Real Analysis Exchange
Summer Symposium 2002 pp. 491–492

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LINEAR RECURRENCE RELATIONS ON MEASURABLE SETS

Let $A \subset \mathbb{R}$ be a measurable set, and let $\{a_n\} \to 0$ be a sequence of nonzero real numbers with the property that $\chi_A(x + a_n) = \chi_A(x)$ for almost every $x \in \mathbb{R}$. It is known, that A must be either null or of full measure. (See [1].) Here we sketch the proof to illustrate the importance of the assumption that the sequence $\{a_n\}$ converges to zero. Set $F(x) = \int_0^x \chi_A(t)dt$. Then it is straight-forward to check that the condition $\chi_A(x + a_n) = \chi_A(x)$ *a.e.* implies that $F(x + a_n + h) - F(x + h) - F(x + a_n) + F(x) \equiv 0$ for all x and h. It follows that for every n, $F(x + a_n) - F(x)$ is differentiable everywhere and its derivative is zero. Thus $F(x + a_n) - F(x) = \gamma_n$ for some constant γ_n. Now for almost every x we have $\chi_A(x) = \lim_{n\to\infty} \frac{F(x+a_n)-F(x)}{a_n} = \lim_{n\to\infty} \frac{\gamma_n}{a_n} =$ a constant. Thus $\chi_A(x) = 0$ or 1 *a.e.*.

Similarly, for two sets, A and B and corresponding nonzero sequences, $\{a_n\} \to 0$ and $\{b_n\} \to 0$, if $\chi_A(x + a_n) + \chi_B(x + b_n) = \chi_A(x) + \chi_B(x)$ for almost every $x \in \mathbb{R}$, then either A and B are null, of full measure, or are complements of each other modulo a null set. In other words $\chi_A(x)+\chi_B(x) =$ a constant almost everywhere. Moreover, in [2] this was generalized to three sets. Finally in [3] the problem of characterizing sets A_1, $A_2, \ldots, A_m$ and zero convergent sequences $\{a_{i,n}\}$ of real numbers that satisfy a linear recurrence relation $\sum_{i=1}^m c_i\chi_{A_i}(x + a_{i,n}) = 0$ almost everywhere has been completely solved.

Question. What is the situation if the sequences do not converge to zero, but are merely bounded, or converge to different values? Also category analogs to any of these questions are unknown.

Key Words: linear recurrence relation, measurable set.
Mathematical Reviews subject classification: 26A12
Received by the editors June 12, 2008

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[3] H. Fejzić, C. Freiling, D. Rinne, *Linear Recurrence relations on measurable sets*, 20pp (to appear).