closed sets in NULL_ν for a Borel σ-finite continuous measure ν on Y,

(iv) all closed subsets of a $\mathbf{\Pi^1_1}$ set $A \subseteq Y$.

We also prove that $\langle \mathrm{MGR}(X), \mathrm{MGR}(Y)\rangle$ and $\langle \mathrm{NULL}_\mu, \mathrm{NULL}_\nu\rangle$ are essentially the only cases of FP in the class of σ-ideals obtained from $\mathrm{MGR}(X)$ and NULL_μ by the operations of Borel isomorphism, product, extension and countable intersection.

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1 Preliminaries

This paper is a contribution to the study of σ-ideals on Polish spaces. Our notation and terminology follow [9].

A *σ-ideal* on a metric space X is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. We always assume that the underlying space is uncountable, I is *proper*; i.e., $X \notin I$, contains all singletons and has a basis consisting of Borel sets. The latter means that every set from I is covered by a Borel set from I; i.e., for any $A \in I$ there is $B \in \mathbf{B}(X) \cap I$ with $A \subseteq B$. If, moreover, every set from I is covered by a $\mathbf{\Sigma^0_2}$ set from I, then we say that I is $\mathbf{\Sigma^0_2}$ supported. The most important example of a $\mathbf{\Sigma^0_2}$ supported σ-ideal on X is $\mathrm{MGR}(X) =$ the collection of all meager subsets of X (provided $X \notin \mathrm{MGR}(X)$ and X has no isolated points, so that $\mathrm{MGR}(X)$ is proper and contains all singletons).

The most important example of a σ-ideal on X which is not $\mathbf{\Sigma^0_2}$ supported is

$\mathrm{NULL}_\mu =$ the collection of all subsets of X, having outer measure zero with respect to a Borel σ-finite continuous measure μ on X.

Given σ-ideals I and J on Polish spaces X and Y, respectively, the pair $\langle I, J\rangle$ satisfies the *Fubini Property* (FP) if for every Borel subset B of $X \times Y$, if all its vertical sections $B_x = \{y : \langle x, y\rangle \in B\}$ are in J, then its horizontal sections $B^y = \{x : \langle x, y\rangle \in B\}$ are in I, for every y outside a set from J. Thus the Fubini theorem and the Kuratowski–Ulam theorem (see [9], 8.41) assert, in particular, that the pairs $\langle \mathrm{NULL}_\mu, \mathrm{NULL}_\nu\rangle$ and $\langle \mathrm{MGR}(X), \mathrm{MGR}(Y)\rangle$ satisfy FP. On the other hand, neither $\langle \mathrm{MGR}(X), \mathrm{NULL}_\nu\rangle$ nor $\langle \mathrm{NULL}_\mu, \mathrm{MGR}(Y)\rangle$ satisfies FP (see Example 3.6).

The aim of this paper is to decide for a number of other pairs of σ-ideals whether they satisfy FP. Special attention is given to the $\mathbf{\Sigma^0_2}$ supported σ-ideals. This class includes such important members as:

$[X]^{\leq\aleph_0} =$ the collection of countable subsets of X,

$\mathcal{K}^*_\sigma =$ the σ-ideal generated by all compact subsets of a non-σ-compact X (if X is the Baire space $\mathbb{N}^\mathbb{N}$, then this is just the collection of σ-bounded subsets of $\mathbb{N}^\mathbb{N}$) (see [9], 21.24),

$\mathcal{E} =$ the σ-ideal generated by all closed measure zero subsets of the Cantor space $2^\mathbb{N}$ (see [3], 2.6).

Note that the three σ-ideals above are generated by $\mathbf{\Pi^1_1}$ (in the Effros Borel structure) hereditary families of closed sets (see [9], 35.G). This makes them much easier to handle, as the following lemma shows.

Lemma 1.1. *Let X and Y be Polish spaces and $\mathcal{F}$ a hereditary $\mathbf{\Pi^1_1}$ family of closed subsets of Y. Let I be an arbitrary σ-ideal and J be the σ-ideal* *generated by $\mathcal{F}$. Then the following conditions are equivalent:*

(i) the pair $\langle I, J\rangle$ satisfies FP,

(ii) for every Borel set $B \subseteq X \times Y$ with all sections B_x in $\mathcal{F}$, we have $\{y : B^y \notin I\} \in J$.

PROOF. To prove the non-trivial direction (ii) $\Rightarrow$ (i), take an arbitrary Borel set $B \subseteq X \times Y$ such that $\forall x(B_x \in J)$. Then, the Burgess–Hillard theorem (see [9], 35.43) tells us that $B \subseteq \bigcup_n B_n$, with B_n Borel and $\forall n \forall x[(B_n)_x \in \mathcal{F}]$. By condition (ii), it follows that for each n we have $\{y : (B_n)^y \notin I\} \in J$. Finally,

$$\{y : B^y \notin I\} \subseteq \bigcup_n \{y : (B_n)^y \notin I\} \in J. \qquad \square$$

What makes it still easier to work with σ-ideals which are $\mathbf{\Sigma^0_2}$ supported is the following basic structural result.

Theorem 1.2 (Kechris, Solecki [10])**.** *Let I be a $\mathbf{\Sigma^0_2}$ supported σ-ideal on a Polish space X. Then precisely one of the following possibilities holds.*

(i) There is a countable family $\{F_n : n \in \mathbb{N}\}$ of closed uncountable subsets of X such that, if for each n we put $I_n = \{B \subseteq X : B \cap F_n \in \mathrm{MGR}(F_n)\}$, then $I = \bigcap_{n\in\mathbb{N}} I_n$.

(ii) There is a homeomorphic embedding $\Phi : 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to X$ such that for any $\alpha \in 2^{\mathbb{N}}$ we have $\Phi[\{\alpha\} \times \mathbb{N}^{\mathbb{N}}] \notin I$.

In fact, a revised form of the above will be especially useful for us.

We say that σ-ideals I and J on spaces X and Y, respectively, are *Borel isomorphic* ($I \equiv_B J$) if there exists a Borel isomorphism $f : X \to Y$ between X and Y such that for $A \in \mathbf{B}(X)$, $A \in I \iff f[A] \in J$; in this case we also write $J = f_*(I)$. According to a theorem of Sikorski (see [9], 15.10), if X and Y are uncountable Borel subsets of Polish spaces and both I and J contain uncountable sets, then $I \equiv_B J$ is equivalent to the fact that the corresponding Boolean algebras $\mathbf{B}(X)/(I \cap \mathbf{B}(X))$ and $\mathbf{B}(Y)/(J \cap \mathbf{B}(Y))$ are isomorphic. Recall that if X is an uncountable Polish space with no isolated points, then the Boolean algebra $\mathbf{B}(X)/(\mathrm{MGR}(X) \cap \mathbf{B}(X))$ is the unique, up to an isomorphism, complete atomless Boolean algebra that contains a countable dense subset (see e.g. [8], 25.4). It follows that all σ-ideals of the form MGR(B) for an uncountable Borel subset of a Polish space are pairwise Borel isomorphic (tacitly assuming, of course, that B has no isolated points and $B \notin \mathrm{MGR}(B)$). It is also well known that all σ-ideals of the form NULL_μ

for a σ-finite continuous Borel measure μ on an uncountable Polish space are pairwise Borel isomorphic (see [9], 17.41).

We say that a σ-ideal I on a Polish space X fulfills the *countable chain condition* (abbreviated is ccc), if there is no uncountable family of disjoint Borel sets outside I. We say that I has *property (M)*, if there exists (equivalently: for every) uncountable Polish space Y there is a Borel function $f : X \to Y$ such that $f^{-1}[\{y\}] \notin I$ for each $y \in Y$ (see [1]).

The revised form of the Kechris–Solecki theorem, announced above, is formulated as follows

Proposition 1.3. *Let I be a $\mathbf{\Sigma^0_2}$ supported σ-ideal on a Polish space X. Then precisely one of the following possibilities holds.*

(i) I is ccc and then it is Borel isomorphic to the σ-ideal $\mathrm{MGR}(2^{\mathbb{N}})$.

(ii) I has property (M).

PROOF. It is not difficult to see that the two possibilities above correspond to the two clauses from 1.2. In case (ii) this is clear (see [1]). So it suffices to show that if $I = \bigcap_{n\in\mathbb{N}} I_n$ for a countable family $\{F_n : n \in \mathbb{N}\}$ of closed uncountable subsets of X, where $I_n = \{B \subseteq X : B \cap F_n \in \mathrm{MGR}(F_n)\}$, then I is Borel isomorphic to the σ-ideal $\mathrm{MGR}(2^{\mathbb{N}})$.

Clearly, for each n, the Boolean algebra $\mathbf{B}(X)/(I_n \cap \mathbf{B}(X))$ is isomorphic to $\mathbf{B}(F_n)/\mathrm{MGR}(F_n)$ and so also to the algebra $\mathbf{B}(2^{\mathbb{N}})/\mathrm{MGR}(2^{\mathbb{N}})$. (Note that since I contains all singletons, F_n has no isolated points.) Next note that the function $\Phi([A]_I) = \langle [A]_{I_n} : n \in \mathbb{N}\rangle$ for $A \in \mathbf{B}(X)$ is a complete embedding of $\mathbf{B}(X)/(I \cap \mathbf{B}(X))$ into the product algebra $\prod\limits_{n\in\mathbb{N}} \mathbf{B}(X)/(I_n \cap \mathbf{B}(X))$ of the algebras $\mathbf{B}(X)/(I_n \cap \mathbf{B}(X))$. It then easily follows that $\mathbf{B}(X)/(I \cap \mathbf{B}(X))$ is a complete atomless Boolean algebra with a countable dense subset and thus it is also isomorphic to $\mathbf{B}(2^{\mathbb{N}})/\mathrm{MGR}(2^{\mathbb{N}})$. Hence, by Sikorski's theorem mentioned above, $I \equiv_B \mathrm{MGR}(2^{\mathbb{N}})$. (For a different proof see [2], 3.1). □

To realize the relevance of the dichotomy above to the study of FP for $\mathbf{\Sigma^0_2}$-supported σ-ideals let us note the following lemma.

Lemma 1.4. *Suppose that I_1, I_2, J_1 and J_2 are σ-ideals on Polish spaces X_1, X_2, Y_1 and Y_2, respectively. If $I_1 \equiv_B I_2$ and $J_1 \equiv_B J_2$, then $\langle I_1, J_1\rangle$ satisfies FP if and only if $\langle I_2, J_2\rangle$ does also.*

PROOF. Straightforward. □

Thus, from the point of view of FP, Borel isomorphic ideals are identical. In particular, the dichotomy above says that there is essentially only one $\mathbf{\Sigma^0_2}$

supported ccc σ-ideal, which will be referred to as *the category σ-ideal* and denoted simply by MGR.

In the first part of the paper we examine FP for pairs $\langle I, J\rangle$ of σ-ideals, at least one of which is $\mathbf{\Sigma^0_2}$ supported. The main results are:

– if J is $\mathbf{\Sigma^0_2}$ supported, then $\langle \text{MGR}, J\rangle$ satisfies FP (theorem 2.4).

– if J is one of the σ-ideals $[Y]^{\leq\aleph_0}$, $\mathcal{K}^*_\sigma$, $\mathcal{E}$, then $\langle \text{NULL}, J\rangle$ satisfies FP (theorems 2.1.(i) and 2.5). Here NULL stands for *the measure σ-ideal*; i.e. any of the Borel isomorphic σ-ideals of the form NULL_μ.

In the second part we concentrate on ccc σ-ideals. We try to find a pair of ccc σ-ideals which satisfies FP but is different from the obvious examples: $\langle \text{MGR}, \text{MGR}\rangle$ and $\langle \text{NULL}, \text{NULL}\rangle$. We show that there is no such pair in the class of σ-ideals obtained from MGR and NULL by the operations of Borel isomorphism, product, extension and countable intersection (Theorem 3.9). The only examples we give require some additional set-theoretic assumptions (Theorem 3.10).

2 Fubini Property for $\mathbf{\Sigma^0_2}$ Supported σ-Ideals.

Let us first deal with the simplest $\mathbf{\Sigma^0_2}$ supported σ-ideal; namely the one formed by countable sets. The idea of the proof of (i) below has been borrowed from [7] (see [7], 1.2).

Theorem 2.1. *Let X and Y be Polish spaces.*

(i) If I is an arbitrary ccc σ-ideal on X, then $\langle I, [Y]^{\leq\aleph_0}\rangle$ satisfies FP.

(ii) If J is any σ-ideal on Y, then $\langle [X]^{\leq\aleph_0}, J\rangle$ does not satisfy FP.

PROOF. (i) Let $B \subseteq X \times Y$ be a Borel set with all sections B_x countable. Then, by the Lusin-Novikov Theorem (see [9], 18.10), B can be written as $\bigcup_n B_n$, where each B_n is the graph of a partial Borel function f_n. Note that $\{y : B^y \notin I\} = \bigcup_n \{y : (B_n)^y \notin I\}$. But for each n, $(B_n)^y = f_n^{-1}[\{y\}]$; so $|\{y : (B_n)^y \notin I\}| \leq \aleph_0$, by the ccc property of I. Thus $|\{y : (B)^y \notin I\}| \leq \aleph_0$.

(ii) It's easy to find a Borel function $f : X \to Y$ with $\forall y \in Y\ |f^{-1}[\{y\}]| > \aleph_0$. Let B be the graph of f and note that B violates FP for $\langle [X]^{\leq\aleph_0}, J\rangle$. $\square$

Note that in proving 2.1 (ii) we use only the fact that $[X]^{\leq\aleph_0}$ has property (M). Thus we have

Proposition 2.2. *Let X and Y be Polish spaces. If I is a σ-ideal on X with property (M) and J is an arbitrary σ-ideal on Y, then the pair $\langle I, J\rangle$ does not satisfy FP.*

Combining this with Proposition 1.3 immediately gives the following.

Theorem 2.3. *Let X and Y be Polish spaces. Let I be a $\mathbf{\Sigma^0_2}$ supported σ-ideal on X. If I is not ccc, then the pair $\langle I, J\rangle$ does not satisfy FP for any σ-ideal J on Y.*

Now let us turn to the case when I is ccc.

Theorem 2.4. *Let X and Y be Polish spaces. Let I and J be $\mathbf{\Sigma^0_2}$ supported σ-ideals on X and Y, respectively. If I is ccc, then the pair $\langle I, J\rangle$ satisfies FP.*

PROOF. By Proposition 1.3 and Lemma 1.4 we can assume that $I = \mathrm{MGR}(X)$. Now suppose, to the contrary, that a Borel set $B \subseteq X \times Y$ is a counterexample to FP for $\langle \mathrm{MGR}(X), J\rangle$; i.e. $\forall x\ B_x \in J$ but $Y' = \{y : B^y \notin \mathrm{MGR}(X)\} \notin J$. Since Y' is Borel (see [9], 16.1), we can extend the topology τ on Y to a new Polish topology τ' with the same Borel sets in which Y' is clopen; so Polish (in the relative topology – see [9], 13.1).

From now on let us work with the topology τ'. Note that J remains $\mathbf{\Sigma^0_2}$ supported.

Let $\{V_n :\ n \in \mathbb{N}\}$ be an open basis for Y. Consider the space $Z = Y' \setminus \bigcup\{V_n :\ V_n \in J\}$. Note that Z is closed and $Z \notin J$ (in fact $Y' \setminus Z \in J$); so Z is a nonempty Polish space. Moreover, for any open subset U of Y if $U \cap Z \neq \emptyset$, then $U \cap Z \notin J$.

Let $C = B \cap (X \times Z)$. Then $\forall z \in Z\ C^z \notin \mathrm{MGR}(X)$; so, by the Kuratowski-Ulam theorem applied to the product $X \times Z$, there is an $x_0 \in X$ such that $C_{x_0} \notin \mathrm{MGR}(Z)$. But $C_{x_0} \in J$; so there is $F \in J \cap \mathbf{\Sigma^0_2}$ with $C_{x_0} \subseteq F$. Then $F \cap Z$ is a non-meager $\mathbf{\Sigma^0_2}$ subset of Z (in the relative topology). It follows that there is an open subset U of Y such $U \cap Z \neq \emptyset$ and $U \cap Z \subseteq F$. But this implies that $F \notin J$ which contradicts the choice of F. □

This completes our discussion of FP for pairs in which both σ-ideals are $\mathbf{\Sigma^0_2}$ supported. We now consider the case, when I is the measure ideal NULL. Of course, we cannot hope to obtain the full analogue of Theorem 2.4, since we have the counterexample $\langle \mathrm{NULL}, \mathrm{MGR}\rangle$ (see 3.6). Nevertheless, some important cases of Theorem 2.4 are still valid.

Theorem 2.5. *Let X and Y be Polish spaces and let μ be a σ-finite Borel continuous measure on X. Then the pair $\langle \mathrm{NULL}_\mu, J\rangle$ satisfies FP whenever J is the σ-ideal generated by any of the following families of closed subsets of Y:*

(i) all compact sets (In this case Y is assumed to be non-σ-compact.) Moreover, if $B \subset X \times Y$ is a Borel set with $B_x \in \mathcal{K}^*_\sigma$ for each $x \in X$, then there exists a set $Z \in \mathcal{K}^*_\sigma$ such that $B_x \subseteq Z$ for μ-almost all $x \in X$.

(ii) all closed sets in NULL_ν *for a σ-finite Borel continuous measure ν on Y,*

(iii) all closed subsets of a $\mathbf{\Pi^1_1}$ set $A \subseteq Y,\ A \neq Y$.

PROOF. First note that in all three cases J is generated by a hereditary $\mathbf{\Pi^1_1}$ family $\mathcal{F}$ of closed subsets of Y; so, by Lemma 1.1, it is enough to consider Borel sets B with all sections B_x in $\mathcal{F}$. We can also assume that the measure μ is finite.

(i) It clearly suffices to prove the "moreover" part. So let $B \subset X \times Y$ be a Borel set with $B_x \in K(Y)$ for each $x \in X$ and consider the function $\varphi : X \to K(Y)$ defined by $\varphi(x) = B_x$ for $x \in X$, where by $K(Y)$ we denote the space of all compact subsets of Y equipped with the Vietoris topology.

Since φ is Borel, Lusin's theorem (see [9], 17.12) implies there are compact sets $F_n \subseteq X,\ n \in \mathbb{N}$, such that $\forall n\ \varphi|F_n$ is continuous and $\mu(X \setminus \bigcup_n F_n) = 0$. Then $\forall n\ \varphi[F_n] \in K(K(Y))$; so by the continuity of the mapping $\bigcup : K(K(Y)) \to K(Y)$ (see [9], 4.29), $\bigcup \varphi[F_n] \in K(Y)$ and it suffices to define $Z = \bigcup_n (\bigcup \varphi[F_n])$.

(ii) and (iii) are easy consequences of the following result.

Lemma 2.6 (folklore?)**.** *If $B \subseteq X \times Y$ is a Borel set such that every section B_x is closed, then $\{y :\ \mu(B^y) > 0\}$ is $\mathbf{\Sigma^0_2}$.*

PROOF. (of 2.6) This seems to be essentially known but for the sake of completeness we shall sketch a simple argument, suggested by the referee. We may assume that $\mu(X)$ is finite. It is enough to prove that in this case the sets $Y_a = \{y : \mu(B^y) \geq a\}$ are closed for every $a > 0$. Let $y_n \in Y_a,\ \lim_n y_n = y$. Then B^y contains every point belonging to infinitely many of the sets B^{y_n}, since the vertical sections of B are closed. That is, B^y contains $\bigcap_k \bigcup_{n \geq k} B^{y_n}$, which implies $\mu(B^y) \geq a$ and $y \in Y_a$. □

Finally note that the part of 2.5.(i) stating FP for $\langle \mathrm{NULL}_\mu, \mathcal{K}^*_\sigma \rangle$ is a special case of Theorem 2.5 (iii), since every Polish space is a $\mathbf{\Pi^0_2}$; so $\mathbf{\Pi^1_1}$, subset of the Hilbert cube.

3 Fubini Property for ccc σ-Ideals.

The two most frequently encountered ccc σ-ideals are the category σ-ideal MGR and the measure σ-ideal NULL. They also provide two classical cases, $\langle \mathrm{MGR}, \mathrm{MGR} \rangle$ and $\langle \mathrm{NULL}, \mathrm{NULL} \rangle$, of the Fubini Property phenomenon. Are there any other natural examples of pairs of ccc σ-ideals satisfying FP? In this section we negatively answer this question restricted to the realm of σ-ideals cooked up from MGR and NULL with the help of the operations of extension, countable intersection, Borel isomorphism and product.

Recall that given σ-ideals I on X and J on Y their *product* $I \otimes J$ is defined as the σ-ideal with the Borel basis consisting of Borel sets $B \subseteq X \times Y$ such that $\{x: B_x \notin J\} \in I$. Notice that the pair $\langle I, J\rangle$ has the Fubini property if and only if, for every Borel set $B \subset X \times Y$, $B \in I \otimes J$ implies $B^* \in J \otimes I$, where $B^* = \{(y,x) : (x,y) \in B\}$.

Since all measure (category, resp.) σ-ideals are Borel isomorphic, MGR $\otimes$ MGR $\equiv_B$ MGR and NULL $\otimes$ NULL $\equiv_B$ NULL. Let $\mathbb{K}$ and $\mathbb{L}$ denote the σ-ideals of meager subsets in $\mathbb{R}$ and of Lebesgue measure zero subsets of $\mathbb{R}$, respectively. Gavalec [6] proved that if $I_1, \dots, I_n \in \{\mathbb{K}, \mathbb{L}\}$, then it makes sense to write $I = I_1 \otimes \dots \otimes I_n$, without brackets, and call I the product of n factors $I_1, \dots, I_n$.

The next proposition summarizes some general properties of the product operation which will be needed in the sequel. The proofs are routine.

Proposition 3.1. *Let X, X', Y, Y' be Polish spaces. If I, $\bar{I}$ and I_n, $n \in \mathbb{N}$, are σ-ideals on X, J, $\bar{J}$ and J_n, $n \in \mathbb{N}$, are σ-ideals on Y, I' is a σ-ideal on X' and J' is a σ-ideal on Y', then*

(i) $I \equiv_B I' \wedge J \equiv_B J' \to I \otimes J \equiv_B I' \otimes J'$,

(ii) $(\bigcap_n I_n) \otimes J = \bigcap_n (I_n \otimes J)$,

(iii) $I \otimes (\bigcap_n J_n) = \bigcap_n (I \otimes J_n)$,

(iv) $I \subseteq I' \to I \otimes J \subseteq I' \otimes J$,

(v) $J \subseteq J' \to I \otimes J \subseteq I \otimes J'$.

Let $\overline{\mathcal{S}}$ be the smallest class of σ-ideals on Polish spaces such that:
(i) $\mathbb{K}$, $\mathbb{L} \in \overline{\mathcal{S}}$,
(ii) for any σ-ideals I_1, I_2, $(I_1 \in \overline{\mathcal{S}} \wedge I_1 \equiv_B I_2) \to I_2 \in \overline{\mathcal{S}}$,
(iii) for any σ-ideals $I_n \in S$, $n \in \mathbb{N}$, on a space X, $\bigcap_n I_n \in \overline{\mathcal{S}}$,
(iv) for any σ-ideals I, $J \in \overline{\mathcal{S}}$, $I \otimes J \in \overline{\mathcal{S}}$.
(v) for any σ-ideals I, $\bar{I}$, $(I \in \overline{\mathcal{S}} \wedge I \subseteq \bar{I}) \to \bar{I} \in \overline{\mathcal{S}}$.

Let $\mathcal{S}$ be the smallest class of σ-ideals satisfying conditions (i)–(iv) above. Since the operations involved in (ii)–(iv) are monotone, we have

$$\bar{I} \in \overline{\mathcal{S}} \iff \exists I \in \mathcal{S}\ I \subseteq \bar{I}$$

The elements of $\mathcal{S}$ can be represented in a useful normal form. Let $\mathcal{P}_n$ denote the family of all σ-ideals I on $\mathbb{R}^n$ which are the products of n factors taken from $\{\mathbb{K}, \mathbb{L}\}$.

Proposition 3.2. *If $I \in \mathcal{S}$ is a σ-ideal on a Polish space X, then there are: positive integers n_k, Borel isomorphisms φ_k between $\mathbb{R}^{n_k}$ and X, and σ-ideals $I_k \in \mathcal{P}_{n_k}$ on $\mathbb{R}^{n_k}$, $k \in \mathbb{N}$, such that $I = \bigcap_k (\varphi_k)_*(I_k)$.*

PROOF. It is enough to show that the class of σ-ideals of the above form is closed under the operations involved in the definition of $\mathcal{S}$. In the case of countable intersection this is obvious. If $\psi : X \to X'$ is a Borel isomorphism between X and another Polish space X', then $\psi_*(I) = \psi_*\big(\bigcap_k (\varphi_k)_*(I_k)\big) = \bigcap_k (\psi \circ \varphi_k)_*(I_k)$.

Finally, let $J = \bigcap_l (\psi_l)_*(J_l)$, where for each l, J_l is the product of m_l factors from $\{\mathbb{K}, \mathbb{L}\}$ and ψ_l are Borel isomorphisms between $\mathbb{R}^{m_l}$ and Y. Then by 3.1 (i)–(iii),

$$\begin{aligned} I \otimes J &= \Big(\bigcap_k (\varphi_k)_*(I_k)\Big) \otimes \Big(\bigcap_l (\psi_l)_*(J_l)\Big) = \bigcap_{k,l} \big((\varphi_k)_*(I_k) \otimes (\psi_l)_*(J_l)\big) \\ &= \bigcap_{k,l} (\vartheta_{k,l})_*(I_k \otimes J_l), \end{aligned}$$

where for each k, l, $\vartheta_{k,l}$ is a certain Borel isomorphism between $\mathbb{R}^{n_k+m_l}$ and $X \times Y$. □

Gavalec [6] proved that $\forall n > 0$, every σ-ideal $I \in \mathcal{P}_n$ is ccc. Combining this with Proposition 3.2 and taking into account that the ccc property is preserved by Borel isomorphisms, countable intersections and extensions, we immediately get the following

Proposition 3.3. *Every σ-ideal $I \in \overline{\mathcal{S}}$ is ccc.*

Our next auxiliary result concerns extensions of ccc σ-ideals. We say that σ-ideals I_1, I_2 on X are *orthogonal* and write $I_1 \perp I_2$, if there is no common extension of I_1, I_2 to a (proper!) σ-ideal on X. This is equivalent to the existence of a Borel set in I_1 whose complement is in I_2.

If I is a σ-ideal on X and $A \in \mathbf{B}(X) \setminus I$, then *the restriction of I to A*, denoted by $I|A$, is the σ-ideal on X given by

$$I|A = \{C \subseteq X : \ C \cap A \in I\}.$$

Clearly, $I \subseteq I|A$ for any $A \in \mathbf{B}(X) \setminus I$ and if I is, moreover, ccc, then every σ-ideal $\bar{I}$ on X extending I is of this form. (It suffices to let $A = X \setminus \bigcup \mathcal{R}$, where $\mathcal{R}$ is any maximal family of pairwise disjoint Borel sets in $\bar{I} \setminus I$.)

Proposition 3.4. *Let I_n, $n \in \mathbb{N}$, and $\bar{I}$ be ccc σ-ideals on a Polish space X. If $\bigcap_n I_n \subseteq \bar{I}$, then there exists a nonempty set $T \subseteq \mathbb{N}$ such that $\bigcap_{n \in T} I_n \subseteq \bar{I}$ and $I_n \not\subseteq \bar{I}$ for each $n \in T$.*

PROOF. Put $I = \bigcap_n I_n$ and find $A \in \mathbf{B}(X) \setminus I$ such that $\bar{I} = I|A$. Let $T = \{n : A \notin I_n\}$. Since $A \notin I$, $T \neq \emptyset$. We have

$$\bigcap_{n \in T} I_n \subseteq (\bigcap_{n \in T} I_n)|A = I|A = \bar{I}.$$

Finally, take $n \in T$. Since $A \notin I_n$, $I_n|A$ is a proper σ-ideal on X. But $I_n \subseteq I_n|A$ and $\bar{I} = I|A = (\bigcap_{n \in T} I_n)|A = \bigcap_{n \in T}(I_n|A) \subseteq I_n|A$; so $I_n|A$ witnesses that $I_n \not\leq \bar{I}$. □

The special role played by the σ-ideals MGR and NULL in the class $\overline{\mathcal{S}}$, as far as the Fubini Property is concerned, strongly depends on closure conditions which are summarized in the following proposition.

Proposition 3.5 (folklore?). *(i) The countable intersection of category (measure, resp.) σ-ideals is a category (measure, resp.) σ-ideal. More precisely, if X is an uncountable Polish space and σ-ideals I_n, $n \in \mathbb{N}$, on X are Borel isomorphic to* $\mathrm{MGR}(2^{\mathbb{N}})$ *(are of the form* NULL_{μ_n} *for certain probability Borel continuous measures μ_n on X, resp.), then the σ-ideal $I = \bigcap_n I_n$ is of the same form.*

(ii) The product of finitely many category (measure, resp.) σ-ideals is a category (measure, resp.) σ-ideal.

(iii) The extension of a category (measure, resp.) σ-ideal is a category (measure, resp.) σ-ideal.

PROOF. (i). The category case has essentially been dealt with in the course of proving Proposition 1.3. In the measure case it is enough to define μ by

$$\mu(A) = \sum_n \frac{1}{2^{n+1}} \cdot \mu_n(A) \quad \text{for } A \in \mathbf{B}(X),$$

to get $\bigcap_n \mathrm{NULL}_{\mu_n} = \mathrm{NULL}_\mu$.

(ii). This follows from the fact that the product of n copies of $\mathbb{K}$ ($\mathbb{L}$, resp) is the σ-ideal of meager (Lebesgue measure zero, resp.) subsets of $\mathbb{R}^n$.

(iii). Let J be a σ-ideal on $\mathbb{R}$ and $I \subseteq J$, where $I \in \{\mathbb{K}, \mathbb{L}\}$. Find $A \in \mathbf{B}(\mathbb{R}) \setminus I$ with $J = I|A$. Then the Boolean algebras $\mathbf{B}(\mathbb{R})/\big(I|A \cap \mathbf{B}(\mathbb{R})\big)$ and $\mathbf{B}(\mathbb{R})/\big(I \cap \mathbf{B}(\mathbb{R})\big)$ are isomorphic. □

We return to the question of which pairs $\langle I, J\rangle$ of members of $\overline{\mathcal{S}}$ have the Fubini Property. Let us first deal with the special case when $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$ for some fixed positive integers n, m. It will turn out that if $\langle I, J\rangle$ does not satisfy FP, then in fact FP is violated in a very strong sense. Here is a general formulation of the relevant definition.

Given σ-ideals I and J on Polish spaces X and Y, respectively, we say that a Borel set $B \subseteq X \times Y$ is a *0-1 counterexample to FP for* $\langle I, J\rangle$, if $\forall x \in X\ B_x \in J$ and $\forall y \in Y\ (X \setminus B^y) \in I$. It is straightforward to prove, that with regard to the existence of 0-1 counterexamples to FP, Borel isomorphic ideals are identical. Also note, that if B is a 0-1 counterexample to FP for $\langle I, J\rangle$, then $\{\langle y, x\rangle : \ \langle x, y\rangle \in (X \times Y) \setminus B\}$ is a 0-1 counterexample to FP for $\langle J, I\rangle$. Thus the existence of such a B contradicts FP for both $\langle I, J\rangle$ and $\langle J, I\rangle$.

The following construction is well-known.

Example 3.6. *Let $C \in \mathbf{B}(\mathbb{R})$ be such that $C \in \mathbb{L}$ and $\mathbb{R} \setminus C \in \mathbb{K}$. Then the set $B \subseteq \mathbb{R} \times \mathbb{R}$ defined by*

$$B = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : x + y \in C\},$$

is a 0-1 counterexample to FP for $\langle \mathbb{K}, \mathbb{L}\rangle$.

It is known that there exists a 0-1 counterexample to FP for $\langle \mathbb{K}\otimes\mathbb{L}, \mathbb{K}\otimes\mathbb{L}\rangle$ (see [7], 2.1). These facts generalize to

Lemma 3.7. *If $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$, then the following conditions are equivalent:*

(i) $\langle I, J\rangle$ does not satisfy FP,

(ii) neither $I \equiv_B J \equiv_B$ MGR nor $I \equiv_B J \equiv_B$ NULL,

(iii) there is a 0-1 counterexample to FP for $\langle I, J\rangle$.

PROOF. Only (ii) $\rightarrow$ (iii) requires proof. Let

$$I = I_1 \otimes \ldots \otimes I_n \text{ and } J = J_1 \otimes \ldots \otimes J_m,$$

with each $I_k,\ J_l \in \{\mathbb{K}, \mathbb{L}\}$. It follows, from Proposition 3.5, that we can find integers k and l such that $I_k \neq J_l$. Assume, w.l.o.g., that $k < l$, $I_k = \mathbb{K}$ and $J_l = \mathbb{L}$. Put $r = \max\{n + l - k, m\}$. Let $I', J' \in \mathcal{P}_r$ be the σ-ideals obtained by substituting for I_k (J_l, resp.) the product of $r - n$ copies of $\mathbb{K}$ ($r - m$ copies of $\mathbb{L}$, resp.). Then

$$I' = I'_1 \otimes \ldots \otimes I'_r \text{ and } J' = J'_1 \otimes \ldots \otimes J'_r,$$

with $I'_l = \mathbb{K}$ and $J'_l = \mathbb{L}$. Moreover, $I \equiv_B I'$ and $J \equiv_B J'$; so it suffices to find a 0-1 counterexample to FP for $\langle I', J'\rangle$.

First note that $I' \perp J'$. For that purpose take a Borel set $A \subseteq \mathbb{R}$ such that $A \in \mathbb{L}$ and $\mathbb{R} \setminus A \in \mathbb{K}$. Define a Borel set $C \subseteq \mathbb{R}^r$ by

$$C = \{(x_1, \ldots, x_r) : \ x_l \in A\}.$$

It is not difficult to prove that $C \in J'$ and $(\mathbb{R}^r \setminus C) \in I'$.

We complete the proof exactly as in 3.6. Namely, the set $B \subseteq \mathbb{R}^r \times \mathbb{R}^r$ defined by $B = \{\langle \bar{x}, \bar{y} \rangle \in \mathbb{R}^r \times \mathbb{R}^r : \bar{x} + \bar{y} \in C\}$, (+ is the ordinary addition in $\mathbb{R}^r$) is a 0-1 counterexample to FP for $\langle I', J' \rangle$. □

Our last auxiliary fact concerns the question of what happens to FP when we pass to larger σ-ideals.

Proposition 3.8. *Let I, $\bar{I}$ and J, $\bar{J}$ be σ-ideals on Polish spaces X and Y, respectively.*

(i) If $I \subseteq \bar{I}$, then FP for $\langle I, J \rangle$ implies FP for $\langle \bar{I}, J \rangle$,

(ii) If $I \subseteq \bar{I}$, $J \subseteq \bar{J}$ and J is, moreover, ccc, then FP for $\langle I, J \rangle$ implies FP for $\langle \bar{I}, \bar{J} \rangle$.

PROOF. (i) follows immediately from the fact, that for any $B \in \mathbf{B}(X \times Y)$, $\{y : B^y \notin \bar{I}\} \subseteq \{y : B^y \notin I\}$.

(ii). By (i), we can assume that $\bar{I} = I$. Fix $A \in \mathbf{B}(Y) \setminus J$ such that $\bar{J} = J|A$.

Take an arbitrary Borel set $B \subseteq X \times Y$ with $\forall x\ B_x \in \bar{J}$. Let $D = B \cap (X \times A)$ and note that $\forall x\ D_x = B_x \cap A \in J$. By FP for $\langle I, J \rangle$, $\{y : D^y \notin I\} \in J$. But $\{y : D^y \notin I\} = \{y : B^y \notin I\} \cap A$; so $\{y : B^y \notin I\} \in \bar{J}$. □

We are now ready to state the main result of this section.

Theorem 3.9. *For every σ-ideals I, $J \in \overline{\mathcal{S}}$ the following conditions are equivalent:*

(i) $\langle I, J \rangle$ satisfies FP,

(ii) either $I \equiv_B J \equiv_B$ MGR or $I \equiv_B J \equiv_B$ NULL.

PROOF. Only the implication (i) → (ii) requires proof. So assume that I, $J \in \overline{\mathcal{S}}$ and $\langle I, J \rangle$ satisfies FP. By 3.2,

$$\bigcap_k (\varphi_k)_*(I_k) \subseteq I \text{ and } \bigcap_l (\psi_l)_*(J_l) \subseteq J$$

for certain positive integers n_k, m_l, Borel isomorphisms $\varphi_k : \mathbb{R}^{n_k} \to X$, $\psi_l : \mathbb{R}^{m_l} \to X$ and σ-ideals $I_k \in \mathcal{P}_{n_k}$, $J_l \in \mathcal{P}_{m_l}$, respectively. By Propositions 3.3 and 3.4, there are nonempty sets T, $W \subseteq \mathbb{N}$ such that $\forall k \in T\ (\varphi_k)_*(I_k) \not\subseteq I$, $\forall l \in W\ (\psi_l)_*(J_l) \not\subseteq J$ and

$$\bigcap_{k \in T} (\varphi_k)_*(I_k) \subseteq I \text{ and } \bigcap_{l \in W} (\psi_l)_*(J_l) \subseteq J.$$

Now suppose, towards a contradiction, that neither $I \equiv_B J \equiv_B$ MGR nor $I \equiv_B J \equiv_B$ NULL. By Proposition 3.5, we may find integers $k \in T$ and

$l \in W$ such that neither $I_k \equiv_B J_l \equiv_B$ MGR nor $I_k \equiv_B J_l \equiv_B$ NULL. Since $(\varphi_k)_*(I_k) \not\leq I$ ($(\psi_l)_*(J_l) \not\leq J$, resp.), let $\bar{I}$ ($\bar{J}$, resp.) be a σ-ideal extending $(\varphi_k)_*(I_k)$ and I ($(\psi_l)_*(J_l)$ and J, resp.). Then, by 3.7, there is a 0-1 counterexample B to FP for $\langle(\varphi_k)_*(I_k), (\psi_l)_*(J_l)\rangle$. Clearly, B is also a 0-1 counterexample to FP for $\langle\bar{I}, \bar{J}\rangle$. But on the other hand, since $\langle I, J\rangle$ satisfies FP, so does $\langle\bar{I}, \bar{J}\rangle$, by 3.8.(ii) and 3.3. This contradiction completes the proof. □

Finally we present the promised examples of pairs of ccc σ-ideals which satisfy FP but are different from $\langle \mathrm{MGR}, \mathrm{MGR}\rangle$ and $\langle \mathrm{NULL}, \mathrm{NULL}\rangle$. Recall that a cardinal κ is called *quasi-measurable* if it is uncountable and there is a proper ω_1-saturated (i.e., there is no uncountable family of disjoint subsets of κ outside the ideal) κ-additive ideal of $\mathcal{P}(\kappa)$ containing singletons (see [4], 9C). It is well-known that "ZFC + Martin's Axiom + there is a quasi-measurable cardinal $\kappa < \boldsymbol{c}$" is equiconsistent with "ZFC + there is a two-valued-measurable cardinal" (see [4], 9G).

The σ-ideal J below has previously been considered by several authors (see e.g. [5] and [7]).

Theorem 3.10. *Assume Martin's Axiom + there is a quasi-measurable cardinal $\kappa < \boldsymbol{c}$. Let $A \subseteq \mathbb{R}$ be a set of cardinality κ and let $\mathcal{J}$ be a proper, ω_1-saturated κ-additive ideal on A containing singletons. If J is the σ-ideal on $\mathbb{R}$ with the basis consisting of Borel sets $B \subseteq \mathbb{R}$ such that $B \cap A \in \mathcal{J}$, then:*

(i) J is ccc,

(ii) the pairs $\langle \mathbb{K}, J\rangle$ and $\langle \mathbb{L}, J\rangle$ satisfy FP,

(iii) the pairs $\langle J, \mathbb{K}\rangle$ and $\langle J, \mathbb{L}\rangle$ satisfy FP,

(iv) $J \not\equiv_B$ MGR and $J \not\equiv_B$ NULL.

Proof. (i) immediately follows from the ω_1-saturation of $\mathcal{J}$.

(ii). Let $I \in \{\mathbb{K}, \mathbb{L}\}$. Suppose, to the contrary, that there is a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$ which is a counterexample to FP for $\langle I, J\rangle$; i.e., $\forall x\ B_x \in J$ but $Y' = \{y : B^y \notin I\} \notin J$. Since Y' is Borel (see [9], 16.1 and 17.25), $Y' \cap A \notin \mathcal{J}$. Let $D = \bigcup_{q \in \mathbb{Q}}(B + \langle q, 0\rangle)$. Then $\forall x \in \mathbb{R}\ D_x \in J$ and $\forall y \in Y'\ \mathbb{R} \setminus D^y \in I$, since $D^y = \bigcup_{q \in \mathbb{Q}}(B^y + q)$ and $B^y \in \mathbf{B}(\mathbb{R}) \setminus I$. Let $C = Y' \cap A$. The rest of the argument is well-known. Since $|C| \leq \kappa < \boldsymbol{c}$ and $\forall y \in C\ (\mathbb{R} \setminus D^y) \in I$, MA implies that $\bigcap_{y \in C} D^y \neq \emptyset$. But if $x \in \bigcap_{y \in C} D^y$, then $C \subseteq D_x \in J$, contradicting the fact that $C \notin J$.

(iii) Let $I \in \{\mathbb{K}, \mathbb{L}\}$. Let $B \subseteq \mathbb{R} \times \mathbb{R}$ be a Borel set with $\forall x\ B_x \in I$. Set $E = \bigcup_{x \in A} B_x$. We have $\{y : \ B^y \notin J\} \subseteq E$, since for each y, $B^y \notin J \to B^y \cap A \neq \emptyset$. But $|A| = \kappa < \boldsymbol{c}$; so MA implies that $E \in I$.

(iv) follows from (ii) and 3.6. □

Of course the result above is not a satisfactory solution to the problem of finding a new pair of *natural* ccc σ-ideals with FP or, even better, a single ccc σ-ideal I having FP with itself and such that MGR $\not\equiv_B I \not\equiv_B$ NULL. The latter is apparently open even if the word "natural" is omitted.

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