INROADS

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 417–430

Charles Coppin,* Department of Mathematics, Lamar University, P.O. Box 10047, Beaumont, TX 77710. email: ccoppin@gt.rr.com

Philip Muth, 1607 Nandina Dr, Apt 1328, Arlington, TX 76014. email: philmuth@yahoo.com

A STUDY OF A STIELTJES INTEGRAL DEFINED ON ARBITRARY NUMBER SETS

Abstract

Our purpose is to study a generalized Stieltjes integral defined on a class of subsets of a closed number interval. We extend the results of previous work by the first author. Among other results, we prove that

- If $M \subseteq [a,b]$ and f and g are functions with domain M such that f is g-integrable over M, and there exist left (right) extensions f^* and g^* of f and g to $[a,b]$, respectively, then f^* is g^*- integrable on $[a,b]$ and
$$\int_a^b f^* dg^* = \int_M f dg$$
- Suppose that F and G are functions with domain including $[a,b]$ such that
  (a) F is G-integrable on $[a,b]$,
  (b) $\overline{M} \subseteq [a,b]$, and $a,b \in M$
  (c) if z belongs to $[a,b]-M$ and ϵ is a positive number, then there is an open interval s containing z such that $|F(x) - F(z)||G(v) - G(u)| < \epsilon$ where each of u, v, and x is in $s \cap [a,b]$, $u < z < v$, and $u \leq x \leq v$.

  Then F is G-integrable on M, and $\int_a^b FdG = \int\limits_M FdG$.

Key Words: Stieltjes integral
Mathematical Reviews subject classification: 26A42
Received by the editors March 19, 2007
Communicated by: Stefan Schwabik
*This research was partially supported by the O'Hara Chemical Sciences Institute at the University of Dallas.

1 Introduction.

The Riemann-Stieltjes integral remains a topic of significant interest. See, for example, D'yachkov [8], Kats [12], Liu and Zhao [13], and Tseytlin [18]. Modifications of the Stieltjes integral abound. One only has to sample some of the most recent papers. For some interesting results, see B. Bongiorno and L. Di Piazza [1], A.G. Das and Gokul Sahu [7], Ch. S. Hönig [11], Supriya Pal, D.K. Ganguly and Lee Peng Yee [15], Š. Schwabik, M. Tvrdỳ, and O. Vejvoda [16], Swapan Kumar Ray and A.G. Das [17], and Ju Han Yoon and Byung Moo Kim [22].

In this paper, we investigate a modified Stieltjes integral defined on arbitrary number sets. A special case of this integral was first defined by Coppin [3] and Vance [21] where the integral was defined over dense subsets of an interval containing the end points of that interval. Coppin and Vance [6] showed necessary and sufficient conditions for f to be g-integrable on a dense subset of $[a, b]$ where $f|M$ and $g|M$ do not have common points of discontinuity. Vance [21] gave a characterization of bounded linear functionals. He proved a representation theorem for bounded linear functionals with domain being the set of all real-valued, quasi-continuous functions defined on a closed interval.

Let Δ denote the set of all dense subsets of $[a, b]$ which contain a and b. Coppin [4] gave conditions where f is g-integrable on M' in Δ provided f is g-integrable on M in Δ and $M \subset M'$. He showed that if f is g-integrable on some uncountable member of Δ, then f is g-integrable on uncountable many members of Δ. In addition, he proved that if M is a countable member M of Δ, then there are real-valued functions f and g with domain $[a, b]$ such that f is g-integrable on M and no other member of Δ. Coppin [5] added to the results of [6] by showing that f is g-integrable on M in Δ and $f|M$ and $g|M$ have no common points of discontinuity if and only if f is g-integrable on each subset of M which is a member of Δ. Also, in [5], it is proved that if $M \in \Delta$, f and g are functions defined on $[a, b]$ which have no common discontinuities from the left at z nor common discontinuities from the right at z and f is g-integrable on M, then f is g-integrable on $M \cup \{z\}$ and $\int_{M\cup\{z\}} f dg = \int_M f dg$. In [5], it is shown that if f and g are functions with domain $[a, b]$ and f and g have no common discontinuities from the left nor common discontinuities from the right, then the set $\{\ w\ :\ w = \int_M f dg\ for\ M \in \Delta\}$ is connected.

In this paper, we study a Stieltjes integral defined over arbitrary number sets not merely those of [3] and [21]. We compare this integral with the partition-refinement Stieltjes integral.

2 Preliminary Definitions.

We give the definitions and conventions used in this paper.

In general, an interval (or an interval of M) is a set $[c,d]_M = [c,d] \cap M$ where c and d belong to M and $c < d$. Two intervals, A and B, are said to be nonoverlapping if and only if $A \cap B$ does not contain an interval. A nonempty collection of intervals is said to be nonoverlapping if and only if each two distinct members of the collection are nonoverlapping.

In this paper, all functions are bounded real-valued functions.

Definition 2.1. If M is a number set, then D is said to be a partition of M if and only if D is a finite collection of non-overlapping subintervals of M. $E(D)$ denotes the set of end points of members of D.

Definition 2.2. If M is a number set and D is a partition of M, then D' is said to be a refinement of D if and only if D' is a partition of M and $E(D) \subseteq E(D')$.

Definition 2.3. If D is a nonempty collection of intervals, then δ is said to be a choice function on D if and only if δ is a function with domain D such that $\delta(d) \in d$ for each d in D.

Definition 2.4. If D is a partition of a number set M, δ is a choice function on D, and f and g are functions with domain including $\cup D$, then

$$\Sigma(f,g,D,\delta) = \sum_{[p,q]_M \in D} f(\delta([p,q]_M)) \cdot [g(q) - g(p)].$$

Definition 2.5. Suppose that M is a number set and f and g are functions with domain including M. Then f is said to be g-integrable on M if and only if there exists a number W (called "an integral of f with respect to g" and denoted by $\int_M f dg$) such that for each $\varepsilon > 0$, there is a partition D of M such that

$$|W - \Sigma(f,g,D',\delta)| < \varepsilon$$

for each refinement D' of D and each choice function δ on D'.

We follow the style of [2] and call the integral of this paper Definition D. Definition C will refer to the definition found on page 305 of [2], the usual partition-refinement version of the Stieltjes integral.

3 A Joint Cauchy Criterion for Limits.

Definition 3.1. Suppose M is a set of numbers. The statement that D is a direction in M (or direction D, if ambiguity exists) means that D is a nonempty collection of intervals of M such that for each two sets S_1 and S_2 in D there is a member S_3 in D such that S_3 is a subset of $S_1 \cap S_2$.

Definition 3.2. Suppose f is a function with domain including a number set M and D is a direction in M. Then the statement that f has a limit according to D means that there is a number L (written $\lim_D f$) such that if $\epsilon > 0$, there is an $S \in D$ such that $|L - f(x)| < \epsilon$ for each $x \in S$.

From McCleod [14], we have the following theorem.

Theorem 3.1. *(Cauchy Criterion for Limits). Suppose D is a direction in M and f is a function with domain including M. Then $\lim_D f$ exists if and only if for every $\epsilon > 0$ there is an $S \in D$ such that $|f(u) - f(v)| < \epsilon$ for all u and v in S.*

We have our own generalization of Theorem 3.1 which, of course, we will find useful later.

Theorem 3.2. *(Joint Cauchy Criterion for Limits). Suppose D is a direction in M, and f are g are bounded functions with domain including M. Then $\lim_D f$ exists or $\lim_D g$ exists if and only if for each $\epsilon > 0$ there is an $S \in D$ such that $|f(u) - f(v)||g(s) - g(r)| < \epsilon$ for each u, v, r and s in S.*

PROOF. ($\Rightarrow$). Suppose that $\lim_D f$ or $\lim_D g$ exists. For the sake of argument, we assume that $\lim_D f$ exists. Because g is bounded, we know there is $A > 0$ such that

$$|g(x)| < A \tag{1}$$

for each $x \in M$. Let $\epsilon > 0$. Because $\lim_D f$ exists, by Theorem 3.1, for $\epsilon/2A > 0$, there is an S in D such that

$$|f(u) - f(v)| < \frac{\epsilon}{2A} \tag{2}$$

for each u and v in S. From (1) above, we have

$$|g(s) - g(r)| < 2A \tag{3}$$

for each r and s in M and, therefore, each r and s in S. From (2) and (3), $|f(u) - f(v)||g(s) - g(r)| < \epsilon$ for each u, v, r, and s in S. $\square$

Proof. ($\Leftarrow$). Suppose that for each $\epsilon > 0$ there is some S in D such that

$$|f(u) - f(v)||g(s) - g(r)| < \epsilon \tag{4}$$

for each u, v, r, and s in S. For the sake of argument, assume that $\lim_D f$ does not exist. Thus, by Definition 3.2, there is $\rho > 0$ such that for any S in D and some u and v in S

$$|f(u) - f(v)| \geq \rho.$$

We will show that this assumption leads to the fact that $\lim_D g$ must exist.

Suppose $\epsilon > 0$. From (4), for $\rho\epsilon > 0$, there is some S in D such that

$$|f(u) - f(v)||g(s) - g(r)| < \rho\epsilon \tag{5}$$

for each u, v, r and s in S. However, there are $u, v \in S$ such that

$$|f(u) - f(v)| \geq \rho. \tag{6}$$

Thus from (5) and (6) we obtain $\rho|g(s)-g(r)| \leq |f(u)-f(v)||g(s)-g(r)| < \rho\epsilon$, or $|g(s) - g(r)| < \epsilon$ for each s, r in S. Therefore by Definition 3.2, we know that $\lim_D g$ exists. $\square$

Corollary 3.3. $\lim_D f$ *exists or* $\lim_D g$ *exists if and only if for each* $\epsilon > 0$ *there is an* $S \in D$ *such that* $|f(u) - f(v)||g(s) - g(r)| < \epsilon$ *for each* u, v, r *and* s *in* S *where* $r \leq u \leq s$ *and* $r \leq v \leq s$.

Proof. ($\Rightarrow$). This follows immediately from Theorem 3.2.

Proof. ($\Leftarrow$). Assume the hypothesis and that both $\lim_D f$ and $\lim_D g$ do not exist.

Then, by Definition 3.2, for some $\epsilon_1 > 0$ and each $S \in D$ there are $u, v \in S$ such that $|f(u) - f(v)| \geq \epsilon_1$. Likewise, for some $\epsilon_2 > 0$ and each $S \in D$ there are $r, s \in S$ such that $|g(s) - g(r)| \geq \epsilon_2$.

For $\epsilon_1\epsilon_2 > 0$, by hypothesis, there is some $S \in D$ where

$$|f(u) - f(v)||g(s) - g(r)| < \epsilon_1\epsilon_2 \tag{7}$$

for each u, v, r and s in S where $r \leq u \leq s$ and $r \leq v \leq s$.

Now, arbitrarily choose $r, s \in S$. We can assume $r < s$. There are $u, v \in S$ such that $r \leq u, v \leq s$ and $|f(u) - f(v)| \geq \epsilon_1$. From (7), we have

$$|g(s) - g(r)|\epsilon_1 < |f(u) - f(v)||g(s) - g(r)| < \epsilon_1\epsilon_2$$

or

$$|g(s) - g(r)| < \epsilon_2$$

for each r, s in S. This is in direct contradiction to the third sentence of this proof. □

4 Transformation from Definition D to Definition C.

Definition 4.1. Suppose $\overline{M} \subseteq [a, b]$. Then a gap G in M (or gap G if no ambiguity exists) is a maximal connected subset of (a, b) which contains no points of M.

Definition 4.2. Suppose M is a set and G is a gap. In this definition, we follow the style of Hewitt and Stromberg [9], page 54, for the meaning of interval. We now define the following directions:

D_G is the collection of all intervals containing a point of G, right end point in M and left end point in M.

D_G^+ is the collection of all intervals with left end point in M and right end point in the gap G.

D_G^- is the collection of all intervals with right end point in M and left end point in the gap G.

Theorem 4.1. *If f is a function with domain including a number set M, G is a gap in M, and $\lim_{D_G} f$ exists, then $\lim_{D_G^+} f$ and $\lim_{D_G^-} f$ exist and*

$$\lim_{D_G} f = \lim_{D_G^+} f = \lim_{D_G^-} f.$$

Proof. Suppose f is a function with domain including a number set M, G is a gap in M, and $\lim_{D_G} f$ exists, which we denote by L.

Let $\epsilon > 0$. Then since $\lim_{D_G} f$ exists, there is an $S \in D_G$ such that $|L - f(x)| < \epsilon$ for each $x \in S$. Now, let S^+ be a member of D_G^+ where $S^+ \subseteq S$. Then $|L - f(x)| < \epsilon$ for each $x \in S^+$. Thus by definition of $\lim_{D_G^+} f$, we know that $\lim_{D_G^+} f = L$. Likewise we can prove that $\lim_{D_G^-} f$ exists and $\lim_{D_G^-} f = L$.

Therefore $\lim_{D_G} f = \lim_{D_G^+} f = \lim_{D_G^-} f$. □

Theorem 4.2. *If f and g are functions with domain $M \subseteq [a,b]$ such that f is g-integrable on M and G is a gap in M, then $\lim_{D_G^-} f$ and $\lim_{D_G^+} f$ exist or $\lim_{D_G^-} g$ exists and $\lim_{D_G^+} g$ exist.*

Proof. In the following argument, the direction D is D_G for some gap G.

Suppose $\epsilon > 0$. Since f is g-integrable on M, there is a number W and a partition P of M such that

$$|W - \sum_{P' \in P} f(x)[g(q) - g(p)]| < \frac{\epsilon}{2} \tag{8}$$

for any refinement P' of P and for all $[p,q]_M$ in P' and any x in $[p,q]_M$.

Let $[c,d]_M$ be the member of P where $G \subseteq [c,d]$. Note that $S = [c,d]_M \in D$. Let r, s, u, v be arbitrary members of S. For the sake of argument, assume $r \leq s$ and $r \leq u, v \leq s$. Let P' be the refinement of P such that $E(P') = E(P) \cup \{r, s\}$.

Let $T = \sum f(x)[g(q) - g(p)]$ where $x = p$ for each $[p,q]_M \in P'$ except in the case when $[p,q]_M = [r,s]_M$ we let $x = u$. Let U be defined in the same manner as T except in the case $[p,q]_M = [r,s]_M$ we let $x = v$.

From (8), we have

$$|W - T| < \frac{\epsilon}{2} \text{ and } |W - U| < \frac{\epsilon}{2}.$$

Adding and applying the triangle inequality for absolute values, we obtain

$$|U - T| < \epsilon.$$

It can easily be shown that

$$U - T = [f(v) - f(u)][g(s) - g(r)].$$

So

$$|[f(v) - f(u)][g(s) - g(r)]| < \epsilon.$$

In summary, for any $\epsilon > 0$ there is $S \in D$ containing G such that for any r and s in M where $[r,s] \subseteq D$ and any u and v in M where $r \leq u \leq s$ and $r \leq v \leq s$ we have that

$$|[f(u) - f(v)][g(s) - g(r)]| < \epsilon.$$

Thus by Corollary 3.3, $\lim_D f$ exists or $\lim_D g$ exists. By Theorem 4.1, $\lim_{D_G^-} f$ and $\lim_{D_G^+} f$ exist or $\lim_{D_G^-} g$ exists and $\lim_{D_G^+} g$ exist. $\square$

Theorem 4.3. *If f is a function with domain $M \subseteq [a,b]$, z is a member of $[a,b] - M$ which is a limit point of the domain of $f|[a,z]$, then there is a number c such that (z,c) is a limit point of the graph of $f|[a,z]$. Similarly, if z is a limit point of the domain of $f|[z,b]$, then there is a number c such that (z,c) is a limit point of the graph of $f|[z,b]$.*

PROOF. The proof is a straight forward application of the Heine-Borel Theorem applied to the vertical interval $\{(z,t) : -B \leq t \leq B\}$ where B is a common positive bound for $|f|$ and $|g|$. □

Definition 4.3. In Theorem 4.3 c is said to be a quasi-end value.

Definition 4.4. Suppose f is a function with domain $M \subseteq [a,b]$. By f^* we mean a function such that

(a) $f^*(x) = f(x)$ for each $x \in M$, and

(b) if $x \in [a,b] - M$ and G is a gap containing x, then $f^*(x)$ is equal to a quasi-end value of f with respect to G. It is understood that when there is more than one choice for $f^*(x)$ then only one choice is made and is the same for each value in G.

f^* will be known as an extension of f to $[a,b]$. If quasi-left end values are used consistently for each gap, then f^* is known as a left extension of f on $[a,b]$. Right extensions are defined in a similar fashion.

Theorem 4.4. *If f and g are functions with domain $M \subseteq [a,b]$, $a \leq r^* \leq x^* \leq s^* \leq b$ where r^*, x^*, s^* are in M, and $\epsilon > 0$, then*

(a) *if $a \in M$, there are left extensions f^* and g^* of f and g to $[a,b]$, respectively, and there are numbers r, s and x in M such that $a \leq r \leq r^*$, $r \leq x \leq x^*$, $x \leq s \leq s^*$ and $|f^*(x^*)[g^*(s^*) - g^*(r^*)] - f(x)[g(s) - g(r)]| < \epsilon$ and*

(b) *if $b \in M$, there are right extensions f^* and g^* of f and g to $[a,b]$, respectively, and there are numbers r, s and x in M such that $r^* \leq r \leq x$, $x^* \leq x \leq s$, $s^* \leq s \leq b$ and $|f^*(x^*)[g^*(s^*) - g^*(r^*)] - f(x)[g(s) - g(r)]| < \epsilon$.*

PROOF. For (a) suppose $a \leq r^* \leq x^* \leq s^* \leq b$ where r^*, x^*, s^* are in M. Suppose $\epsilon > 0$ and B is a positive common bound of $|f|$ and $|g|$. Let $\epsilon' = min\{\epsilon/6B, \sqrt{\epsilon/6}\}$. Since $a \leq r^*$, let $z = inf(M \cap [a, r^*])$. If $z \in M$, let $r = z$. If not, z is a limit point of M. In the latter case, by Theorem 4.3, there is a point with abscissa z which is a limit point of the graph of $g|[a,z]$.

Thus, there is a member r of M such that $a \leq r \leq r^*$ and $g(r) = g^*(r^*) + \delta_1$ where $|\delta_1| < \epsilon'$. Similarly, there is a member $x \in M$ such that $r \leq x \leq x^*$ and $f(x) = f^*(x^*) + \delta_2$ where $|\delta_2| < \epsilon'$. In like manner, there is a member $s \in M$ such that $r \leq x \leq x^*$ and $g(s) = g^*(s^*) + \delta_3$ where $|\delta_3| < \epsilon'$. Then

$$\begin{aligned}
&|f^*(x^*)[g^*(s^*) - g^*(r^*)] - f(x)[g(s) - g(r)]| \\
&\quad = |[f(x) + \delta_2][g(s) + \delta_3 - g(r) - \delta_1] - f(x)[g(s) - g(r)]| \\
&\quad = |f(x)[g(s) - g(r)] + f(x)[\delta_3 - \delta_1] + \delta_2[g(s) - g(r)] + \delta_2[\delta_3 - \delta_1] - \\
&\qquad f(x)[g(s) - g(r)]| \leq |f(x)[\delta_3 - \delta_1]| + \delta_2[g(s) - g(r)]| + \delta_2[\delta_3 - \delta_1]| \\
&\quad < B\frac{2\epsilon}{6B} + \frac{\epsilon}{6B}2B + \frac{2\epsilon}{6} = \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = \epsilon.
\end{aligned}$$

Thus, $|f^*(x^*)[g^*(s^*) - g^*(r^*)] - f(x)[g(s) - g(r)]| < \epsilon$.
The proof of (b) is similar to (a). □

Theorem 4.5. *If $M \subseteq [a,b]$, f and g are functions with domain M such that f is g-integrable over M, and there are left (right) extensions f^* and g^* of f and g to $[a,b]$, respectively, then f^* is g^*- integrable on $[a,b]$ and*

$$\int_a^b f^* dg^* = \int_M f dg$$

Proof. Suppose $M \subseteq [a,b]$ and f and g are functions with domain M such that f is g-integrable on M. Let $W = \int_M f dg$ and f^*, g^* be left (right) extensions of f and g, respectively. For the sake of argument we assume left extensions of f and g. There is no loss of generality if $a, b \in M$. Suppose $\rho > 0$. Thus, there is a partition D of M such that

$$|W - \sum f(x)[g(q) - g(p)]| < \frac{\rho}{2} \tag{9}$$

for any refinement D' of D and for all $[p,q]_M$ in D' and any $x \in [p,q]_M$.

Now, we construct D' and δ. Let P be a partition of $[a,b]$ such that $E(P) = E(D)$ and let P' be an arbitrary refinement of P. Now, we will construct a refinement D' of D such that

$$|\sum(f, g, D', \delta) - \sum(f^*, g^*, P', \delta')| < \frac{\rho}{2}$$

where δ' is any choice function on P' and δ is a specific choice function on D' yet to be described.

Let N be the number of elements in P'. Denote $P' = \{[u^*_{k-1}, u^*_k]\}_{k=1}^N$. We start by choosing ϵ in the preceding theorem to be $\rho/2N$. Consider $[u^*_0, u^*_1]$ of P' and $x^* = \delta'([u^*_0, u^*_1])$. Then, by Theorem 4.4, we obtain numbers u_0, u_1, and x_0 in M such that $a \le u_0 \le u^*_0 \le x \le x^* \le u_1 \le u^*_1 \le b$ and

$$|f^*(x^*_0)[g^*(u^*_1) - g^*(u^*_0)] - f(x_0)[g(u_1) - g(u_0)]| < \frac{\rho}{2N}.$$

Now, consider numbers u_1, u^*_2, and x^*_1. There are numbers x_1 and u_2 such that $u_1 \le x_1 \le x^*_1 \le u_2 \le u^*_2$ and

$$|f^*(x^*_1)[g^*(u^*_2) - g^*(u^*_1)] - f(x_1)[g(u_2) - g(u_1)]| < \frac{\rho}{2N}.$$

Then, we continue to apply the process for $k = 2$ to $k = N$ to generate the following inequalities:

$$|f^*(x^*_{k-1})[g^*(u^*_k) - g^*(u^*_{k-1})] - f(x_1)[g(u_k) - g(u_{k-1})]| < \frac{\rho}{2N}.$$

for $k = 1$ to N.

Adding the above N inequalities and with application of the triangle inequality, we obtain the following:

$$|\sum(f, g, D', \delta) - \sum(f^*, g^*, P', \delta')| < \frac{\rho}{2} \tag{10}$$

where $D' = \{[u_{k-1}, u_k]\}_{k=1}^N$ and $\delta([u_{k-1}, u_k]) = x_k$ for $k = 1$ to N. Now, we have D' and δ.

Adding (9) and (10), we obtain

$$|W - \sum(f^*, g^*, P', \delta')| < \rho.$$

where P' is any refinement of P and δ' is any choice function on P'.

Therefore f^* is g^*-integrable on $[a, b]$ and $\int_a^b f^* dg^* = \int_M f dg$. □

Theorem 4.6. *Suppose that F and G are functions with domain including $[a, b]$ such that*

(a) F is G-integrable on $[a, b]$,

(b) $\overline{M} = [a, b]$, $a, b \in M$,

(c) if z belongs to $[a, b] - M$ and ϵ is a positive number, then there is an open interval s containing z such that $|F(x) - F(z)||G(v) - G(u)| < \epsilon$ where each of u, v, and x is in $s \cap [a, b]$, $u < z < v$, and $u \le x \le v$.

Then F is G-integrable on M and $\int_a^b FdG = \int\limits_M FdG$.

Proof. Suppose $\epsilon > 0$. Since F is G-integrable on $[a,b]$, there is a partition D of $[a,b]$ such that, if D' is a refinement of D, then

$$\Big|\int_a^b FdG - \sum(F,G,D',\delta)\Big| < \frac{\epsilon}{2}$$

for each choice function δ on D'.

For the sake of argument let us take the case that an element of D has an end point not belonging to M.

Suppose $A = E(D) \cap M^c$ which can be written as $A = \{x_1, x_2, x_3, \dots, x_N\}$. By parts (b) and (c) of the hypothesis, there is a collection $G = \{(r_i, s_i) : i = 1, 2, \dots, N\}$ of disjoint open subintervals of $[a,b]$ with end points in M, each of which contains exactly one element of A, contains no point of $E(D) \cap M$, and, if x_i belongs to A, then

$$|F(x) - F(x_i)||G(v) - G(u)| < \frac{\epsilon}{2N} \tag{11}$$

for each u, v and x in $(r_i, s_i) \cap [a,b]$ where $u < x_i < v$, $u \le x \le v$ for $i = 1, 2, \dots, N$.

Let D' denote the refinement of D where $E(D') = E(D) \cup \{r_1, s_1, r_2, s_2, \dots, r_N, s_N\}$. Let P denote a partition of M such that $E(P) = E(D') \cap M$. Suppose that P' is any refinement of P. For $i = 1, 2, \dots, N$, let $[c_i, d_i]_M$ denote the element of P' such that $c_i < x_i < d_i$.

From (11), since c_i,d_i and x_i are in $(r_i, s_i) \cap [a,b]$, we have

$$\begin{aligned}\big|F(x)[G(d_i) - G(c_i)] &- F(x_i)[G(x_i) - G(c_i)] \\ &- F(x_i)[G(d_i) - G(x_i)]\big| < \frac{\epsilon}{2N}\end{aligned} \tag{12}$$

where x is any number in $[c_i, d_i]_M$, $i = 1, 2, \dots, N$. Since there are N elements in A, from (12) we have

$$\begin{aligned}\Big|\sum_{i=1}^N F(x)[G(d_i) - G(c_i)] &- \sum_{i=1}^N F(x_i)[G(x_i) - G(c_i)] \\ &- \sum_{i=1}^N F(x_i)[G(d_i) - G(x_i)]\Big| < \frac{\epsilon}{2}.\end{aligned} \tag{13}$$

Let D'' denote a refinement of D such that $E(D'') = E(P') \cup E(D)$. Let $Q_{P'} = \{[c_i, d_i]_M\}_{i=1}^N$ and $Q_{D''} = \{[c_i, x_i]\}_{i=1}^N \cup \{[x_i, d_i]\}_{i=1}^N$.

Let ρ be any choice function on P' and let δ' be the choice function on D'' defined as $\delta'([p,q]) = \rho([p,q]_M)$ for each $[p,q]_M$ in $P' - Q_{P'}$, each $[p,q]$ in $D'' - Q_{D''}$ and $\delta'([c_i, x_i]) = \delta'([x_i, d_i]) = x_i, i = 1, 2, \ldots, N$. Thus, (13) becomes

$$\left|\sum(F, G, Q_{P'}, \rho) - \sum(F, G, Q_{D''}, \delta')\right| < \frac{\epsilon}{2}. \tag{14}$$

We also have

$$\sum(F, G, D'' - Q_{D''}, \delta') = \sum(F, G, P' - Q_{P'}, \rho) \tag{15}$$

and

$$\sum(F, G, D'', \delta') = \sum(F, G, Q_{D''}, \delta') + \sum(F, G, D'' - Q_{D''}, \delta') \tag{16}$$

and

$$\sum(F, G, P', \rho) = \sum(F, G, Q_{P'}, \rho) + \sum(F, G, P' - Q_{P'}, \rho). \tag{17}$$

Substituting (15) into (16), we obtain

$$\sum(F, G, D'', \delta') = \sum(F, G, Q_{D''}, \delta') + \sum(F, G, P' - Q_{P'}, \rho). \tag{18}$$

Computing the difference between the left sides of (17) and (18) and substituting into (14) yields

$$\left|\sum(F, G, P', \rho) - \sum(F, G, D'', \delta')\right| < \frac{\epsilon}{2}. \tag{19}$$

Then, we have from (4)

$$\left|\int_a^b F dG - \sum(F, G, D'', \delta')\right| < \frac{\epsilon}{2}. \tag{20}$$

Combining (19) and (20), we have

$$\left|\int_a^b F dG - \sum(F, G, P', \rho)\right| < \epsilon$$

for each choice function ρ on P'.

Therefore, by definition, F is G-integrable on M and, by the uniqueness of the integral, $\int_a^b FdG = \int_M FdG$. $\square$

Acknowledgment. The authors wish to thank the referees for their constructive critique of the first draft. It is indeed appreciated.

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