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THE APPROXIMATE VARIATIONAL INTEGRAL

Abstract

The concept of the GAP-integral was introduced by the authors [7]. In this paper we characterize the Variational integral by the GAP-integral and present some significant convergence theorems for the GAP-integral.

1 Introduction.

The Approximately Continuous Perron integral was introduced by Burkill [1] and its Riemann-type definition was given by Bullen [2] . Schwabik [8] presented a generalized version of the Perron integral leading to the new approach to a generalized ordinary differential equation. The authors [7] introduced the concept of the Generalized Approximately Continuous Perron integral (GAP) and established some fundamental properties of the integral. The Variational integral is a kind of non-absolute integral originally defined by R. Henstock [4]. Kubota [5] has shown some elementary properties of the integral including Cauchy and Harnack extensions. In the present paper, we shall establish a characterization of the Variational integral by the GAP-integral and define the Approximate Variational integral. An attempt has been made to establish some significant convergence theorems of the GAP-integral using the Approximate Variational integral.

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2 Preliminaries.

Definition 2.1. *A collection Δ of closed subintervals of $[a,b]$ is called an approximate full cover (AFC) if for every $x \in [a,b]$ there exists a measurable set $D_x \subset [a,b]$ such that $x \in D_x$ and D_x has density 1 at x, with $[u,v] \in \Delta$ whenever $u,v \in D_x$ and $u \le x \le v$.*

For all approximate full covers that occur in this paper the sets $D_x \subset [a,b]$ are the same. Then the relations $\Delta_1 \subseteq \Delta_2$ or $\Delta_1 \cap \Delta_2$ for approximate full covers Δ_1, Δ_2 are clear.
A division of $[a,b]$ obtained by $a = x_0 < x_1 < \cdots < x_n = b$ and $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is called a Δ-division if Δ is an approximate full cover with $[x_{i-1}, x_i]$ coming from Δ or more precisely, if we have $x_{i-1} \le \xi_i \le x_i$ and $x_{i-1}, x_i \in D_{\xi_i}$ for all i. We call ξ_i the associated point of $[x_{i-1}, x_i]$ and x_i $(i = 0, 1, \ldots, n)$ the division points.
A division of $[a,b]$ given by $a \le y_1 \le \zeta_1 \le z_1 \le y_2 \le \zeta_2 \le z_2 \le \cdots \le y_m \le \zeta_m \le z_m \le b$ is called a Δ-partial division if Δ is an approximate full cover with $([y_i, z_i], \zeta_i) \in \Delta$, for $i = 1, 2, \ldots, m$.
The next Cousin-type lemma from [3] makes it possible to give a Riemann-type definition of the GAP-integral.

Lemma 2.2. *If Δ is an approximate full cover of $[a,b]$, then there exists a tagged partition P of $[a,b]$ such that $P \subseteq \Delta$.*

In [7], the GAP-integral is defined as follows:

Definition 2.3. *A function $U : [a,b] \times [a,b] \to R$ is said to be Generalized Approximate Perron (GAP)-integrable to a real number A if for every $\epsilon > 0$ there is an AFC Δ of $[a,b]$ such that for every Δ-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$ we have*

$$|(D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\} - A| < \epsilon$$

and we write $A = (GAP) \int_a^b U$.

The set of all functions U which are GAP-integrable on $[a,b]$ is denoted by GAP$[a,b]$. We use the notation

$$S(U,D) = (D) \sum \{U(\tau, \beta) - U(\tau, \alpha)\}$$

for the Riemann-type sum corresponding to the function U and the Δ-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$. Note that the integral is uniquely determined.

Remark 2.4. *Setting $U(\tau, t) = f(\tau)t$ where $f : [a,b] \to R$ and $t, \tau \in [a,b]$, we obtain the ap-Henstock integral [3] .*

With the notion of partial division we have proved the following theorem in [7].

Theorem 2.5. (Saks-Henstock Lemma) *Let $U : [a,b]\times[a,b] \to R$ be GAP-integrable over $[a,b]$. Then, given $\epsilon > 0$, there is an approximate full cover Δ of $[a,b]$ such that for every Δ-division $D = \{([\alpha_{j-1},\alpha_j],\tau_j); j = 1,2,\ldots,q\}$ of $[a,b]$ we have*

$$|\sum_{j=1}^{q}\{U(\tau_j,\alpha_j) - U(\tau_j,\alpha_{j-1})\} - (GAP)\int_a^b U| < \epsilon.$$

Then, if $\{([\beta_j,\gamma_j],\zeta_j); j = 1,2,\ldots,m\}$ represents a Δ-partial division of $[a,b]$, we have

$$|\sum_{j=1}^{m}[\{U(\zeta_j,\gamma_j) - U(\zeta_j,\beta_j)\} - (GAP)\int_{\beta_j}^{\gamma_j} U]| < \epsilon.$$

In [7], the indefinite GAP-integral is defined as follows:

Definition 2.6. *Let $U \in GAP[a,b]$. The function $\phi : [a,b] \to R$ defined by $\phi(s) = (GAP)\int_a^s U, a < s \leq b$, $\phi(a) = 0$ is called the indefinite GAP-integral of U.*

Given a function $\phi : [a,b] \to R$ then for $[\alpha,\beta] \subset [a,b]$, we put $\phi(\alpha,\beta) = \phi(\beta) - \phi(\alpha)$.

3 The Approximate Variational Integral.

Definition 3.1. *An interval function ψ is said to be non-negative if $\psi(x,y) \geq 0$ and superadditive if $\psi(x,y) + \psi(y,z) \leq \psi(x,z)$ when $x < y < z$.*
A function $U : [a,b]\times[a,b] \to R$ is said to be approximately variationally integrable on $[a,b]$ with the primitive ϕ if for every $\epsilon > 0$ there is an approximate full cover Δ of $[a,b]$ and a non-negative superadditive interval function ψ with $\psi(a,b) < \epsilon$ such that whenever $([\alpha,\beta],\tau) \in \Delta$ we have

$$|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}| \leq \psi(\alpha,\beta).$$

Theorem 3.2. *A function $U : [a,b]\times[a,b] \to R$ is approximately variationally integrable on $[a,b]$ if and only if $U \in GAP[a,b]$.*

Proof. Suppose that U is approximately variationally integrable on $[a,b]$ with the primitive ϕ. Then for every $\epsilon > 0$ there is an approximate full cover Δ of $[a,b]$ and a non-negative superadditive interval function ψ with $\psi(a,b) < \epsilon$ such that whenever $([\alpha,\beta],\tau) \in \Delta$ we have

$$|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}| \leq \psi(\alpha,\beta).$$

Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\begin{aligned}|\phi(a,b) - \sum\{U(\tau,\beta) - U(\tau,\alpha)\}| &\leq \sum|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}| \\ &\leq \sum\psi(\alpha,\beta) \leq \psi(a,b) < \epsilon.\end{aligned}$$

Hence $U \in GAP[a,b]$. Now we suppose that $U \in GAP[a,b]$. Let $\psi(x,y) = \sup\sum|\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)|$ where the supremum is over all Δ-division $D = ([\alpha,\beta],\tau)$ of $[x,y]$. Since $U \in GAP[a,b]$, given $\epsilon > 0$, there is an approximate full cover Δ of $[a,b]$ such that for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$\sum|\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)| < \epsilon.$$

It is clear that $\psi(x,y) \geq 0$, $\psi(x,y) + \psi(y,z) \leq \psi(x,z)$ when $x < y < z$ and $\psi(a,b) = \sup\sum|\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)| < \epsilon$ where the supremum is over all Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$. Then ψ satisfies the required condition and U is approximately variationally integrable on $[a,b]$. □

Theorem 3.3. (Generalized Basic Convergence Theorem) *Let (i) $U_n : [a,b] \times [a,b] \to R$ be GAP-integrable on $[a,b]$ with the primitives ϕ_n, $n = 1, 2, \ldots$, (ii) there be an approximate full cover Δ' of $[a,b]$ such that*

$$\lim_{n \longrightarrow \infty}[U_n(\tau,t_2) - U_n(\tau,t_1)] = U(\tau,t_2) - U(\tau,t_1)$$

for each $\tau \in [a,b]$ and every interval-point pair $([t_1,t_2],\tau) \in \Delta'$, (iii) ϕ_n converge point-wise to a limit function ϕ. Then $U \in GAP[a,b]$ with the primitive ϕ if and only if for every $\epsilon > 0$ there exists a function $M(\tau)$ defined on $[a,b]$ taking integer values such that for infinitely many $m(\tau) \geq M(\tau)$ there is an approximate full cover Δ and a non-negative superadditive interval function ψ with $\psi(a,b) < \epsilon$ such that whenever $([\alpha,\beta],\tau) \in \Delta$ we have

$$|\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)| \leq \psi(\alpha,\beta).$$

Proof. Suppose that $U \in GAP[a,b]$ with the primitive ϕ. Then U is also approximately variationally integrable on $[a,b]$; i.e. there is an approximate full cover Δ_0 of $[a,b]$ and a non-negative superadditive interval function ψ_0 with $\psi_0(a,b) < \epsilon$ such that for any Δ_0-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}| \leq \psi_0(\alpha,\beta).$$

Again, since each U_n is also approximately variationally integrable on $[a,b]$, there is an approximate full cover Δ_n of $[a,b]$ and a non-negative superadditive interval function ψ_n with $\psi_n(a,b) < \epsilon\ 2^{-n}$ such that for any Δ_n-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have $|\phi_n(\alpha,\beta) - \{U_n(\tau,\beta) - U_n(\tau,\alpha)\}| \leq \psi_n(\alpha,\beta)$. Given $\epsilon > 0$, for every fixed Δ'-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, there exists an integer $M(\tau)$ such that whenever $m(\tau) \geq M(\tau)$ we have

$$|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \epsilon \quad \textit{for every } \tau \in [a,b].$$

Without any loss of generality, we may assume that $\Delta' = \Delta_1 \cap \Delta_2 \cap \cdots \cap \Delta_{m(\tau)}$. For each $\tau \in [a,b]$, we choose any integer $m(\tau) \geq M(\tau)$ and we take $\Delta = \Delta' \cap \Delta_0$. Also let $\psi(x,y) = \psi_0(x,y) + \sum_{n=1}^{\infty} \psi_n(x,y)$. Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, we have

$$\begin{aligned} |\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)| &\leq |\phi_{m(\tau)}(\alpha,\beta) - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}| \\ &\quad + |\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| \\ &\quad + |\{U(\tau,\beta) - U(\tau,\alpha)\} - \phi(\alpha,\beta)| \\ &\leq \psi_{m(\tau)}(\alpha,\beta) + \epsilon + \psi_0(\alpha,\beta) \leq \psi(\alpha,\beta) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $|\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)| \leq \psi(\alpha,\beta)$. Conversely, suppose that the condition is satisfied. Then for every $\epsilon > 0$ there is a function $M(\tau)$ defined on $[a,b]$ taking integer values such that for infinitely many $m(\tau) \geq M(\tau)$ there is an approximate full cover Δ_0 of $[a,b]$ and a non-negative superadditive interval function ψ with $\psi(a,b) < \epsilon$ such that for any Δ_0-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have $|\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)| \leq \psi(\alpha,\beta)$. Also for every fixed Δ'-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we can find $m(\tau) \geq M(\tau)$ such that $|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \epsilon$ for every $\tau \in [a,b]$. Using the same notations as in the first part, we choose $\Delta = \Delta' \cap \Delta_0,\ \tau \in [a,b]$. Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, we have

$$\begin{aligned} &|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}| \\ &\le |\phi(\alpha,\beta) - \phi_{m(\tau)}(\alpha,\beta)| + |\phi_{m(\tau)}(\alpha,\beta) - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}| \\ &+ |\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| \\ &\le \psi(\alpha,\beta) + \psi_{m(\tau)}(\alpha,\beta) + \epsilon. \end{aligned}$$

Therefore, by definition, U is approximately variationally integrable on $[a,b]$ with the required interval function provided by the right hand side of the above inequality. Hence $U \in GAP[a,b]$ with the primitive ϕ. □

In [6] we have proved the Basic Convergence theorem for the GAP-integral which is stated as follows:

Theorem 3.4. (Basic Convergence Theorem) *Let* (i) $U_n : [a,b]\times[a,b] \to R$ *be GAP-integrable on* $[a,b]$ *with the primitives* ϕ_n, $n = 1, 2, \dots$, (ii) *there be an approximate full cover* Δ' *of* $[a,b]$ *such that*

$$\lim_{n\longrightarrow\infty}[U_n(\tau,t_2) - U_n(\tau,t_1)] = U(\tau,t_2) - U(\tau,t_1)$$

for each $\tau \in [a,b]$ *and every interval-point pair* $([t_1,t_2],\tau) \in \Delta'$, (iii) ϕ_n *converge point-wise to a limit function* ϕ. *Then* $U \in GAP[a,b]$ *with the primitive* ϕ *if and only if for every* $\epsilon > 0$ *there is a function* $M(\tau)$ *defined on* $[a,b]$ *taking integer values such that for infinitely many* $m(\tau) \ge M(\tau)$ *there is an approximate full cover* Δ *such that for any* Δ*-division* $D = ([\alpha,\beta],\tau)$ *of* $[a,b]$ *we have*

$$|\sum\{\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)\}| < \epsilon.$$

Remark 3.5. *Theorem 3.4 immediately follows from Theorem 3.3.*

Definition 3.6. *A sequence of functions* $\{\phi_n\}$ *is said to be oscillation convergent to* ϕ *on* $[a,b]$ *if* $[a,b]$ *is the union of a sequence of closed sets* X_i, $i = 1, 2, \dots$ *and for every* i *and* $\epsilon > 0$ *there is an integer* N *and a non-negative superadditive interval function* ψ *with* $\psi(a,b) < \epsilon$ *such that for infinitely many* $n \ge N$ *there is an approximate full cover* Δ_n *of* $[a,b]$ *such that for any* Δ_n*-division* $D = ([\alpha,\beta],\tau)$ *of* $[a,b]$ *tagged in* X_i*, for each* i*, we have*

$$|\phi_n(\alpha,\beta) - \phi(\alpha,\beta)| \le \psi(\alpha,\beta).$$

Theorem 3.7. (Oscillation Convergence Theorem) *Let (i) $U_n : [a,b] \times [a,b] \to R$ be GAP-integrable on $[a,b]$ with the primitives ϕ_n, $n = 1, 2, \ldots$, (ii) there be an approximate full cover Δ' of $[a,b]$ such that*

$$\lim_{n \longrightarrow \infty} [U_n(\tau, t_2) - U_n(\tau, t_1)] = U(\tau, t_2) - U(\tau, t_1)$$

for each $\tau \in [a,b]$ and every interval-point pair $([t_1, t_2], \tau) \in \Delta'$, (iii) the primitives ϕ_n be oscillation convergent to ϕ on $[a,b]$, (iv) the primitives ϕ_n converge uniformly to ϕ on $[a,b]$. Then $U \in GAP[a,b]$ with the primitive ϕ and

$$\lim_{n \longrightarrow \infty} (GAP) \int_a^b U_n = (GAP) \int_a^b U.$$

PROOF. Let $\epsilon > 0$ be given. In view of (iii) above, for every i and j there exists an integer $N(i,j)$ such that for infinitely many $n \geq N(i,j)$ there is an approximate full cover Δ_{ij} of $[a,b]$ and a non-negative superadditive interval function ψ_{ij} with $\psi_{ij}(a,b) < \epsilon \ 2^{-i-j}$ such that for any Δ_{ij}-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$ with $\tau \in X_i$ we have

$$|\phi_n(\alpha, \beta) - \phi(\alpha, \beta)| \leq \psi_{ij}(\alpha, \beta).$$

Take $n = n(i,j)$ so that the above inequality holds. We may assume that for each i, $\{\phi_{n(i,j)}\}$ is a subsequence of $\{\phi_{n(i-1,j)}\}$. Now consider $\phi_{n(j)} = \phi_{n(j,j)}$ in place of ϕ_n and write $Y_1 = X_1$ and $Y_i = X_i - (X_1 \cup X_2 \cup \cdots \cup X_{i-1})$ for $i = 2, 3, \ldots$. Put $M(\tau) = n(i)$ when $\tau \in Y_i$. We note that there are infinitely many $m(\tau) \geq M(\tau)$, namely all $n(j) \geq n(i)$. If $m(\tau)$ takes values in $\{n(j) : j \geq i\}$ when $m(\tau) \geq M(\tau) = n(i)$, we put $\Delta = \Delta_{m(\tau)}$ and define

$$\psi(\alpha, \beta) = \sum_{i,j} \psi_{ij}(\alpha, \beta).$$

Obviously, ψ is non-negative, superadditive and

$$\psi(a,b) = \sum_{i,j} \psi_{ij}(a,b) < \sum_{i,j} \epsilon \ 2^{-i-j} \leq \epsilon.$$

Then for any Δ-division $D = ([\alpha, \beta], \tau)$ of $[a,b]$ with $\tau \in Y_i$, for some i, we have

$$|\phi_{m(\tau)}(\alpha, \beta) - \phi(\alpha, \beta)| \leq \psi_{ij}(\alpha, \beta) \leq \psi(\alpha, \beta).$$

Hence the condition of Generalized Basic Convergence theorem is satisfied. Therefore $U \in GAP[a,b]$ with the primitive ϕ and

$$\lim_{n \longrightarrow \infty} (GAP) \int_a^b U_n = (GAP) \int_a^b U.$$

$\square$

In [6] we have proved the Mean Convergence theorem for the GAP-integral which is stated as follows:

Theorem 3.8. (Mean Convergence Theorem) *Let* (i) $U_n : [a,b] \times [a,b] \to R$ *be GAP-integrable on* $[a,b]$ *with the primitives* ϕ_n, $n = 1, 2, \ldots$, (ii) *there be an approximate full cover* Δ' *of* $[a,b]$ *such that*

$$\lim_{n \longrightarrow \infty} [U_n(\tau, t_2) - U_n(\tau, t_1)] = U(\tau, t_2) - U(\tau, t_1)$$

for each $\tau \in [a,b]$ *and every interval-point pair* $([t_1, t_2], \tau) \in \Delta'$, (iii) $[a,b]$ *be the union of a sequence of closed sets* X_i, $i = 1, 2, \ldots$ *and for every* i *and* $\epsilon > 0$ *there exist an integer* N *and an approximate full cover* Δ *of* $[a,b]$ *such that for any* Δ*-division* $D = ([\alpha, \beta], \tau)$ *of* $[a,b]$ *tagged in* X_i, *for each* i, *we have*

$$|\sum \{\phi_n(\alpha, \beta) - \phi(\alpha, \beta)\}| < \epsilon,$$

for some function ϕ, *whenever* $n \geq N$, (iv) *the primitives* ϕ_n *converge uniformly to* ϕ *on* $[a,b]$. *Then* $U \in GAP[a,b]$ *with the primitive* ϕ *and*

$$\lim_{n \longrightarrow \infty} (GAP) \int_a^b U_n = (GAP) \int_a^b U.$$

Remark 3.9. *Theorem 3.8 immediately follows from Theorem 3.7.*

In [6] , we have proved the following lemma.

Lemma 3.10. *Let* $U, V : [a,b] \times [a,b] \to R$ *be such that* $U, V \in GAP[a,b]$ *and if there be an approximate full cover* Δ_0 *of* $[a,b]$ *such that*

$$U(\tau, t) - U(\tau, \tau) \leq V(\tau, t) - V(\tau, \tau)$$

for every interval-point pair $([\tau, t], \tau) \in \Delta_0$ *where* $\tau < t$ *and*

$$U(\tau, \tau) - U(\tau, t) \leq V(\tau, \tau) - V(\tau, t)$$

for every interval-point pair $([t,\tau],\tau) \in \Delta_0$ *where* $t < \tau$, *then*

$$(GAP)\int_a^b U \leq (GAP)\int_a^b V$$

holds.

We have proved the Monotone Convergence theorem for the GAP-integral in [6]. We now give an alternative proof of the same theorem using the approximate variational integral.

Theorem 3.11. (Monotone Convergence Theorem) *Let* (i) $U, U_n : [a,b]\times[a,b] \to R$, $n = 1,2,\dots$ *be such that* $U_n \in GAP[a,b]$ *for all* $n = 1,2,\dots$ *with* $\sup\ (GAP)\int_a^b U_n < \infty$, (ii) *there be an approximate full cover* Δ_0 *of* $[a,b]$ *such that* $U_n(\tau,t) - U_n(\tau,\tau) \leq U_{n+1}(\tau,t) - U_{n+1}(\tau,\tau)$ *for every interval-point pair* $([\tau,t],\tau) \in \Delta_0$ *where* $\tau < t$ *and* $U_n(\tau,\tau) - U_n(\tau,t) \leq U_{n+1}(\tau,\tau) - U_{n+1}(\tau,t)$ *for every interval-point pair* $([t,\tau],\tau) \in \Delta_0$ *where* $t < \tau$, $(n = 1,2,\dots)$, (iii) *there be an approximate full cover* Δ' *of* $[a,b]$ *such that* $\lim_{n\longrightarrow\infty}[U_n(\tau,t_2) - U_n(\tau,t_1)] = U(\tau,t_2) - U(\tau,t_1)$ *for each* $\tau \in [a,b]$ *and every interval-point pair* $([t_1,t_2],\tau) \in \Delta'$. *Then* $U \in GAP[a,b]$ *and* $\lim_{n\longrightarrow\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U$.

PROOF. Let $\epsilon > 0$ be arbitrary. Let each $U_n \in GAP[a,b]$ with the primitive ϕ_n for each positive integer n. Then since each U_n is also approximately variationally integrable on $[a,b]$, there is an approximate full cover Δ_n of $[a,b]$ and a non-negative superadditive interval function ψ_n with $\psi_n(a,b) < \epsilon\ 2^{-n}$ such that for any Δ_n-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$ we have

$$|\phi_n(\alpha,\beta) - \{U_n(\tau,\beta) - U_n(\tau,\alpha)\}| \leq \psi_n(\alpha,\beta).$$

By (iii), given $\epsilon > 0$, for every fixed Δ'-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, there exists an integer $M(\tau)$ such that whenever $m(\tau)$ is an integer with $m(\tau) \geq M(\tau)$ we have

$$|\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| < \epsilon$$

for every $\tau \in [a,b]$. Since $\{(GAP)\int_a^b U_n\}$ is non-decreasing and bounded above by Lemma [3.10], $\{(GAP)\int_\alpha^\beta U_n\}$ is also non-decreasing and bounded above [7], where $[\alpha,\beta] \subset [a,b]$; i.e. $\{\phi_n(\alpha,\beta)\}$ is non-decreasing and bounded above, therefore, $\lim\limits_{n\longrightarrow\infty}\phi_n(\alpha,\beta)$ exists. Let $\lim\limits_{n\longrightarrow\infty}\phi_n(\alpha,\beta) = \phi(\alpha,\beta)$. Then for every $\epsilon > 0$ and for every $\tau \in [a,b]$ there exists a function $M(\tau)$ defined on $[a,b]$

taking integer values such that whenever $m(\tau) \geq M(\tau)$ we have $|\phi_{m(\tau)}(\alpha,\beta) - \phi(\alpha,\beta)| < \epsilon$. Define $\psi(\alpha,\beta) = \phi(\alpha,\beta) - \phi_{M(\tau)}(\alpha,\beta)$. Then ψ is non-negative, superadditive and $\psi(a,b) = \phi(a,b) - \phi_{M(\tau)}(a,b) < \epsilon$. For each $\tau \in [a,b]$, we choose any integer $m(\tau) \geq M(\tau)$ and we take $\Delta = \Delta' \cap \Delta_0 \cap \Delta_{m(\tau)}$. Then for any Δ-division $D = ([\alpha,\beta],\tau)$ of $[a,b]$, we have

$$\begin{aligned}
&|\phi(\alpha,\beta) - \{U(\tau,\beta) - U(\tau,\alpha)\}| \\
&\leq |\phi(\alpha,\beta) - \phi_{m(\tau)}(\alpha,\beta)| + |\phi_{m(\tau)}(\alpha,\beta) - \{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\}| \\
&\quad + |\{U_{m(\tau)}(\tau,\beta) - U_{m(\tau)}(\tau,\alpha)\} - \{U(\tau,\beta) - U(\tau,\alpha)\}| \\
&\leq \{\phi(\alpha,\beta) - \phi_{M(\tau)}(\alpha,\beta)\} + \psi_{m(\tau)}(\alpha,\beta) + \epsilon \leq \psi(\alpha,\beta) + \psi_{m(\tau)}(\alpha,\beta) + \epsilon.
\end{aligned}$$

Therefore, by definition, U is approximately variationally integrable on $[a,b]$ with the required interval function provided by the right hand side of the above inequality. Hence $U \in GAP[a,b]$ and $\lim_{n\longrightarrow\infty}(GAP)\int_a^b U_n = (GAP)\int_a^b U$. □

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