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STABILITY OF TWO TYPES OF CUBIC FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN SPACES

Abstract

We prove the generalized stability of the cubic type functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

and another functional equation

$$f(ax+y)+f(x+ay)=(a+1)(a-1)^2[f(x)+f(y)]+a(a+1)f(x+y),$$

where a is an integer with $a \neq 0, \pm 1$ in the framework of non-Archimedean normed spaces.

1 Introduction and Preliminaries.

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional

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equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If there exists an affirmative answer we say that the equation $\mathcal{E}$ is stable [3]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [3, 5, 11] and monographs [2, 6, 9, 12] and references therein.

The functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

is called a *cubic type functional equation*, since the function $f(x)=cx^3$ is a solution of this functional equation. In particular, every solution of a cubic type functional equation is said to be a *cubic type mapping*. The stability problem for a cubic type functional equation was proved by K.W. Jun and H.M. Kim [7] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space.

The functional equation

$$f(ax+y)+f(x+ay)=(a+1)(a-1)^2[f(x)+f(y)]+a(a+1)f(x+y) \tag{1.2}$$

is another cubic type functional equation. The stability problem for this functional equation for integer a with $a \neq 0, \pm 1$ and in the framework of quasi-Banach spaces was proved by K.W. Jun and H.M. Kim [8].

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r|\,|s|$, and $|r+s| \leq \max\{|r|,|s|\}$ for all $r,s \in K$. Clearly $|1|=|-1|=1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0|=0$. Let X be a vector space over a field K with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii)

$$\|rx\|=|r|\|x\| \quad (r \in K, x \in X);$$

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq \max\{\|x\|,\|y\|\} \quad (x,y \in X).$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n-x_m\| \leq \max\{\|x_{j+1}-x_j\| : m \leq j \leq n-1\} \qquad (n>m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [4] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

In [1], the authors investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. In [10], the stability of Cauchy and quadratic functional equations were investigated in the context of non-Archimedean normed spaces. In this paper, by following some ideas from [7, 8, 10], we establish the stability of cubic type functional equations (1.1) and (1.2) in the setting of non-Archimedean normed spaces.

Throughout the paper, we assume that G is an abelian (additive) group and X is a complete non-Archimedean normed space.

2 Stability of the Functional Equation (1.1).

In this section, we prove the stability of functional equation (1.1).

Theorem 2.1. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|8|^n} = 0 \qquad (x, y \in G) \tag{2.1}$$

and set

$$\widetilde{\varphi}(x) := \sup\left\{\frac{\varphi(2^j x, 0)}{|8|^j} : j \in \mathbb{N}\right\} \qquad (x \in G). \tag{2.2}$$

Suppose that $f : G \to X$ is a mapping satisfying

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \le \varphi(x, y) \tag{2.3}$$

for all $x, y \in G$. Then there exists a unique mapping $T : G \to X$ satisfying (1.1) such that

$$\|f(x) - T(x)\| \le \frac{1}{|16|}\widetilde{\varphi}(x) \qquad (x \in G). \tag{2.4}$$

Proof. Set $y = 0$ in (2.3) to get

$$\|f(2x) - 8f(x)\| \le \frac{1}{|2|}\varphi(x,0) \qquad (x \in G). \tag{2.5}$$

Let $x \in G$. Replacing x by $2^n x$ in (2.5) we obtain

$$\left\| \frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n} \right\| \le \frac{\varphi(2^n x, 0)}{|2| \,.\, |8|^{n+1}} \qquad (x \in G). \tag{2.6}$$

It follows from (2.6) and (2.1) that the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is Cauchy. Since X is complete, we conclude that $\left\{\frac{f(2^n x)}{8^n}\right\}$ is convergent. Set $T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$.

Using induction one can show that

$$\left\| \frac{f(2^n x)}{8^n} - f(x) \right\| \le \frac{1}{|16|} \max\left\{ \frac{\varphi(2^k x, 0)}{|8|^k} : 0 \le k < n \right\} \tag{2.7}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (2.7) and using the fact that

$$\lim_{n\to\infty} \max\left\{ \frac{\varphi(2^j x, 0)}{|8|^j} : 0 \le j < n \right\} = \sup\left\{ \frac{\varphi(2^j x, 0)}{|8|^j} : j \in \mathbb{N} \right\}$$

one obtains

$$\|f(x) - T(x)\| \le \frac{1}{|16|}\widetilde{\varphi}(x) \qquad (x \in G). \tag{2.8}$$

Replacing x and y by $2^n x$ and $2^n y$, respectively, in (2.3) we get

$$\begin{aligned} \Big\| \frac{f(2^n(2x+y))}{8^n} + \frac{f(2^n(2x-y))}{8^n} - 2\frac{f(2^n(x+y))}{8^n} \\ -2\frac{f(2^n(x-y))}{8^n} - 12\frac{f(2^n x)}{8^n} \Big\| \\ \le \frac{\varphi(2^n x, 2^n y)}{|8|^n} \qquad (x, y \in G). \end{aligned}$$

Taking the limit as $n \to \infty$ and using (2.1) we obtain

$$T(2x+y) + T(2x-y) = 2T(x+y) + 2T(x-y) + 12T(x) \qquad (x, y \in G).$$

If T' is another cubic type mapping satisfying (2.4), then

$$\|T(x)-T'(x)\| = \lim_{k\to\infty} |8|^{-k} \|T(2^k x) - T'(2^k x)\|$$
$$\leq \lim_{k\to\infty} |8|^{-k} \max\left\{\|T(2^k x) - f(2^k x)\|, \|f(2^k x) - T'(2^k x)\|\right\}$$
$$\leq \frac{1}{|16|} \lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(2^j x, 0)}{|8|^j} : k \leq j < n+k\right\}$$
$$= 0 \qquad (x \in G),$$

since, by (2.1),

$$\lim_{k\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(2^j x, 0)}{|8|^j} : k \leq j < n+k\right\}$$
$$= \lim_{k\to\infty} \sup\left\{\frac{\varphi(2^j x, 0)}{|8|^j} : k \leq j < \infty\right\} = 0.$$

Therefore $T = T'$. This completes the proof of the uniqueness of T. □

Corollary 2.2. *Let $|2| < 1$, and let $\rho : [0,\infty) \to [0,\infty)$ be defined by*

$$\rho(t) = \begin{cases} \frac{|8|^n}{n+1} & t = |2|^n r,\ n \in \mathbb{N} \cup \{0\},\ r > 0 \\ t & \text{otherwise} \end{cases}$$

Suppose that $\delta > 0$, G is a normed space and $f : G \to X$ fulfills the inequality

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\|$$
$$\leq \delta\left(\rho(\|x\|) + \rho(\|y\|)\right) \qquad (x, y \in G).$$

Then there exists a unique mapping $T : G \to X$ satisfying (1.1) such that

$$\|f(x) - T(x)\| \leq \frac{1}{|16|} \delta \rho(\|x\|) \qquad (x \in G). \tag{2.9}$$

PROOF. By defining $\varphi : G \times G \to [0,\infty)$ by $\varphi(x,y) := \delta\left(\rho(\|x\|) + \rho(\|y\|)\right)$ we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|8|^n} = \lim_{n\to\infty} \frac{\delta}{|8|^n} \left(\rho(\|2^n x\|) + \rho(\|2^n y\|)\right) = 0 \qquad (x, y \in G)$$

$$\widetilde{\varphi}(x) = \sup\left\{\frac{\varphi(2^j x, 0)}{|8|^j} : j \in \mathbb{N}\right\} = \varphi(x, 0).$$

By applying Theorem 2.1 we conclude the required result. □

Remark 2.3. The hypotheses in Corollary 2.2 gives us an example for which the crucial assumption $\sum_{n=1}^{\infty} \frac{\varphi(2^n x, 0)}{|8|^n} < \infty$ in the main theorem of [7] does not hold on balls of X of the radius $r_0 > 0$. (An analogous statement is true for the situation described in Section 3). Hence our results in the setting of non-Archimedean normed spaces differs from those of [7, 8].

3 Stability of the Functional Equation (1.2).

In this section, we establish the stability of functional equation (1.2).

Theorem 3.1. *Let a be an integer with $a \neq 0, \pm 1$, let $\psi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi(a^n x, a^n y)}{|a|^{3n}} = 0 \qquad (x, y \in G)$$

and set

$$\widetilde{\psi}(x) = \sup\left\{ \frac{\psi(a^j x, 0)}{|a|^{3j}} : j \in \mathbb{N} \right\} \qquad (x \in G).$$

Suppose that $f : G \to X$ is a mapping satisfying

$$\|f(ax+y) + f(x+ay) - (a+1)(a-1)^2[f(x)+f(y)] - a(a+1)f(x+y)\| \\ \leq \psi(x, y) \qquad (x, y \in G). \tag{3.1}$$

Then there exists a unique mapping $Q : G \to X$ satisfying (1.2) such that

$$\left\| f(x) + \frac{(a^2-1)}{a^2+a+1} f(0) - Q(x) \right\| \leq \frac{1}{|a|^3} \widetilde{\psi}(x) \qquad (x \in G). \tag{3.2}$$

Proof. Set $y = 0$ in (3.1) and divide by $|a|^3$ to get

$$\left\| \frac{f(ax)}{a^3} - f(x) - \frac{(a+1)(a-1)^2}{a^3} f(0) \right\| \leq \frac{1}{|a|^3} \psi(x, 0) \qquad (x \in G). \tag{3.3}$$

Hence

$$\|F(x) - \frac{F(ax)}{a^3}\| \leq \frac{1}{|a|^3} \psi(x, 0) \qquad (x \in G),$$

where $F(x) = f(x) + \frac{(a^2-1)}{a^2+a+1} f(0)$. Replace x by $a^n x$ in (3.3) and divide by $|a|^{3n}$ to obtain

$$\left\| \frac{F(a^n x)}{a^{3n}} - \frac{F(a^{n+1}x)}{a^{3(n+1)}} \right\| \le \frac{1}{|a|^3} \frac{\psi(a^n x, 0)}{|a|^{3n}} \qquad (x \in G).$$

Hence the sequence $\left\{ \frac{F(a^n x)}{a^{3n}} \right\}$ is Cauchy. We can therefore define a mapping $Q : G \to X$ by

$$Q(x) := \lim_{n\to\infty} \frac{F(a^n x)}{a^{3n}} = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}} \qquad (x \in G).$$

Using the same method as in the proof of Theorem 2.1 we conclude that $Q(x)$ is the unique cubic type mapping satisfying (3.2). □

Corollary 3.2. *Let $a > 1$ be a constant natural number and let $\tau : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\tau(|a|t) \le \tau(|a|)\tau(t) \qquad (t \ge 0),$$
$$\tau(|a|) < |a|^3$$
$$\tau(0) = 0.$$

Suppose that δ is a nonnegative real number, G is a normed space and $f : G \to X$ fulfills the inequality

$$\begin{aligned} &\|f(ax+y) - f(x+ay) - (a+1)(a-1)^2[f(x)+f(y)] - a(a+1)f(x+y)\| \\ &\le \delta\left(\tau(\|x\|) + \tau(\|y\|)\right) \qquad (x, y \in G). \end{aligned}$$

Then there exists a unique mapping $Q : G \to X$ satisfying (1.2) such that

$$\|f(x) - Q(x)\| \le \frac{1}{|a|^3} \delta\tau(\|x\|) \qquad (x \in G).$$

Proof. Defining $\psi : G \times G \to [0,\infty)$ by $\psi(x,y) := \delta\big(\tau(\|x\|) + \tau(\|y\|)\big)$ we have

$$\lim_{n\to\infty} \frac{\psi(a^n x, a^n y)}{|a|^{3n}} \le \lim_{n\to\infty} \left(\frac{\tau(|a|)}{|a|^3} \right)^n \psi(x,y) = 0 \qquad (x, y \in G)$$

$$\widetilde{\psi}(x) = \lim_{n\to\infty} \max\left\{ \frac{\psi(a^j x, 0)}{|a|^{3j}} : 0 \le j < n \right\} = \psi(x, 0).$$

Clearly $f(0) = 0$. Applying Theorem 3.1 we conclude the required result. □

Remark 3.3. The classical example of the function τ is the mapping $\tau(t) = t^p,\ t \in [0,\infty)$, where $p > 3$ and $|a| \neq 1$.

Remark 3.4. We can formulate similar statements to Theorem 2.1 and Theorem 3.1 in which we deal with the Hyers type sequences $\left\{8^n f(\frac{x}{2^n})\right\}$ and $\left\{a^{3n} f(\frac{x}{a^n})\right\}$ respectively, under suitable conditions on the functions φ and ψ.

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