Real Analysis Exchange
Vol. 26(1), 2000/2001, pp. 445–448

Toby C. O'Neil, Department of Pure Mathematics, Faculty of Maths and Computing, The Open University, Walton Hall, Milton Keynes, MK7 6AA, UK. e-mail: `t.c.oneil@open.ac.uk`

GRAPHS OF CONTINUOUS FUNCTIONS FROM $\mathbb{R}$ TO $\mathbb{R}$ ARE NOT PURELY UNRECTIFIABLE

Abstract

We present an elementary proof that the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ is not purely unrectifiable. As a consequence of our method, we observe that all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ meet the graph of some monotonic function in a set of positive linear measure.

In an unpublished note, Gy. Petruska showed that a typical continuous function meets any function of bounded variation at a nowhere dense null set; this improves on the well-known fact (which is an immediate consequence of Jarník's result [2] that a typical continuous function has at a.e. point no approximate derivative) that a typical continuous function meets any Lipschitz function in a null set. As a natural counterpart to this result, we show that the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ meets the *rotated* graph of a Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$ in a set of positive linear measure. In a similar way, we also observe that the graph of any continuous function meets the graph of a monotone function in a set of positive linear measure. This should be contrasted with the result of Humke and Laczkovich [1] which states that a typical continuous graph meets any monotone graph over a strongly porous subset of the real axis.

Recall that a subset, E, of the plane is purely unrectifiable if for all Lipschitz maps $f\colon \mathbb{R} \to \mathbb{R}^2$, the linear measure of $f(\mathbb{R}) \cap E$ is zero. This is equivalent to saying that E meets all rotatations of graphs of Lipschitz functions from $\mathbb{R} \to \mathbb{R}$ in a set of zero linear measure.

We give a simple proof that:

Theorem. *If $f\colon [0,1] \to \mathbb{R}$ is a continuous function then the graph of f is not purely unrectifiable.*

Key Words: rectifiable, continuous
Mathematical Reviews subject classification: 28A27
Received by the editors February 23, 2000

PROOF. We may suppose that $f(0) < f(1)$: The same argument works in the case when $f(a) < f(b)$ for some $a < b$ and a symmetric one applies if $f(a) > f(b)$ for some $a < b$. And if f is constant, then the statement is obvious.

Since $f(0) < f(1)$ and f is continuous, the projection of the graph of f onto the y-axis must be a closed interval containing $[f(0), f(1)]$. Define the set E to be those points, x, of the graph of f with y-coordinate lying in the interval $[f(0), f(1)]$, and for which the line segment orthogonal to the y-axis connecting x to the y-axis intersects the graph of f only at x. Since the orthogonal projection of E onto the y-axis contains the interval $[f(0), f(1)]$, it necessarily has positive linear measure. It thus suffices to show that E lies on the rotated graph of a Lipschitz function. Fix a point $x \in E$. Let $X(x)$ denote the open, infinite, upper-left quadrant of the plane with vertex x and boundary consisting of the semi-infinite vertical ray $\{x + (0, \alpha) : \alpha > 0\}$ and the semi-infinite horizontal ray to the left of x which ends at x. Since f is continuous, it follows that $X(x)$ cannot intersect the graph of f. We note that the double-sided infinite cone which contains $X(x)$ may only intersect E at x. It is now immediate that E lies on the graph of a Lipschitz function rotated through $\pi/4$; see either [3, Lemma 15.13] or [4] for further details. □

Remarks.

1. The method of the proof also shows that f intersects the graph of a monotonic function in a set of positive linear measure; for the restriction of f to $f^{-1}(E)$ in the proof of the theorem is increasing (and bounded) and so may be extended to a monotone function from $\mathbb{R}$ to $\mathbb{R}$.

2. The method fails completely if we drop either the requirement that f be continuous or only ask that f be continuous on a set of positive measure. A simple example of this is given by the lower envelope of the von Koch curve: this is the function on the unit interval whose graph consists of the closest point of the von Koch curve to the x-axis. (We assume that the curve lies above the unit interval.) Since the von Koch curve is purely unrectifiable, it follows that the graph of this function is also purely unrectifiable. Lusin's theorem now allows us to find a subset of the unit interval of positive measure on which this function is continuous.

3. An immediate corollary of this result is that the graph of 1-dimensional Brownian motion contains rectifiable pieces. Unfortunately it seems to give no insight into understanding Brownian motion in the plane.

References

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