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ON GENERALIZED CONTINUOUS MULTIFUNCTIONS AND THEIR SELECTIONS

Abstract

In this paper a generalized concept of continuous multifunctions has been studied. The main goal of this paper is to study some properties concerning a new type of multifunction along with its selections.

1 Introduction.

In recent years a considerable amount of research work has been done relating to many types of generalized continuous multifunctions. The notion of quasicontinuity [12] has been studied most intensively. The quasicontinuity is closely related to other types of continuity introduced by several authors (see [1], [3], [5], [9]). The notion of upper and lower $\mathcal{E}$-continuous multifunctions was first introduced by M. Matejdes [8]. In this paper we are interested in the existence of $\mathcal{E}$-cluster multifunctions, as explored by M. Matejdes in [8], [9], and [10]. An attempt has been made to investigate some properties of $\mathcal{E}$-cluster multifunctions together with its selections.

Throughout the paper X, Y are topological spaces. For a subset A of a topological space $Cl(A)$ denotes the closure of A and $\emptyset$ the empty set. Here

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$\mathbb{R}$ is the space of real numbers with the usual topology and $\mathbb{N}$ stands for the set of natural numbers. A multifunction is a mapping from X to $P(Y) \setminus \{\emptyset\}$ where $P(Y)$ is the power set of Y. We use capital letters F, G, H, etc. to denote multifunctions. For a multifunction $F : X \longrightarrow P(Y) \setminus \{\emptyset\}$ we write simply $F : X \longrightarrow Y$. A single-valued mapping $f : X \longrightarrow Y$ can be considered as a multifunction as $x \mapsto \{f(x)\}$, $x \in X$. A multifunction $S : X \longrightarrow Y$ is a submultifunction of $F : X \longrightarrow Y$ if $S(x) \subseteq F(x)$ for all $x \in X$. For a multifunction $F : X \longrightarrow Y$ with $A \subseteq Y$, we write $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$.

Definition 1. ([1]) A multifunction $F : X \longrightarrow Y$ is said to be upper (lower) semi-continuous at $x \in X$ if for each open set V in Y with $F(x) \subseteq V$ ($F(x) \cap V \neq \emptyset$) there exists a neighbourhood U of x such that $U \subseteq F^+(V)$ ($U \subseteq F^-(V)$). A multifunction is called upper (lower) semicontinuous on X if it is so at each point of X.

Definition 2. ([10]) Let $\mathcal{E}$ be a non-empty family of non-empty subsets of X. A point $y \in Y$ is called an $\mathcal{E}$-cluster point of a multifunction $F : X \longrightarrow Y$ at $x \in X$ if for every open neighbourhood U of x and for every open neighbourhood V of y there is $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^-(V)$. The set of all $\mathcal{E}$-cluster points of F at $x \in X$ will be denoted by $\mathcal{E}_F(x)$ and is called $\mathcal{E}$-cluster set of F at x.

Definition 3. ([8]) A multifunction $F : X \longrightarrow Y$ is said to be upper (lower) $\mathcal{E}$-continuous at $x \in X$ if for each open neighbourhood U of x and each open set V in Y with $F(x) \subseteq V$ ($F(x) \cap V \neq \emptyset$) there is a set $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^+(V)$ ($E \subseteq F^-(V)$). A multifunction is called upper (lower) $\mathcal{E}$-continuous on X if it is so at every point of X.

For a single-valued mapping $f : X \longrightarrow Y$, upper and lower $\mathcal{E}$-continuity are same as $\mathcal{E}$-continuity. Let

1. $\mathcal{O} = \{E \subseteq X : E \neq \emptyset$ and open in $X\}$,
2. $\mathcal{B}_r = \{E \subseteq X : E$ is second category with the Baire property$\}$,
3. $\mathcal{B} = \{E \subseteq X : E$ is either non-empty open or second category with the Baire property$\}$
4. $\mathcal{B}^* = \{E \subseteq X : E$ is not nowhere dense with the Baire property$\}$.

In the case $\mathcal{E} = \mathcal{O}$ ($= \mathcal{B}_r = \mathcal{B} = \mathcal{B}^*$), we have the upper (lower) $\mathcal{E}$-continuity as the usual notion of upper (lower) quasi-continuity ([13]) (Baire continuity [9], B-continuity [9], and B^*-continuity [5], respectively).

2 Subcontinuity and Weak-Subcontinuity.

The notion of subcontinuity for a single-valued mapping $f : X \longrightarrow Y$ was introduced by R. V. Fuller in [4]. A multifunction $F : X \longrightarrow Y$ is said to be subcontinuous at $x \in X$ ([14]) if whenever $\{x_\alpha\}_\alpha$ is a net in X converging to x and $\{y_\alpha\}_\alpha$ is a net in Y with $y_\alpha \in F(x_\alpha)$ for each α, then $\{y_\alpha\}_\alpha$ has a convergent subnet. A multifunction is called subcontinuous on X if it is so at every point of X. Clearly any multifunction $F : X \longrightarrow Y$ is subcontinuous on X when Y is compact.

For a multifunction $F : X \longrightarrow Y$, $G_r(F) = \{(x, y) \in X \times Y : y \in F(x)\}$ is called the graph of F. A multifunction $F : X \longrightarrow Y$ is said to have a closed graph ([7]) if $G_r(F)$ is closed in $X \times Y$. It is proved in [14] that a subcontinuous multifunction with a closed graph is upper semi-continuous. The reader is also referred to the comprehensive information in [7].

Definition 4. A multifunction $F : X \longrightarrow Y$ is said to be weak-subcontinuous at $x \in X$ if for every net $\{x_\alpha\}_\alpha$ in X converging to x there is a net $\{y_\alpha\}_\alpha$ in Y with $y_\alpha \in F(x_\alpha)$ for each α such that $\{y_\alpha\}_\alpha$ has a convergent subnet. A multifunction is called weak-subcontinuous on X if it is so at all points of X.

For a single-valued mapping weak-subcontinuity and subcontinuity are equivalent to each other. However, for a multifunction subcontinuity implies weak-subcontinuity but the converse is not true.

Example 5. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be given by $F(x) = [0, \infty) = \{y \in \mathbb{R} : y \geq 0\}$ for all $x \in \mathbb{R}$. Let $\{x_\alpha\}_\alpha$ be a net in $\mathbb{R}$ converging to $x \in \mathbb{R}$. Let $y_\alpha = 0$ for all α. Clearly 0 is a cluster point of $\{y_\alpha\}_\alpha$. So F is weak-subcontinuous at x and hence it is weak-subcontinuous on $\mathbb{R}$. Let $x_n = 0$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}_n$ converges to 0 in $\mathbb{R}$. But the sequence $\{n\}_n$ does not have any convergent subsequence. So F is not subcontinuous at 0 and hence not subcontinuous on $\mathbb{R}$.

3 $\mathcal{E}$-Cluster Multifunctions.

On the assumption that $\mathcal{E}_F(x) \neq \emptyset$ for all $x \in X$ we can define a multifunction $x \mapsto \mathcal{E}_F(x)$ for each $x \in X$ ([10]). This is called $\mathcal{E}$-cluster multifunction of F.

Example 6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & x \text{ is rational,} \\ 0 & x \text{ is irrational.} \end{cases}$ Here $\mathcal{O}_f(x) = \emptyset$ for all $x \in \mathbb{R}$, $\mathcal{B}_{r_f}(x) = \mathcal{B}_f(x) = \begin{cases} \emptyset & x \text{ is rational,} \\ \{0\} & x \text{ is irrational} \end{cases}$ and $\mathcal{B}^\star_f(x) =$

$\{0,1\}$ for all $x \in \mathbb{R}$. We want to find the conditions under which $\mathcal{E}_F(x) \neq \emptyset$ for all $x \in X$. Using the concept of $\mathcal{E}$-cluster point we can characterize the lower $\mathcal{E}$-continuity as follows:

Theorem 7. *$F : X \longrightarrow Y$ is lower $\mathcal{E}$-continuous at $x \in X$ if and only if $Cl(F(x)) \subseteq \mathcal{E}_F(x)$.*

PROOF. Let $F : X \longrightarrow Y$ be lower $\mathcal{E}$-continuous at $x \in X$, $y \in Cl(F(x))$ and U, V be open neighbourhoods of x in X and y in Y respectively. Then $F(x) \cap V \neq \emptyset$. By the lower $\mathcal{E}$-continuity of F at x, there is an $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^-(V)$. Therefore $y \in \mathcal{E}_F(x)$ and $Cl(F(x)) \subseteq \mathcal{E}_F(x)$.

Conversely suppose that $Cl(F(x)) \subseteq \mathcal{E}_F(x)$. Let U be an open neighbourhood of x in X and V be open in Y with $F(x) \cap V \neq \emptyset$. Suppose $y \in F(x) \cap V$. Then y is an $\mathcal{E}$-cluster point of F at x. Thus there is $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^-(V)$. Hence F is lower $\mathcal{E}$-continuous at x. □

Remark 8. If $F : X \longrightarrow Y$ is lower $\mathcal{E}$-continuous on X then $\mathcal{E}_F(x) \neq \emptyset$ for all $x \in X$. This follows immediately from Theorem 7. The converse of this is not true as shown in the following example.

Example 9. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 0 & x = 1, 2, \ldots, n \ (n \text{ finite}), \\ 1 & \text{otherwise.} \end{cases}$

Here f fails to be quasicontinuous at each of the points $1, 2, \ldots, n$ but $\mathcal{O}_f(x) \neq \emptyset$ for all $x \in \mathbb{R}$ since $\mathcal{O}_f(x) = \{1\}$ for all $x \in \mathbb{R}$.

Theorem 10. *If $F : X \longrightarrow Y$ is lower $\mathcal{E}$-continuous on X then $\mathcal{E}_F : X \longrightarrow Y$ is lower $\mathcal{E}$-continuous on X and F is a submultifunction of $\mathcal{E}_F$.*

PROOF. Let $F : X \longrightarrow Y$ be lower $\mathcal{E}$-continuous on X. Then clearly from Theorem 7, F is a submultifunction of $\mathcal{E}_F$. Let $x \in X$, U be an open neighbourhood of x in X, and V be open in Y with $\mathcal{E}_F(x) \cap V \neq \emptyset$. Suppose $y \in \mathcal{E}_F(x) \cap V$. Then $y \in \mathcal{E}_F(x)$ and $y \in V$. Thus there is an $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^-(V) \subseteq \mathcal{E}_F^-(V)$. Hence $\mathcal{E}_F$ is lower $\mathcal{E}$-continuous at x and consequently $\mathcal{E}_F$ is lower $\mathcal{E}$-continuous on X. □

Remark 11. The lower $\mathcal{E}$-continuity of $\mathcal{E}_F$ on X does not necessarily imply the lower $\mathcal{E}$-continuity of F on X. In Example 9, $\mathcal{O}_f$ is quasicontinuous on $\mathbb{R}$ but f is not so.

Theorem 12. *$\mathcal{E}_F : X \longrightarrow Y$ has a closed graph.*

PROOF. Let $(x, y) \in Cl(G_r(\mathcal{E}_F))$ and U, V be open neighbourhoods of x in X and y in Y respectively. Then $G_r(\mathcal{E}_F) \cap (U \times V) \neq \emptyset$. Suppose $(x', y') \in$

$G_r(\mathcal{E}_F) \cap (U \times V)$. Then $x' \in U$ and $y' \in \mathcal{E}_F(x') \cap V$. So there is an $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^-(V)$. Hence $y \in \mathcal{E}_F(x)$ and so $(x, y) \in G_r(\mathcal{E}_F)$. Therefore $G_r(\mathcal{E}_F)$ is closed in $X \times Y$. □

Remark 13. $\mathcal{E}_F : X \longrightarrow Y$ has closed values. This follows from Theorem 12.

Remark 14. $\mathcal{E}_F : X \longrightarrow Y$ has compact values when Y is compact.

Theorem 15. *If $\mathcal{E}_F : X \longrightarrow Y$ is non-empty valued and Y is compact, then $\mathcal{E}_F$ is upper semi-continuous.*

Proof. This follows immediately from the fact that a subcontinuous multifunction with a closed graph is upper semi-continuous. □

4 Densely Lower $\mathcal{E}$-Continuous Forms.

Densely continuous forms have been studied very intensively (see [6]). Such a form ϕ_f is defined for any single-valued function $f : X \longrightarrow Y$ having a dense set of continuity points $C(f)$. This ϕ_f is a multifunction (possibly empty valued) such that $Gr(\phi_f) = Cl(Gr(f\ |_{C(f)}))$ in $X \times Y$, where $f\ |_{C(f)}$ is the restriction of f on $C(f)$ ([6]). Densely lower $\mathcal{E}$-continuous forms $\mathcal{E}^l_F$ can also be generated by a multifunction $F : X \longrightarrow Y$ whose set $C^l_{\mathcal{E}}(F)$ of all lower $\mathcal{E}$-continuity points is dense. Then $\mathcal{E}^l_F$ is a multifunction (possibly empty valued) such that $Gr(\mathcal{E}^l_F) = Cl(Gr(F\ |_{C^l_{\mathcal{E}}(F)}))$ in $X \times Y$. Clearly $\mathcal{E}^l_F : X \longrightarrow Y$ has closed graph and hence, has closed values. We omit the simple proof of the following lemma.

Lemma 16. *Let $F : X \longrightarrow Y$ be a multifunction having dense $C^l_{\mathcal{E}}(F)$. Then $\mathcal{E}^l_F(x) = \{y \in Y :$ there are nets $\{x_\alpha\}_\alpha$ in $C^l_{\mathcal{E}}(F)$ converging to x and $\{y_\alpha\}_\alpha$ in Y with $y_\alpha \in F(x_\alpha)$ for each α such that y is a cluster point of $\{y_\alpha\}_\alpha\}$ for all $x \in X$. Therefore $F(x) \subseteq \mathcal{E}^l_F(x)$ for all $x \in C^l_{\mathcal{E}}(F)$.*

It follows that $\mathcal{E}^l_F$ is a cluster multifunction generated by the cluster system $\mathcal{E}^l = \{A : \emptyset \neq A \subseteq C^l_{\mathcal{E}}(F)\}$.

Theorem 17. *If $F : X \longrightarrow Y$ is weak-subcontinuous on X and $C^l_{\mathcal{E}}(F)$ is dense then $\mathcal{E}^l_F$ is a non-empty valued submultifunction of $\mathcal{E}_F$.*

Proof. Let $x \in X$. There exists a net $\{x'_\alpha\}_\alpha$ in $C^l_{\mathcal{E}}(F)$ converging to x and F is weak-subcontinuous at x. Therefore there is a net $\{y'_\alpha\}_\alpha$ in Y with $y'_\alpha \in F(x'_\alpha)$ for each α such that $\{y'_\alpha\}_\alpha$ has a cluster point, which we label y'. By Lemma 16, $y' \in \mathcal{E}^l_F(x)$ and so $\mathcal{E}^l_F(x) \neq \emptyset$.

Let $y \in \mathcal{E}^l_F(x)$. Then there are nets $\{x_\alpha\}_\alpha$ in $C^l_{\mathcal{E}}(F)$ converging to x and $\{y_\alpha\}_\alpha$ in Y with $y_\alpha \in F(x_\alpha)$ for each α such that y is a cluster point of $\{y_\alpha\}_\alpha$.

Let U, V be open neighbourhoods of x in X and y in Y respectively. Then there is an index α such that $x_\alpha \in U$ and $y_\alpha \in V \cap F(x_\alpha)$. Now F is lower $\mathcal{E}$-continuous at x_α and $V \cap F(x_\alpha) \neq \emptyset$. Thus there is $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^-(V)$. Hence $y \in \mathcal{E}_F(x)$ and consequently $\mathcal{E}_F^l(x) \subseteq \mathcal{E}_F(x)$. □

Remark 18. In Theorem 17, $\mathcal{E}_F^l$ is in general a proper submultifunction of $\mathcal{E}_F$ as the following example illustrates.

Example 19. Consider the closed interval $[0,1]$ with the subspace topology of the usual topology on $\mathbb{R}$ and let $T = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let $F : [0,1] \to [0,1]$ be given by $F(x) = \begin{cases} \{0,1\} & x \in T, \\ \{0\} & \text{otherwise.} \end{cases}$ Then $\mathcal{E}_F(0) = \{0,1\}$ and $\mathcal{E}_F^l(0) = \{0\}$ where $\mathcal{E} = \{A \subseteq [0,1] : A \text{ is not finite}\}$.

Theorem 20. *If $F : X \longrightarrow Y$ is weak-subcontinuous on X and if the set $C_{\mathcal{O}}^l(F)$ of all lower quasicontinuity points is dense and open then $\mathcal{O}_F^l$ is lower quasicontinuous on X.*

Proof. Let $x \in X$, U be an open neighbourhood of x in X, and V be open in Y such that $\mathcal{O}_F^l(x) \cap V \neq \emptyset$. By Theorem 17, $\mathcal{O}_F^l(x) \subseteq \mathcal{O}_F(x)$. Then $\mathcal{O}_F(x) \cap V \neq \emptyset$. Suppose $y \in \mathcal{O}_F(x) \cap V$. Then there is a $G \in \mathcal{O}$ with $G \subseteq U$ such that $G \subseteq F^-(V)$. Since $C_{\mathcal{O}}^l(F)$ is dense and open, $H = C_{\mathcal{O}}^l(F) \cap G \in \mathcal{O}$. Let $h \in H$. Then $F(h) \cap V \neq \emptyset$ and by Lemma 16, $F(h) \subseteq \mathcal{O}_F^l(h)$. So $\mathcal{O}_F^l(h) \cap V \neq \emptyset$. Hence $\mathcal{O}_F^l$ is lower quasicontinuous at x and therefore lower quasicontinuous on X. □

5 Selection of $\mathcal{E}$-Cluster Multifunctions.

A single-valued mapping $f : X \longrightarrow Y$ is called a selection of a multifunction $F : X \longrightarrow Y$ if $f(x) \in F(x)$ for all $x \in X$. M. Matejdes proved the following theorem in [8].

Theorem 21. *Let X be a T_1-space and Y be a compact metric space. If $F : X \longrightarrow Y$ is upper Baire continuous with compact values then F admits a quasicontinuous selection.*

J. Cao and W.B. Moors give the following extension of Theorem 21 in [2].

Theorem 22. *Let Y be a regular T_1-space. If $F : X \longrightarrow Y$ is upper Baire continuous with compact values then F admits a quasicontinuous selection.*

Using Remark 14 and Theorems 15 and 22, it easily follows that:

Theorem 23. *Let X be a Baire space and Y be a compact T_2-space. If $\mathcal{E}_F : X \longrightarrow Y$ has non-empty values then $\mathcal{E}_F$ admits a quasicontinuous selection.*

M. Matejdes proved the following theorem in [11] which is an elegant generalization of the result of [8].

Theorem 24. *Let Y be a T_2-space and $F : X \longrightarrow Y$ be a compact-valued upper $\mathcal{E}$-continuous multifunction. Then F has a compact-valued submultifunction for which any selection is $\mathcal{E}$-continuous.*

Theorem 25. *Let Y be a compact T_2-space. If $\mathcal{E}_F : X \longrightarrow Y$ has non-empty values then $\mathcal{E}_F$ has a compact-valued submultifunction for which any selection is quasicontinuous.*

Proof. By Remark 14 and Theorem 15, $\mathcal{E}_F : X \longrightarrow Y$ is compact-valued and upper semi-continuous and hence upper quasicontinuous on X. Again by Theorem 24, $\mathcal{E}_F$ has a compact-valued submultifunction for which any selection is quasicontinuous. □

Note that a multifunction $F : X \longrightarrow Y$ is said to have an $\mathcal{E}$-closed graph [11] if $\mathcal{E}_F : X \longrightarrow Y$ is a submultifunction of F. Now if the set of lower $\mathcal{E}$-continuity points of $F : X \longrightarrow Y$ is dense and if Y is compact then $\mathcal{E}_F : X \longrightarrow Y$ is non-empty valued. So from Theorem 25, it readily follows that:

Theorem 26. *Let Y be a compact T_2-space and let the set of lower $\mathcal{E}$-continuity points of $F : X \longrightarrow Y$ be dense. If F has an $\mathcal{E}$-closed graph then F admits a quasicontinuous selection.*

Remark 27. The compactness in Theorems 25 and 26 cannot be omitted as illustrated by the example function $f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

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