INROADS

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 475–482

David Rose, Department of Mathematics and Computer Science, Florida Southern College, Lakeland, Florida 33801. email: `drose@flsouthern.edu`
Ben Thurston, 913 Van Drive, Auburndale, FL 33823.
email: `benthurston27@yahoo.com`

MAXIMALLY RESOLVABLE LOWER DENSITY SPACES

Abstract

We show that all measure based and all category based lower density topologies on $\mathbb{R}$ are maximally resolvable in **ZFC** and extraresolvable under **ZFC + MA**. It is also noted that both maximal resolvability and extraresolvability are feeble topological properties.

1 Introduction.

Recently several new measure based and category based lower density topologies on $\mathbb{R}$ have been found and investigated by various authors [1, 24, 23, 9, 7, 21, 25]. An excellent history and brief overview of density topologies predating 2002 can be found in Chapter 15 of the *Handbook of Measure Theory* [22]. The main purpose of this note is to show that all measure and category based lower density topologies on $\mathbb{R}$ are both maximally resolvable in **ZFC** and extraresolvable under **ZFC + MA**. A space is resolvable [8] if a dense set exists whose complement is also dense. That is, a dense codense set exists. Certainly, since every nonempty open set must nontrivially intersect every dense set, a space X with topology τ cannot have more than $\Delta(\tau)$ pairwise disjoint dense subsets where $\Delta(\tau)$, the dispersion character of (X, τ), is the least cardinal number for any nonempty open set. A topological space (X, τ) having a family $\mathcal{D}$ of $\Delta(\tau)$ pairwise disjoint dense subsets is said to be $\Delta(\tau)$-resolvable [20]. J. G. Ceder [3] defined a space (X, τ) to be maximally

Key Words: measure based lower density space, category based lower density space, maximal resolvability, extraresolvability, feeble topological property, feeble homeomorphism
Mathematical Reviews subject classification: Primary: 28A05, 54A10; Secondary: 54C08
Received by the editors October 28, 2007
Communicated by: Krzysztof Ciesielski

resolvable if it is $\Delta(\tau)$-resolvable for some pairwise disjoint family $\mathcal{D}$ of dense subsets having the additional property that each dense set in the family meets each nonempty open set with an intersection of at least cardinality $\Delta(\tau)$. We now show that this additional property is redundant in spaces with infinite dispersion character such as T_1 spaces without isolated points. In particular it was noted in [5] that T_1 resolvable spaces have no isolated points.

Lemma 1. *If (X,τ) is any topological space with infinite dispersion character $\Delta(\tau)$ then (X,τ) is maximally resolvable if and only if there exist $\Delta(\tau)$ many pairwise disjoint dense subsets.*

PROOF. The necessity is clear. For the sufficiency, suppose that $\mathcal{D}$ is a pairwise disjoint family of dense subsets of (X,τ) with $|\mathcal{D}| = \Delta(\tau) \geq \aleph_0$. Since by Theorem 5.2.4 of [4] $\Delta(\tau) = \Delta(\tau)^2$, members of $\mathcal{D}$ can be indexed by pairs of ordinals less than $\Delta(\tau)$. In particular $\mathcal{D} = \{D_{(\alpha,\beta)} \mid \alpha, \beta < \Delta(\tau)\}$. For each $\alpha < \Delta(\tau)$ let $D_\alpha = \cup_\beta D_{(\alpha,\beta)}$. Then each D_α is dense in (X,τ) and $\mathcal{D}^* = \{D_\alpha \mid \alpha < \Delta(\tau)\}$ is a pairwise disjoint family of $\Delta(\tau)$ dense subsets of (X,τ). Now for each $\alpha < \Delta(\tau)$, if $U \in \tau$ and $U \neq \varnothing$, then $U \cap D_{(\alpha,\beta)} \neq \varnothing$ for each $\beta < \Delta(\tau)$. So $|U \cap D_\alpha| = |\cup_\beta U \cap D_{(\alpha,\beta)})| = \sum_\beta |U \cap D_{(\alpha,\beta)}| \geq \sum_\beta 1 = \Delta(\tau)$. Therefore (X,τ) is maximally resolvable in the sense of Ceder [3]. $\square$

V. I. Malykhin defined a topological space (X,τ) to be extraresolvable [12] (see also [13] or [6]) if there exists a family $\mathcal{D}$ containing more than $\Delta(\tau)$ dense subsets such that $C, D \in \mathcal{D}$ with $C \neq D$ implies $C \cap D$ is nowhere dense. By showing all measure or category based lower density topologies on $\mathbb{R}$ are both maximally resolvable in **ZFC** and extraresolvable under **ZFC + MA**, we extend slightly the paper [17], which is dedicated to finding the commonalities shared by all lower density topologies generally. We also extend the results of J. Luukkainen [11], A. Bella [2], and E. Wagner-Bojakowska [20].

A topology τ on $\mathbb{R}$ is a lower density topology if τ has for a base $\{B \in \mathcal{A} \mid B \subseteq \phi(B)\}$ where for some σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}$ and σ -ideal $\mathcal{I} \subseteq \mathcal{A}$, there is an operator $\phi : \mathcal{A} \to \mathcal{A}$ satisfying for all $A, B \in \mathcal{A}$, $\phi(\emptyset) = \emptyset$ and $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(A \cap B) = \phi(A) \cap \phi(B)$, $A \sim B \Rightarrow \phi(A) = \phi(B)$ (where $A \sim B$ means that $A \triangle B \in \mathcal{I}$), and $A \sim \phi(A)$. It is shown in [17] that regardless of how the operator ϕ is defined the base for τ is closed under arbitrary union and hence $\tau = \{B \in \mathcal{A} \mid B \subseteq \phi(B)\}$ if $\mathcal{A}$ and $\mathcal{I}$ are either the family of Lebesgue measurable sets $\mathcal{L}$ and the ideal of null sets $\mathcal{N}$ or the family of Baire sets $\mathcal{B}$ and the ideal of first category sets $\mathcal{M}$. Thus $\tau_{\mathcal{L}} \subseteq \mathcal{L}$ for every measure based density topology $\tau_{\mathcal{L}}$ and $\tau_{\mathcal{B}} \subseteq \mathcal{B}$ for every category based density topology $\tau_{\mathcal{B}}$.

We say that two density topologies are of the same type if they are induced by lower density operators on the same σ-algebra with set equivalence relative to the same σ-ideal. In the third section below we show that both maximal resolvability and extraresolvability are feeble topological properties. It will follow that if one measure or category based density space is extraresolvable then all are. In fact, under **ZFC + MA**, Bella [2] showed that the Lebesgue density topology is extraresolvable and with the same set-theoretic assumptions Wagner-Bojakowska [20] showed that the I-density topology is extraresolvable. The same reasoning works also for maximal resolvability of all density topologies in **ZFC** since Luukkainen [11] showed that the Lebesgue density topology is maximally resolvable and Wagner-Bojakowska [20] showed that the I-density topology is maximally resolvable. However, in the second section below we avoid using feeble homeomorphism to find instead a unified approach in **ZFC** to maximal resolvability for both types of density topologies using Bernstein sets. Recall that a subset of $\mathbb{R}$ is a Bernstein set if neither it nor its complement contains any perfect (nonempty, closed and dense-in-itself relative to the natural topology) set. In what follows $\tau_{\mathcal{L}}$ and $\tau_{\mathcal{B}}$ generically represent measure based and category based lower density topologies respectively on $\mathbb{R}$.

2 Maximal Resolvability.

Oxtoby in his excellent monograph [14] called attention to many similarities between measure and category in topological spaces. At least in the case of maximal resolvability perfect sets are the common denominator. In particular Theorem 6.3.6 of [4] states that for each $A \in (\mathcal{L} - \mathcal{N}) \cup (\mathcal{B} - \mathcal{M})$ there exists a perfect set $P \subseteq A$. Now if $U \in \tau_{\mathcal{L}} \cup \tau_{\mathcal{B}}$ and $U \neq \emptyset$ then $U \subseteq \phi(U) \Rightarrow \phi(U) \neq \emptyset \Rightarrow U \in (\mathcal{L} - \mathcal{N}) \cup (\mathcal{B} - \mathcal{M})$ and thus $P \subseteq U$ for some perfect set P. This establishes firstly that $\Delta(\tau_{\mathcal{L}}) = \Delta(\tau_{\mathcal{B}}) = c$ since $|P| = c$ for each perfect set P (Theorem 6.2.3 of [4]). Secondly, Bernstein sets in $\mathbb{R}$ are both $\tau_{\mathcal{L}}$-dense and $\tau_{\mathcal{B}}$-dense. For if B is a Bernstein set and U is a nonempty set in $\tau_{\mathcal{L}} \cup \tau_{\mathcal{B}}$ and $P \subseteq U$ is a perfect set then $P \nsubseteq \mathbb{R} - B \Rightarrow U \cap B \neq \emptyset$. Since $\mathbb{R} - B$ is also a Bernstein set, it follows that both $(\mathbb{R}, \tau_{\mathcal{L}})$ and $(\mathbb{R}, \tau_{\mathcal{B}})$ are resolvable. To conclude maximal resolvability it is enough in light of the foregoing Lemma to show that a family of c pairwise disjoint Bernstein sets exists. Then each type of density topology on $\mathbb{R}$ yields a T_1 c-resolvable space without isolated points. But in fact it follows from either Exercise 1 page 105 or Theorem 7.3.4 of [4] that a family $\mathcal{F}$ of pairwise disjoint Bernstein sets with $|\mathcal{F}| = c$ exists. This establishes the following.

Theorem 2. ***(ZFC)*** *The space $\mathbb{R}$ endowed with any measure or category based lower density topology is maximally resolvable.*

Corollary 3. *The three category density topologies of Lazarow, Johnson and*

Wilczyński [10] and the category density topology of Wiertelak [21] are maximally resolvable. Also, the simple density topology of Aversa and Wilczyński [1], the complete density topology of Wilczyński and Wojdowski [23], and the $\mathcal{A}_d$ density topology of Wojdowski [24, 25] are maximally resolvable.

3 Extraresolvability.

Recall that a topological space (X, τ) is extraresolvable [12] if there is a family of more than $\Delta(\tau)$ dense subsets having the property that any pair of distinct dense subsets in the family have nowhere dense intersection. A. Bella [2] showed that unlike the Euclidean topology on $\mathbb{R}$, it is consistent with **ZFC** that the Lebesgue density topology is extraresolvable. Specifically, in [2] it was shown that the Lebesgue density topology is extraresolvable under **ZFC** + **MA**. Elżbieta Wagner-Bojakowska [20] proved a similar result for the I-density topology. As stated earlier we make use of feeble homeomorphism to extend extraresolvability to all measure or category based density topologies under **ZFC** + **MA**.

Definition 4. A subset U of a space X is feebly open if either $U = \emptyset$ or $\mathrm{int}U \neq \emptyset$ where $\mathrm{int}U$ is the interior of U. A function $f : X \to Y$ between topological spaces is feebly continuous if the subset $f^{-1}(V) \subseteq X$ is feebly open for each open subset $V \subseteq Y$. A bijection $h : X \to Y$ is a feeble homeomorphism if both h and h^{-1} are feebly continuous. A property is feebly topological if it is preserved by feeble homeomorphisms.

Strongly irresolvable spaces, those spaces for which every open subspace is irresolvable, were studied in [18] where it was shown that strong irresolvability is feebly topological. Here we show that maximal resolvability and extraresolvability are also feeble topological properties. It is clear that feebly continuous surjections carry dense sets to dense sets. In fact by Proposition 6.3 of [18], a bijection is a feeble homeomorphism if and only if dense sets are both directly and inversely preserved by the bijection. Also, Proposition 6.5 of [18] states that feeble homeomorphisms directly and inversely preserve nowhere dense sets. Of course bijections preserve cardinalities of subsets and cardinalities of families of subsets.

Theorem 5. *Maximal resolvability and extraresolvability are preserved by feeble homeomorphism.*

Proof. From the foregoing remarks and the Lemma above, it is enough to show that dispersion character is preserved by feeble homeomorphism. To this end, let $f : (X, \tau) \to (Y, \sigma)$ be a feeble homeomorphism between topological spaces. For all $V \in \sigma$ with $V \neq \varnothing$, we have $\Delta(\tau) \leq |\text{int} f^{-1}(V)| \leq |f^{-1}(V)| = |V|$. It follows that $\Delta(\tau) \leq \Delta(\sigma)$. By symmetry of argument, since f^{-1} is also a feeble homeomorphism we have $\Delta(\sigma) \leq \Delta(\tau)$ so that $\Delta(\tau) = \Delta(\sigma)$. □

Corollary 6. *If ϕ_1 and ϕ_2 are lower density operators of the same type then if either of the induced density topologies τ_{ϕ_1} or τ_{ϕ_2} is maximally resolvable then both are maximally resolvable. Also, if either τ_{ϕ_1} or τ_{ϕ_2} is extraresolvable then both are extraresolvable.*

Proof. By Theorem 4 of [17], the identity map $i : (\mathbb{R}, \tau_{\phi_1}) \to (\mathbb{R}, \tau_{\phi_2})$ is a feeble homeomorphism. □

Remark 7. The full force of feeble homeomorphism is not really needed in the proof of the corollary above since the identity function preserves sets. The fact that dense sets are preserved is enough here because it is known, and is found in Corollary 8.1 of [17], that the ideal coincides with the family of nowhere dense sets in the density space for operators of either type. In other words $\mathcal{N}$, the ideal of null sets, is the family of nowhere dense sets in every measure based density topology and $\mathcal{M}$, the ideal of first category sets, is the family of nowhere dense sets for every category based density topology. Hence for any two lower density operators of the same type, the identity function between their induced density spaces preserves nowhere dense sets. In any case, we have the following.

Theorem 8. ***(ZFC + MA)*** *The space $\mathbb{R}$ endowed with any measure or category based lower density topology is extraresolvable.*

Acknowledgement. We are very grateful to the referee for offering some corrections in the original manuscript, suggesting consideration of extraresolvability, and for providing some important references.

References

[1] V. Aversa & W. Wilczyński, *Simple density topology*, Rend. Circ. Mat. Palermo, Serie II-Tomo LIII, (2004), 344–352.

[2] A. Bella, *The density topology is extraresolvable*, Atti Sem. Mat. Fis. Univ. Modena, **48** (2000), 495–498.

[3] J. G. Ceder, *On maximally resolvable spaces*, Fund. Math., **55** (1964), 87–93.

[4] K. Ciesielski, *Set Theory for the Working Mathematician*, Cambridge University Press, (1997).

[5] W. W. Comfort and S. García-Ferreira, *Resolvability: a selective survey and some new results*, Topology Appl., **74** (1996), 149–167.

[6] W. W. Comfort and S. García-Ferreira, *Dense subsets of maximally almost periodic groups*, Proc. Amer. Math. Soc., **129(2)** (2000), 593–599.

[7] J. Hejduk and G. Horbaczewska, *On $\mathcal{I}$-density topologies with respect to a fixed sequence*, Reports on Real Analysis, Conference at Rowy (2003), 78–85.

[8] E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J., **10** (1943), 309–333.

[9] G. Horbaczewska, *The family of $\mathcal{I}$-density Type Topologies*, Comment. Math. Carolinae, **46(4)** (2005), 735–745.

[10] E. Łazarow, R. A. Johnson, and W. Wilczyński, *Topologies related to sets having the Baire property*, Demonstratio Math., **22(1)** (1989), 179–191.

[11] J. Luukkainen, *The density topology is maximally resolvable*, Real Anal. Exchange, **25(1)** (1999/2000), 419–420.

[12] V. I. Malykhin, *Irresolvability is not descriptively good*, privately circulated, 1996.

[13] V. I. Malykhin, *Something new about totally bounded groups*, Mat. Zametki, **65** (1999), 474–476. (Russian).

[14] J. C. Oxtoby, *Measure and Category*, Second Edition, Springer-Verlag (1980).

[15] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński, *A Category analogue of the density topology*, Fund. Math., **125** (1985), 167–173.

[16] W. Poreda, E. Wagner-Bojakowska, and W. Wilczyński, *Remarks on I-density and I-approximately continuous functions*, Comment. Math. Univ. Carolinae, **26(3)** (1985), 553–563.

[17] D. A. Rose, D. Janković, and T. R. Hamlett, *Lower density topologies*, Ann. New York Acad. Sci., Papers on General Topology and Applications, **704** (1993), 309–321.

[18] David Rose, Kari Sizemore, and Ben Thurston, *Strongly irresolvable spaces*, Int. J. Math. Math. Sci., **2006** (2006), 1–12.

[19] F. D. Tall, *The density topology*, Pac. J. Math., **67(1)** (1976), 275–284.

[20] Elżbieta Wagner-Bojakowska, *Resolvability of I-density topology*, Zeszyty Naukowe, **1(1)** (2001), 35–37.

[21] Renata Wiertelak, *A generalization of the density topology with respect to category*, Real Anal. Exchange, **32(1)** (2006/2007), 273–286.

[22] W. Wilczyński, *Density Topologies*, Chapter 15 in *Handbook of Measure Theory*, Ed. Endre Pap, Elsevier Sci. (2002), 675–702.

[23] W. Wilczyński and W. Wojdowski, *Complete density topology*, Indagationes Mathematicae, **18(2)** (2007), 295–302.

[24] W. Wojdowski, *A generalization of density topology*, Real Anal. Exchange, **32(2)** (2006/2007), 1–10.

[25] W. Wojdowski, *On generalization of the density topology on the real line*, (to appear).