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UNIFORM APPROXIMATION BY BIVARIATE STEP FUNCTIONS QUASICONTINUOUS WITH RESPECT TO SINGLE COORDINATES

Abstract

Quasicontinuity with respect to one coordinate and symmetrical quasicontinuity strengthen the concept of classical quasicontinuity of a bivariate function f from a product space $X \times Y$ into a topological space Z.

For certain spaces X, Y, we show that a function f from $X \times Y$ into a metric space Z is quasicontinuous with respect to the first coordinate if and only if it is the uniform limit of step functions quasicontinuous with respect to the first coordinate. This applies in particular to arbitrary $X \subseteq \mathbb{R}^m$, $m \geq 0$, and every $Y \subseteq \mathbb{R}^n$, $n \geq 1$, without isolated points.

A second result concerns spaces X, Y such that every continuous $f : X \times Y \to Z$ is the uniform limit of symmetrically quasicontinuous step functions. It comprises all $X, Y \subseteq \mathbb{R}$ without isolated points.

1 Introduction.

A function f from a topological space X into a topological space Y is called *quasicontinuous* at $x \in X$ if, for every $W \in \mathcal{U}(f(x))$ and every $U \in \mathcal{U}(x)$, there exists a nonempty open set $G \subseteq U$ such that $f(G) \subseteq W$, where $\mathcal{U}(p)$ is the family of all open neighbourhoods of p in the respective space (see [2]). In [3] it is shown that f is quasicontinuous in the global sense if and only if,

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for every open set $W \subseteq Y$, $f^{-1}(W)$ belongs to the family $SO(X) = \{A \subseteq X : A \subseteq \mathrm{cl}(\mathrm{int}(A))\}$ of *semi-open subsets* of X.

A function $\varphi : X \to Y$ is called a *semi-open step function* if there exists a partition $\{P_i : i \in I\}$ of X into semi-open subsets such that φ is constant on every P_i, $i \in I$. Semi-open step functions are quasicontinuous, because all unions of semi-open sets are semi-open themselves. A main result of [6] says that every quasicontinuous function $f : X \to \mathbb{R}$ can be expressed as a uniform limit of semi-open step functions. Since the system of quasicontinuous functions from X into $\mathbb{R}$ is closed with respect to uniform limits, it can be characterized as the closure of the family of semi-open step functions under uniform limits.

Motivated by considerations of real-valued functions on $\mathbb{R}^n$ in [2], more specific concepts of quasicontinuity of a bivariate function f from the product space $X \times Y$ into a third space Z are defined in [4]. f is called *quasicontinuous with respect to the first coordinate* at $(x, y) \in X \times Y$ if, for every $W \in \mathcal{U}(f(x, y))$ and every $U \in \mathcal{U}(x, y)$, there exists a nonempty open set $G \subseteq U$ such that $f(G) \subseteq W$ and $x \in \pi_1(G)$, π_1 denoting the projection onto the first coordinate. *Quasicontinuity with respect to the second coordinate* at (x, y) is defined analogously. f is called *symmetrically quasicontinuous* at (x, y) if it is quasicontinuous with respect to both coordinates at (x, y).

In [7] corresponding subfamilies of $SO(X \times Y)$ and concepts of step functions are defined.

$$
\begin{aligned}
SO_1(X,Y) &= \big\{A \subseteq X \times Y : \forall\,(x,y) \in A \,\big(\, y \in \mathrm{cl}((\mathrm{int}(A))_x)\,\big)\big\} \\
&= \big\{A \subseteq X \times Y : \forall\,(x,y) \in A \;\forall\, U \in \mathcal{U}(x,y)\; \exists\, V \text{ open in } X \times Y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(\, V \subseteq U \cap A \,\wedge\, x \in \pi_1(V)\,\big)\big\}, \\
SO_2(X,Y) &= \big\{A \subseteq X \times Y : \forall\,(x,y) \in A \,\big(\, x \in \mathrm{cl}((\mathrm{int}(A))^y)\,\big)\big\}, \\
SOS(X,Y) &= SO_1(X,Y) \cap SO_2(X,Y),
\end{aligned}
$$

where $A_x = \{y \in Y : (x, y) \in A\}$ and $A^y = \{x \in X : (x, y) \in A\}$ are vertical and horizontal sections of A, respectively. Then f is quasicontinuous with respect to the first coordinate (symmetrically quasicontinuous) in the global sense if $f^{-1}(W) \in SO_1(X, Y)$ ($f^{-1}(W) \in SOS(X, Y)$) for every open set $W \subseteq Z$. A map $\varphi : X \times Y \to Z$ is called an *SO_1-step function* (an *SOS-step function*) if there exists a partition $\{P_i : i \in I\}$ of $X \times Y$ into sets from $SO_1(X, Y)$ (from $SOS(X, Y)$) such that φ is constant on every P_i, $i \in I$. SO_1-step functions (SOS-step functions) obviously are quasicontinuous with respect to the first coordinate (symmetrically quasicontinuous), because $SO_1(X, Y)$ ($SOS(X, Y)$) is closed under arbitrary unions. It is easily seen that the family of all functions from $X \times Y$ into a metric space Z that are quasicontinuous with respect to the first coordinate (symmetrically quasicontinuous) is closed under uniform limits (see also [5, Proposition 2]).

The present paper is motivated by Stronska's questions from [7]: *Given a function $f : X \times Y \to \mathbb{R}$ quasicontinuous with respect to the first coordinate (symmetrically quasicontinuous), is f the uniform limit of a sequence of SO_1-step functions (SOS-step functions)?*

Quasicontinuity with respect to the first coordinate includes quasicontinuity on the one hand and continuity on the other hand as extremal cases. Indeed, if $X = \{x\}$ is a singleton, then we are in the case of quasicontinuity. Hence, by the above mentioned result from [6], every $f : \{x\} \times Y \to \mathbb{R}$ quasicontinuous with respect to the first coordinate appears as the uniform limit of SO_1-step functions. If $Y = \{y\}$ is a singleton, then every $f : X \times \{y\} \to \mathbb{R}$ quasicontinuous with respect to the first coordinate is continuous. Uniform approximation by SO_1-step functions then amounts to approximation by step functions on partitions into open sets. If X is paracompact, such an approximation of every continuous function $f : X \times \{y\} \to \mathbb{R}$ is possible if and only if X is *strongly zero-dimensional* (see [5]), that is, if $A, B \subseteq X$ can be separated by a continuous function $h : X \to [0, 1]$ in so far as $h(A) \equiv 0$ and $h(B) \equiv 1$, then there exists a set $U \subseteq X$ which is both open and closed such that $A \subseteq U \subseteq X \setminus B$ (see [1, p. 361]). Consequently, if X is paracompact and not strongly zero-dimensional and if Y contains an isolated point, then there even exists a continuous function $f : X \times Y \to \mathbb{R}$ that cannot be expressed as a uniform limit of a sequence of SO_1-step functions. A fortiori, it is impossible to approximate f by SOS-step functions.

In the following section we shall present a class of pairs of spaces (X, Y) such that, given any metric space Z, every $f : X \times Y \to Z$ quasicontinuous with respect to the first coordinate is the uniform limit of some sequence of SO_1-step functions (Theorem 1). We shall see that that this class includes all pairs (X, Y) formed by any subspace $X \subseteq \mathbb{R}^m$ and any subspace $Y \subseteq \mathbb{R}^n$ without isolated points, in particular the pair $(\mathbb{R}^m, \mathbb{R}^n)$ itself (Section 3).

Our knowledge on the approximation by SOS-step functions is much more restricted. We do not even know if every symmetrically quasicontinuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a uniform limit of SOS-step functions. Section 4 concerns the approximation of continuous functions f from $X \times Y$ into a metric space Z by SOS-step functions. The affirmative result given there applies to all spaces $X, Y \subseteq \mathbb{R}$ without isolated points (Section 5).

Throughout this paper $\mathbb{R}^k$, $k \geq 0$, is considered with its natural topology and with the Euclidean norm $\|\cdot\|$.

The *diameter* of a subset A of some metric space (X, d) is defined as $\operatorname{diam}(A) = \sup_{x_1, x_2 \in A} d(x_1, x_2)$ if $A \neq \emptyset$ and by $\operatorname{diam}(\emptyset) = 0$. The closed ball of radius $r > 0$ centred at $x \in X$ is denoted by $B(x, r)$.

A family $\mathcal{F}$ of subsets of a given set X is called a *packing* if its members are disjoint. $\mathcal{F}$ is a *cover* of X if $X \subseteq \bigcup \mathcal{F}$. $\mathcal{F}$ is a *partition* of X if it is both a packing and a cover of X. We call $\mathcal{F}$ *open* if it consists of open sets.

We say that a set $A \subseteq X$ is *finer* than a family $\mathcal{F} \subseteq 2^X$ if $A \subseteq F$ for some $F \in \mathcal{F}$. $\mathcal{F}_2 \subseteq 2^X$ is a *refinement* of $\mathcal{F}_1 \subseteq 2^X$ if every set from $\mathcal{F}_2$ is finer than $\mathcal{F}_1$. A sequence $(\mathcal{F}_n)_{n=1}^\infty$ of families $\mathcal{F}_n \subseteq 2^X$ is called a *chain* if $\mathcal{F}_{n+1}$ is a refinement of $\mathcal{F}_n$, $n \geq 1$.

2 Uniform Limits of SO_1-Step Functions.

Theorem 1. *Let X and Y be topological spaces satisfying the packing property*

$$\textbf{(P)}\ \begin{cases} \textit{There exists a chain } (\mathcal{G}_j)_{j=1}^\infty \textit{ of open packings in } X \times Y \textit{ such that, for} \\ \textit{every open set } U \subseteq X \times Y \textit{ and every } x \in \pi_1(U) \textit{, there is } G \in \bigcup_{j=1}^\infty \mathcal{G}_j \\ \textit{such that } G \subseteq U \textit{ and } x \in \pi_1(G). \end{cases}$$

and let a function f from $X \times Y$ into a metric space (Z, d) be quasicontinuous with respect to the first coordinate. Then there exist a chain $K = K(f) = (\mathcal{P}_n)_{n=1}^\infty$ of partitions of $X \times Y$ into sets from $SO_1(X, Y)$ and a sequence of SO_1-step functions φ_n defined on the partitions $\mathcal{P}_n$, $n \geq 1$, which uniformly converge to f.

This is possible with finite partitions $\mathcal{P}_n$ if (Z, d) is totally bounded.

If, in addition, X and Y are compact and metrizable, then one can choose K such that, given any continuous function $g : X \times Y \to Z$, there is a sequence of SO_1-step functions ψ_n defined on the partitions $\mathcal{P}_n$, $n \geq 1$, which uniformly converge to g.

Theorem 1 is stronger than a characterization of bivariate functions quasicontinuous with respect to the first coordinate as uniform limits of SO_1-step functions. It is similar to the main theorems of [6] and [5], where other concepts of generalized continuity are considered.

Approximation of f by step functions φ_n, $n \geq 1$, defined on the partitions of the chain $(\mathcal{P}_n)_{n=1}^\infty$ is a successive procedure where φ_{n+1} uses information of φ_n in so far as the steps $P^{(n+1)} \in \mathcal{P}_{n+1}$ of φ_{n+1} are obtained from the steps $P^{(n)} \in \mathcal{P}_n$ of φ_n by subdivision. In other words, the spaces $S(\mathcal{P}_n; Z)$ of all step functions on $\mathcal{P}_n$ with values in Z increase with n. The function space $A(K; Z)$ of all uniform limits of step functions on partitions of K with values in Z is the closure of $\bigcup_{n=1}^\infty S(\mathcal{P}_n; Z)$ with respect to uniform limits, the spaces $S(\mathcal{P}_n; Z)$ themselves being closed.

If Z is a normed space K gives rise to a linear approximation scheme formed by the linear subspaces $S(\mathcal{P}_n; Z)$. The set of all functions $f : X \times Y \to Z$

quasicontinuous with respect to the first coordinate, which in general is not a linear space, is represented as the union of closed linear spaces $A(K(f);Z)$.

The case of finite partitions $\mathcal{P}_n$ is particularly interesting, because then the functions of $S(\mathcal{P}_n;Z)$ depend on finitely many parameters $z_P \in Z$, $P \in \mathcal{P}_n$, only. So $S(\mathcal{P}_n;Z)$ is finite-dimensional if Z is normed.

The last claim of Theorem 1 says that one can choose $K = K(f)$ such that $A(K;Z)$ comprises the space of all continuous functions from $X \times Y$ into Z.

The following lemma can be seen as an analogue of Lemma 1 from [6]. Then the proof of Theorem 1 is similar to that of Theorems 1 and 2 from [6] and of Theorem 8 from [5].

Lemma 2. *Let X, Y be topological spaces satisfying property* **(P)***, let (Z,d) be a metric space, let $f : X \times Y \to Z$ be quasicontinuous with respect to the first coordinate, and let $\mathcal{P}$ be a partition of $X \times Y$ into sets $P \in SO_1(X,Y)$ corresponding to f in so far as*

$$P \cap f^{-1}(W) \in SO_1(X,Y) \quad \text{for every open set} \quad W \subseteq Z. \tag{1}$$

Then, given any locally finite open cover $\mathcal{C}$ of $X \times Y$ and any open cover $\mathcal{D}$ of Z, there exists a partition $\mathcal{Q} = \{Q(P,C,D) : P \in \mathcal{P}, C \in \mathcal{C}, D \in \mathcal{D}\}$ of $X \times Y$ into sets $Q(P,C,D) \in SO_1(X,Y)$ satisfying

$$Q(P,C,D) \cap f^{-1}(W) \in SO_1(X,Y) \quad \text{for every open set} \quad W \subseteq Z \tag{2}$$

and

$$Q(P,C,D) \subseteq P \cap \mathrm{cl}(C) \cap f^{-1}(\mathrm{cl}(D)). \tag{3}$$

PROOF. *Step 1. Construction of $\mathcal{Q}$.*

1.0. Preliminaries. The sets $Q(P,C,D)$ will be defined as disjoint unions of sets $R(P,C,D)$ and $S(P,C,D)$ from partitions $\mathcal{R}$ and $\mathcal{S}$ of complementary parts of $X \times Y$.

We assume $\mathcal{D}$ to be locally finite. (Otherwise we can replace $\mathcal{D}$ by a locally finite open cover $\mathcal{D}' = \{D' : D \in \mathcal{D}\}$ such that $D' \subseteq D$, since Z is paracompact (see [1, p. 300]).)

1.1. Construction of $\mathcal{R}$. Let

$$\mathcal{H} = \left\{P \cap C \cap f^{-1}(D) : P \in \mathcal{P}, C \in \mathcal{C}, D \in \mathcal{D}\right\}.$$

Given $G_1, G_2 \in \bigcup_{j=1}^{\infty} \mathcal{G}_j$, G_1 is called a *predecessor* of G_2 if $G_1 \supseteq G_2$ and $G_1 \neq G_2$. We use the packings from **(P)** for introducing

$$\mathcal{G} \;=\; \Big\{G \in \textstyle\bigcup_{j=1}^{\infty} \mathcal{G}_j : G \text{ is finer than } \mathcal{H}, \text{ but no predecessor of } G \text{ is finer than } \mathcal{H}\Big\}.$$

The members of $\mathcal{G}$ are disjoint, because two distinct sets from $\bigcup_{j=1}^{\infty} \mathcal{G}_j$ have a nonempty intersection only if one is a predecessor of the other.

Since $\mathcal{G}$ is a refinement of $\mathcal{H}$, there exist functions $\varrho : \mathcal{G} \to \mathcal{P}$, $\sigma : \mathcal{G} \to \mathcal{C}$, and $\tau : \mathcal{G} \to \mathcal{D}$ such that

$$G \subseteq \varrho(G) \cap \sigma(G) \cap f^{-1}(\tau(G)) \quad \text{for all} \quad G \in \mathcal{G}.$$

We define the open partition $\mathcal{R}$ of $\bigcup \mathcal{G}$ by

$$\begin{aligned} \mathcal{R} &= \{R(P,C,D) : P \in \mathcal{P}, C \in \mathcal{C}, D \in \mathcal{D}\} \quad \text{where} \\ R(P,C,D) &= \bigcup \{G \in \mathcal{G} : \varrho(G) = P, \sigma(G) = C, \tau(G) = D\}. \end{aligned}$$

Clearly, $\mathcal{G}$ is a refinement of $\mathcal{R}$ and $\mathcal{R}$ is a refinement of $\mathcal{H}$, namely

$$R(P,C,D) \subseteq P \cap C \cap f^{-1}(D) \quad \text{for all} \quad P \in \mathcal{P}, C \in \mathcal{C}, D \in \mathcal{D}. \tag{4}$$

Moreover, $\mathcal{R}$ satisfies the following.

$$\begin{aligned} &\text{If } G \in \textstyle\bigcup_{j=1}^{\infty} \mathcal{G}_j \text{ is finer than } \mathcal{H}, \text{ then there exists} \\ &R(P,C,D) \in \mathcal{R} \text{ such that } G \subseteq R(P,C,D). \end{aligned} \tag{5}$$

Indeed, if $G \in \mathcal{G}$, then $G \subseteq R(\varrho(G), \sigma(G), \tau(G))$. Otherwise, since G has only finitely many predecessors, one of them, say G', belongs to $\mathcal{G}$ and we obtain $G \subseteq G' \subseteq R(\varrho(G'), \sigma(G'), \tau(G'))$.

1.2. Construction of $\mathcal{S}$. Let $(x,y) \in (X \times Y) \setminus \bigcup \mathcal{R} = (X \times Y) \setminus \bigcup \mathcal{G}$. There is a unique $P_0 \in \mathcal{P}$ with $(x,y) \in P_0$. By the local finiteness of $\mathcal{C}$ and $\mathcal{D}$,

$$\begin{aligned} \{C \in \mathcal{C} : (x,y) \in \mathrm{cl}(C)\} &= \{C_1, \ldots, C_k\} \quad \text{and} \\ \{D \in \mathcal{D} : f(x,y) \in \mathrm{cl}(D)\} &= \{D_1, \ldots, D_l\} \end{aligned}$$

are finite. Say $(x,y) \in C_1$ and $f(x,y) \in D_1$. We obtain open neighbourhoods

$$\begin{aligned} U_0 &= (X \times Y) \setminus \textstyle\bigcup_{C \in \mathcal{C} \setminus \{C_1, \ldots, C_k\}} \mathrm{cl}(C) \in \mathcal{U}(x,y) \quad \text{and} \\ W_0 &= Z \setminus \textstyle\bigcup_{D \in \mathcal{D} \setminus \{D_1, \ldots, D_l\}} \mathrm{cl}(D) \in \mathcal{U}(f(x,y)), \end{aligned}$$

respectively. Next we show that

$$\begin{aligned} &\exists\, C_0 \in \{C_1, \ldots, C_k\} \;\; \exists\, D_0 \in \{D_1, \ldots, D_l\} \\ &\forall\, W \in \mathcal{U}(f(x,y)) \;\; \forall\, U \in \mathcal{U}(x,y) \;\; \exists\, V \text{ open in } X \times Y : \\ &V \subseteq R(P_0, C_0, D_0) \cap U \cap f^{-1}(W) \;\wedge\; x \in \pi_1(V). \end{aligned} \tag{6}$$

On the contrary, suppose that, for all $C_r \in \{C_1, \ldots, C_k\}$ and all $D_s \in \{D_1, \ldots, D_l\}$, there exist $W_{r,s} \in \mathcal{U}(f(x,y))$ and $U_{r,s} \in \mathcal{U}(x,y)$ such that, for every open $V \subseteq X \times Y$, $V \not\subseteq R(P_0, C_r, D_s) \cap U_{r,s} \cap f^{-1}(W_{r,s})$ or $x \notin \pi_1(V)$. Then the neighbourhoods

$$\begin{array}{rcrcll} \tilde{W} & = & W_0 \cap \bigcap_{1 \le r \le k,\, 1 \le s \le l} W_{r,s} & \in & \mathcal{U}(f(x,y)) & \text{and} \\ \tilde{U} & = & U_0 \cap \bigcap_{1 \le r \le k,\, 1 \le s \le l} U_{r,s} & \in & \mathcal{U}(x,y) & \end{array}$$

satisfy

$$\begin{aligned} &\forall V \text{ open in } X \times Y \;\; \forall C_r \in \{C_1, \ldots, C_k\} \;\; \forall D_s \in \{D_1, \ldots, D_l\} : \\ &V \not\subseteq R(P_0, C_r, D_s) \cap \tilde{U} \cap f^{-1}\big(\tilde{W}\big) \;\vee\; x \notin \pi_1(V). \end{aligned} \tag{7}$$

Application of (1) to $P_0 \in \mathcal{P}$ and arbitrary $W \in \mathcal{U}(f(x,y))$ yields $(x,y) \in P_0 \cap f^{-1}(W) \in SO_1(X,Y)$. Hence, for every $W \in \mathcal{U}(f(x,y))$ and every $U \in \mathcal{U}(x,y)$, there exists an open set $V \subseteq X \times Y$ such that $V \subseteq U \cap P_0 \cap f^{-1}(W)$ and $x \in \pi_1(V)$. For the particular neighbourhoods $C_1 \cap \tilde{U} \in \mathcal{U}(x,y)$ and $D_1 \cap \tilde{W} \in \mathcal{U}(f(x,y))$, we find an open $\tilde{V} \subseteq X \times Y$ such that $\tilde{V} \subseteq P_0 \cap \big(C_1 \cap \tilde{U}\big) \cap f^{-1}\big(D_1 \cap \tilde{W}\big)$ and $x \in \pi_1\big(\tilde{V}\big)$. Application of **(P)** to $\tilde{V}$ gives $G \in \bigcup_{j=1}^{\infty} \mathcal{G}_j$ with $G \subseteq \tilde{V}$ and $x \in \pi_1(G)$. Hence

$$G \subseteq P_0 \cap \big(C_1 \cap \tilde{U}\big) \cap f^{-1}\big(D_1 \cap \tilde{W}\big) \quad \text{and} \quad x \in \pi_1(G). \tag{8}$$

Thus G is finer than $\mathcal{H}$, since in particular $G \subseteq P_0 \cap C_1 \cap f^{-1}(D_1)$, and (5) and (4) yield

$$\exists\, P \in \mathcal{P} \;\; \exists\, C \in \mathcal{C} \;\; \exists\, D \in \mathcal{D} : \;\; G \subseteq R(P, C, D) \subseteq P \cap C \cap f^{-1}(D). \tag{9}$$

Combining (8) with (9) we obtain $\emptyset \neq G \subseteq P_0 \cap P$, which gives $P = P_0$, because $\mathcal{P}$ is a partition. Similarly, $G \subseteq \tilde{U} \cap C \subseteq U_0 \cap C$ and $G \subseteq f^{-1}\big(\tilde{W}\big) \cap f^{-1}(D) \subseteq f^{-1}(W_0 \cap D)$, which yield $C = C_{r_0} \in \{C_1, \ldots, C_k\}$ and $D = D_{s_0} \in \{D_1, \ldots, D_l\}$ by the definitions of U_0 and W_0, respectively. Hence, again by (8) and (9),

$$G \subseteq R(P_0, C_{r_0}, D_{s_0}) \cap \tilde{U} \cap f^{-1}\big(\tilde{W}\big) \quad \text{and} \quad x \in \pi_1(G).$$

This contradiction with (7) completes the verification of (6).

Putting $\overline{\varrho}(x,y) = P_0$, choosing $\overline{\sigma}(x,y) = C_0$ and $\overline{\tau}(x,y) = D_0$ according to (6), and then considering (x,y) to be a variable in $(X \times Y) \setminus \bigcup \mathcal{R}$, we obtain functions $\overline{\varrho}, \overline{\sigma}, \overline{\tau}$ from $(X \times Y) \setminus \bigcup \mathcal{R}$ into $\mathcal{P}, \mathcal{C}, \mathcal{D}$ such that

$$\begin{aligned} &\forall\, (x,y) \in (X \times Y) \setminus \textstyle\bigcup \mathcal{R} \;\; \forall\, U \in \mathcal{U}(x,y) \;\; \forall\, W \in \mathcal{U}(f(x,y)) \\ &\exists\, V \text{ open in } X \times Y : \\ &V \subseteq R\big(\overline{\varrho}(x,y), \overline{\sigma}(x,y), \overline{\tau}(x,y)\big) \cap U \cap f^{-1}(W) \;\wedge\; x \in \pi_1(V). \end{aligned} \tag{10}$$

Finally, we define the partition $\mathcal{S}$ of $(X \times Y) \setminus \bigcup \mathcal{R} = (X \times Y) \setminus \bigcup \mathcal{G}$ by

$$\begin{aligned} \mathcal{S} &= \{S(P,C,D) : P \in \mathcal{P}, C \in \mathcal{C}, D \in \mathcal{D}\} \quad \text{where} \\ S(P,C,D) &= \{(x,y) : \overline{\varrho}(x,y) = P,\, \overline{\sigma}(x,y) = C,\, \overline{\tau}(x,y) = D\}. \end{aligned}$$

1.3. Definition of $\mathcal{Q}$. We put

$$\mathcal{Q} = \big\{Q(P,C,D) = R(P,C,D) \cup S(P,C,D) : P \in \mathcal{P}, C \in \mathcal{C}, D \in \mathcal{D}\big\}.$$

Step 2. Properties of $\mathcal{Q}$. $\mathcal{R}$ and $\mathcal{S}$ being partitions of complementary subsets of $X \times Y$, $\mathcal{Q}$ is a partition of $X \times Y$.

2.1. PROOF OF (2). It is to show that, for every fixed $(x,y) \in Q(P,C,D) \cap f^{-1}(W)$ and every fixed $U \in \mathcal{U}(x,y)$, there exists an open set $V \subseteq X \times Y$ such that $V \subseteq U \cap (Q(P,C,D) \cap f^{-1}(W))$ and $x \in \pi_1(V)$.

Case 1. $(x,y) \in R(P,C,D)$. $R(P,C,D) \cap f^{-1}(W) \in SO_1(X,Y)$, because $R(P,C,D)$ is open and $f^{-1}(W) \in SO_1(X,Y)$. Thus there is an open set $V \subseteq X \times Y$ such that $V \subseteq U \cap (R(P,C,D) \cap f^{-1}(W)) \subseteq U \cap (Q(P,C,D) \cap f^{-1}(W))$ and $x \in \pi_1(V)$.

Case 2. $(x,y) \in S(P,C,D)$. Then $\overline{\varrho}(x,y) = P$, $\overline{\sigma}(x,y) = C$, $\overline{\tau}(x,y) = D$. By (10), there exists an open set $V \subseteq X \times Y$ such that $V \subseteq R(P,C,D) \cap U \cap f^{-1}(W) \subseteq U \cap (Q(P,C,D) \cap f^{-1}(W))$ and $x \in \pi_1(V)$. This completes the proof of (2).

The particular choice $W = Z$ in (2) yields $Q(P,C,D) \in SO_1(X,Y)$.

2.2. PROOF OF (3). By (4), it suffices to show that $S(P,C,D) \subseteq P \cap \mathrm{cl}(C) \cap f^{-1}(\mathrm{cl}(D))$. We fix $(x,y) \in S(P,C,D)$, which gives $\overline{\varrho}(x,y) = P$, $\overline{\sigma}(x,y) = C$, $\overline{\tau}(x,y) = D$. The proof will be complete once we have shown that

$$(x,y) \in P, \quad (x,y) \in \mathrm{cl}(C), \quad f(x,y) \in \mathrm{cl}(D). \tag{11}$$

The first inclusion follows from $\overline{\varrho}(x,y) = P$. ($\mathcal{S}$ is a refinement of $\mathcal{P}$.) Properties (10) and (4) yield

$$\begin{aligned} &\forall U \in \mathcal{U}(x,y) \ \ \forall W \in \mathcal{U}(f(x,y)) \ \ \exists V \text{ open in } X \times Y : \\ &\emptyset \neq V \subseteq R(P,C,D) \cap U \cap f^{-1}(W) \subseteq (C \cap U) \cap f^{-1}(D \cap W). \end{aligned} \tag{12}$$

Hence $(x,y) \in \mathrm{cl}(C)$, because every $U \in \mathcal{U}(x,y)$ has a nonempty intersection with C, and $f(x,y) \in \mathrm{cl}(D)$, since all $W \in \mathcal{U}(f(x,y))$ have nonempty intersections with D. This completes the proof. □

PROOF OF THEOREM 1. Let $\mathcal{P}_0 = \{X \times Y\}$. This partition satisfies assumption (1) of the lemma, because f is quasicontinuous with respect to the first coordinate. Given $\mathcal{P}_{n-1}$, $n \geq 1$, we shall construct the partition $\mathcal{P}_n$ subject to the following conditions.

(i) $\mathcal{P}_n$ is a refinement of $\mathcal{P}_{n-1}$.

(ii) The members P of $\mathcal{P}_n$ belong to $SO_1(X,Y)$ and satisfy property (1).

(iii) $\mathcal{P}_n$ is finite if (Z,d) is totally bounded.

(iv) There exists a step function φ_n on $\mathcal{P}_n$ such that

$$\sup_{(x,y)\in X\times Y} d(f(x,y),\varphi_n(x,y)) \le 2^{-n}.$$

(v) If $X\times Y$ is compact and metrized by some fixed metric $d^\times$ (which obviously is the case if X and Y both are compact and metrizable), then, given any continuous function $g : X\times Y \to Z$, there is a step function $\psi_n : X\times Y \to Z$ on $\mathcal{P}_n$ such that

$$\sup_{(x,y)\in X\times Y} d(g(x,y),\psi_n(x,y)) \le \omega(g;2^{-n}),$$

where $\omega(g;2^{-n}) = \sup d(g(x_1,y_1),g(x_2,y_2))$ is the modulus of continuity, the *sup* being taken over all

$$(x_1,y_1),(x_2,y_2)\in X\times Y,\ d^\times((x_1,y_1),(x_2,y_2)) \le 2^{-n}.$$

We obtain $\mathcal{P}_n$ by applying the lemma to $\mathcal{P}=\mathcal{P}_{n-1}$ and the following covers $\mathcal{C}$ and $\mathcal{D}$. If $X\times Y$ is compact and metrized by $d^\times$ we fix a finite open cover $\mathcal{C}=\{C_1,\dots,C_k\}$ of $X\times Y$ with $\operatorname{diam}(C_r)\le 2^{-n}$, $1\le r\le k$. Otherwise we use $\mathcal{C}=\{X\times Y\}$. $\mathcal{D}$ has to be a cover of Z by open sets of diameter at most 2^{-n}. If Z is totally bounded we use a finite cover $\mathcal{D}$ with that property.

By the lemma, the resulting partition $\mathcal{Q}=\mathcal{P}_n$ satisfies (i) and (ii). If (Z,d) is totally bounded, then $\mathcal{P}_{n-1}$, $\mathcal{C}$, and $\mathcal{D}$ are finite and so is $\mathcal{P}_n$.

By claim (3) of the lemma, every $Q\in\mathcal{P}_n$ is contained in some $f^{-1}(\operatorname{cl}(D))$, $D\in\mathcal{D}$. Hence $\operatorname{diam}(f(Q))\le\operatorname{diam}(\operatorname{cl}(D))=\operatorname{diam}(D)\le 2^{-n}$. For every $Q\in\mathcal{P}_n$, we fix some $(x_Q,y_Q)\in Q$ (provided that $Q\ne\emptyset$) and define $\varphi_n : X\times Y\to Z$ by $\varphi_n(Q)\equiv f(x_Q,y_Q)$. This yields (iv), namely

$$\begin{aligned}
&\sup_{(x,y)\in X\times Y} d(f(x,y),\varphi_n(x,y)) = \sup_{Q\in\mathcal{P}_n}\sup_{(x,y)\in Q} d(f(x,y),\varphi_n(x,y))\\
&= \sup_{Q\in\mathcal{P}_n}\sup_{(x,y)\in Q} d(f(x,y),f(x_Q,y_Q)) \le \sup_{Q\in\mathcal{P}_n}\operatorname{diam}(f(Q)) \le 2^{-n}.
\end{aligned}$$

Finally, if $X\times Y$ is compact and metrized by $d^\times$ and if a continuous $g : X\times Y\to Z$ is given, then we consider $\psi_n : X\times Y\to Z$ with $\psi_n(Q)\equiv$

$g(x_Q, y_Q)$. Again by claim (3), every $Q \in \mathcal{P}_n$ is a subset of some $\mathrm{cl}(C)$, $C \in \mathcal{C}$. Thus $\mathrm{diam}(Q) \leq \mathrm{diam}(\mathrm{cl}(C)) = \mathrm{diam}(C) \leq 2^{-n}$ and

$$\begin{aligned} \sup_{(x,y)\in X\times Y} d(g(x,y), \psi_n(x,y)) &= \sup_{Q\in\mathcal{P}_n} \sup_{(x,y)\in Q} d(g(x,y), \psi_n(x,y)) \\ &= \sup_{Q\in\mathcal{P}_n} \sup_{(x,y)\in Q} d(g(x,y), g(x_Q, y_Q)) \\ &\leq \sup_{Q\in\mathcal{P}_n} \omega(g; \mathrm{diam}(Q)) \leq \omega(g; 2^{-n}), \end{aligned}$$

which proves (v).

Note that (v) gives $\lim_{n\to\infty} \sup_{(x,y)\in X\times Y} d(g(x,y), \psi_n(x,y)) = 0$, since $X \times Y$ is compact and g is continuous. This completes the proof of Theorem 1. □

3 Subspaces of $\mathbb{R}^m \times \mathbb{R}^n$ Satisfying Property (P).

Theorem 3. *Let the subspace $X \subseteq \mathbb{R}^m$, $m \geq 0$, be arbitrary and let $Y \subseteq \mathbb{R}^n$, $n \geq 1$, be a subspace without isolated points. Then (X, Y) has property* **(P)**.

The proof is prepared by a lemma.

Lemma 4. *Let $A \subseteq \mathbb{R}^k$, $k \geq 1$, be a subset without isolated points. Then there is a basis $\{b_1, \ldots, b_k\}$ of $\mathbb{R}^k$ such that, for every $a \in A$ and every $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $\|a_\varepsilon - a\| < \varepsilon$ and $a_\varepsilon - a \notin \bigcup_{i=1}^k \mathrm{span}(\{b_1, \ldots, b_k\} \setminus \{b_i\})$.*

PROOF. Let $\{a_1, a_2, \ldots\} \subseteq A$ be a countable dense subset of A. We denote the Grassmann manifold of all $(k-1)$-dimensional linear subspaces of $\mathbb{R}^k$ by $G_{k-1}(\mathbb{R}^k)$. For arbitrary $1 \leq j_1 < j_2 < \infty$, $G(j_1, j_2) = \{H \in G_{k-1}(\mathbb{R}^k) : a_{j_1} - a_{j_2} \in H\}$ is a nowhere dense subset of the compact space $G_{k-1}(\mathbb{R}^k)$. Hence $D = G_{k-1}(\mathbb{R}^k) \setminus \bigcup_{1\leq j_1<j_2<\infty} G(j_1, j_2)$ is dense in $G_{k-1}(\mathbb{R}^k)$ by the Baire category theorem. We pick hyperplanes $H_1, \ldots, H_k \in D$ that are close to the coordinate hyperplanes of $\mathbb{R}^k$. Then we obtain a basis $\{b_1, \ldots, b_k\}$ by choosing a vector $b_i \neq 0$ in every straight line $\bigcap_{j\in\{1,\ldots,k\}\setminus\{i\}} H_j$, $1 \leq i \leq k$.

For showing the claim, we assume $a \in A$ and $\varepsilon > 0$ to be fixed. We pick $a^* \in \{a_1, a_2, \ldots\}$ with $\|a^* - a\| < \frac{\varepsilon}{2}$. Let $I = \{i \in \{1, \ldots, k\} : a^* - a \notin H_i\}$. We choose $\delta \in \left(0, \frac{\varepsilon}{2}\right]$ such that $B(a^* - a, \delta) \cap \bigcup_{i\in I} H_i = \emptyset$. Now we pick $a_\varepsilon \in (\{a_1, a_2, \ldots\} \setminus \{a^*\}) \cap B\big(a^*, \delta)$. Clearly,

$$\|a_\varepsilon - a\| \leq \|a_\varepsilon - a^*\| + \|a^* - a\| < \delta + \tfrac{\varepsilon}{2} \leq \varepsilon.$$

It remains to show that

$$a_\varepsilon - a \notin \textstyle\bigcup_{i=1}^k \mathrm{span}(\{b_1, \ldots, b_k\} \setminus \{b_i\}) = \bigcup_{i=1}^k H_i.$$

On the contrary, suppose that $a_\varepsilon - a \in H_{i_0}$ for some $i_0 \in \{1, \dots, k\}$.

If $i_0 \in I$ we obtain $a_\varepsilon - a \in B(a^* - a, \delta) \cap \bigcup_{i \in I} H_i$, because

$$a_\varepsilon - a = (a^* - a) + (a_\varepsilon - a^*) \in B(a^* - a, \|a_\varepsilon - a^*\|) \subseteq B(a^* - a, \delta).$$

This contradicts $B(a^* - a, \delta) \cap \bigcup_{i \in I} H_i = \emptyset$.

In the opposite case, that is $i_0 \notin I$, we have $a_\varepsilon - a, a^* - a \in H_{i_0}$, which gives $a_\varepsilon - a^* \in H_{i_0}$. However, $\{a_\varepsilon, a^*\} = \{a_{j_1}, a_{j_2}\} \subseteq \{a_1, a_2, \dots\}$ with $j_1 < j_2$. Hence $H_{i_0} \in G(j_1, j_2)$, a contradiction with the choice of $H_{i_0} \in D$. □

PROOF OF THEOREM 3. Application of the lemma to $A = \{0\} \times Y \subseteq \mathbb{R}^{m+n}$ gives a basis $\{b_1, \dots, b_{m+n}\}$ of $\mathbb{R}^{m+n}$. The open parallelepipeds

$$G(k_1, \dots, k_{m+n}) = \left\{ \textstyle\sum_{i=1}^{m+n} \lambda_i b_i : k_i < \lambda_i < k_i + 1 \text{ for } 1 \le i \le m+n \right\}$$

with $k_i \in \mathbb{Z}$ cover a dense subset of $\mathbb{R}^{m+n}$. We define

$$\mathcal{G}_j = \left\{ \left(2^{-j} G(k_1, \dots, k_{m+n})\right) \cap (X \times Y) : (k_1, \dots, k_{m+n}) \in \mathbb{Z}^{m+n} \right\}.$$

This gives a chain $(\mathcal{G}_j)_{j=1}^\infty$ of open packings in $X \times Y$.

For showing the remainder of **(P)**, we fix an open set $U \subseteq X \times Y$ and a point $x \in \pi_1(U)$. We pick $y_0 \in Y$ with $(x, y_0) \in U$. Let $(x, y_0) = \sum_{i=1}^{m+n} \lambda_i^{(0)} b_i$ be the representation with respect to the fixed basis. Since U is open, there is $j \ge 1$ such that the set $H = \mathrm{cl}(\bigcup \{G \in \mathcal{G}_j : (x, y_0) \in \mathrm{cl}(G)\})$ is contained in U. In the metric space $X \times Y$ we fix a ball $B((x, y_0), \delta) \subseteq H$, $\delta > 0$. By the definition of A and by the claim of the lemma, there exists a sequence $(y_k)_{k=1}^\infty \subseteq Y$ such that $\lim_{k \to \infty} y_k = y_0$ and $(0, y_k - y_0) \notin \bigcup_{i=1}^{m+n} \mathrm{span}(\{b_1, \dots, b_{m+n}\} \setminus \{b_i\})$. If $(0, y_k - y_0) = \sum_{i=1}^{m+n} \mu_i^{(k)} b_i$ this means that $\lim_{k \to \infty} \mu_i^{(k)} = 0$ for $1 \le i \le m+n$ and $\mu_i^{(k)} \ne 0$ for $1 \le i \le m+n$, $k \ge 1$.

Clearly, $\|(0, y_k - y_0)\| \le \delta$ for $k \ge k_0$. For every $i \in \{1, \dots, m+n\}$, we can choose k_i such that $\lambda_i^{(0)} + \mu_i^{(k)} \notin 2^{-j}\mathbb{Z}$ for $k \ge k_i$. (If $\lambda_i^{(0)} \in 2^{-j}\mathbb{Z}$ we use that $\mu_i^{(k)}$ is small, but not zero. If $\lambda_i^{(0)} \notin 2^{-j}\mathbb{Z}$ we use that $\mu_i^{(k)}$ is small.)

We fix $k = \max\{k_0, \dots, k_{m+n}\}$. By $\|(0, y_k - y_0)\| \le \delta$,

$$(x, y_k) = (x, y_0) + (0, y_k - y_0) \in B((x, y_0), \delta) \subseteq H \subseteq U.$$

Moreover, the properties $\lambda_i^{(0)} + \mu_i^{(k)} \notin 2^{-j}\mathbb{Z}$, $1 \le i \le m+n$, yield

$$(x, y_k) = (x, y_0) + (0, y_k - y_0) = \textstyle\sum_{i=1}^{m+n} \left(\lambda_i^{(0)} + \mu_i^{(k)}\right) b_i \in \bigcup \mathcal{G}_j.$$

The definition of H finally shows that there exists $G \in \mathcal{G}_j$ such $(x, y_k) \in G \subseteq H \subseteq U$, in particular $x \in \pi_1(G)$. This completes the proof. □

4 Approximation of Continuous Functions by SOS-Step Functions.

Theorem 5. *Let X and Y be topological spaces such that $X \times Y$ possesses a chain $(\mathcal{R}_j)_{j=1}^{\infty}$ of partitions into sets from $SOS(X,Y)$ such that, for every $(x,y) \in X \times Y$ and every $U \in \mathcal{U}(x,y)$, there exists $R \in \bigcup_{j=1}^{\infty} \mathcal{R}_j$ such that $(x,y) \in R \subseteq U$. Let (Z,d) be a metric space.*

(a) Given a continuous function $f : X \times Y \to Z$, there exists a chain $K = K(f) = (\mathcal{P}_n)_{n=1}^{\infty}$ of partitions of $X \times Y$ into sets from $SOS(X,Y)$ and a sequence of SOS-step functions φ_n defined on the partitions $\mathcal{P}_n$, $n \geq 1$, which uniformly converge to f.

This is possible with finite partitions $\mathcal{P}_n$ if (Z,d) is totally bounded.

(b) If X and Y are compact and metrizable, then there exists a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of finite partitions of $X \times Y$ into sets from $SOS(X,Y)$ such that, given any continuous function $g : X \times Y \to Z$, there is a sequence of SOS-step functions ψ_n defined on the partitions $\mathcal{P}_n$, $n \geq 1$, which uniformly converge to g.

PROOF. As in the proof of Lemma 2, we call $R_1 \in \bigcup_{j=1}^{\infty} \mathcal{R}_j$ a *predecessor* of $R_2 \in \bigcup_{j=1}^{\infty} \mathcal{R}_j$ if $R_1 \supseteq R_2$ and $R_1 \neq R_2$. Let $\mathcal{P}_0 = \{X \times Y\}$.

PROOF OF (a). We shall define K inductively such that, for all $n \geq 1$,

(i) the partition $\mathcal{P}_n$ is a refinement of $\mathcal{P}_{n-1}$,

(ii) every $P \in \mathcal{P}_n$ is a union of sets from $\bigcup_{j=1}^{\infty} \mathcal{R}_j$ (and hence $P \in SOS(X,Y)$),

(iii) $\mathcal{P}_n$ is finite if (Z,d) is totally bounded, and

(iv) there exists a step function φ_n defined on $\mathcal{P}_n$ such that

$$\sup_{(x,y)\in X\times Y} d(f(x,y),\varphi_n(x,y)) \leq 2^{-n}.$$

Suppose $\mathcal{P}_{n-1}$ to be given. We fix an open cover $\mathcal{D}$ of Z such that $\operatorname{diam}(D) \leq 2^{-n}$ for all $D \in \mathcal{D}$. If (Z,d) is totally bounded $\mathcal{D}$ can assumed to be finite. $\mathcal{C} = \{f^{-1}(D) : D \in \mathcal{D}\}$ is an open cover of $X \times Y$. Let

$$\begin{aligned}\overline{\mathcal{R}} \quad = \quad \Big\{ R \in \textstyle\bigcup_{j=1}^{\infty} \mathcal{R}_j : \; & R \text{ is finer than } \mathcal{P}_{n-1} \text{ and } \mathcal{C}, \text{ but no} \\ & \text{predecessor of } R \text{ is finer than } \mathcal{P}_{n-1} \text{ and } \mathcal{C} \Big\}.\end{aligned}$$

For each $R \in \overline{\mathcal{R}}$, we pick $\varrho(R) \in \mathcal{P}_{n-1}$ and $\sigma(R) \in \mathcal{C}$ such that $R \subseteq \varrho(R) \cap \sigma(R)$, this way defining functions ϱ, σ from $\overline{\mathcal{R}}$ into $\mathcal{P}_{n-1}$ and $\mathcal{C}$, respectively. Now we put

$$\begin{aligned} Q(P,C) &= \bigcup \{R \in \overline{\mathcal{R}} :\ \varrho(R) = P, \sigma(R) = C\} \quad \text{and} \\ \mathcal{P}_n &= \{Q(P,C) :\ P \in \mathcal{P}_{n-1}, C \in \mathcal{C}\}. \end{aligned}$$

First we show that $\mathcal{P}_n$ is a partition of $X \times Y$. Any two distinct members of $\bigcup_{j=1}^{\infty} \mathcal{R}_j$ have a nonempty intersection only if one is a predecessor of the other. Hence $\overline{\mathcal{R}}$ is a packing and so is $\mathcal{P}_n$. For showing the covering property we fix $(x,y) \in X \times Y$. We pick $P \in \mathcal{P}_{n-1}$ with $(x,y) \in P$. By the induction hypothesis (ii) on $\mathcal{P}_{n-1}$, there is $R_1 \in \bigcup_{j=1}^{\infty} \mathcal{R}_j$ such that $(x,y) \in R_1 \subseteq \mathcal{P}_{n-1}$. Similarly, we choose $C \in \mathcal{C}$ such that $(x,y) \in C$. By the supposition on $(\mathcal{R}_j)_{j=1}^{\infty}$, there exists $R_2 \in \bigcup_{j=1}^{\infty} \mathcal{R}_j$ with $(x,y) \in R_2 \subseteq C$. Then $R_0 = R_1 \cap R_2 \in \{R_1, R_2\}$ is finer than $\mathcal{P}_{n-1}$ and $\mathcal{C}$. Thus R_0 itself or one of its predecessors belongs to $\overline{\mathcal{R}}$ and contains (x,y), because it covers R_0. So $(x,y) \in \bigcup \overline{\mathcal{R}} = \bigcup \mathcal{P}_n$. Hence, $\mathcal{P}_n$ is a partition of $X \times Y$. The other claims of (i)-(iii) are obvious.

In every $Q \in \mathcal{P}_n$ we fix $(x_Q, y_Q) \in Q$ (provided that $Q \neq \emptyset$) and define φ_n by $\varphi_n(Q) \equiv f(x_Q, y_Q)$. Since Q is covered by some $C = f^{-1}(D) \in \mathcal{C}$, we obtain

$$\operatorname{diam}(f(Q)) \leq \operatorname{diam}(f(f^{-1}(D))) \leq \operatorname{diam}(D) \leq 2^{-n}.$$

As in the verification of claim (iv) of the proof of Theorem 1, this yields (iv) and completes the proof of part (a).

Proof of (b). We assume a metric $d^{\times}$ on $X \times Y$ to be fixed. Now we construct K such that, for every $n \geq 1$,

(i) the partition $\mathcal{P}_n$ is a refinement of $\mathcal{P}_{n-1}$,

(ii) every $P \in \mathcal{P}_n$ is a union of sets from $\bigcup_{j=1}^{\infty} \mathcal{R}_j$ (and hence $P \in SOS(X,Y)$),

(iii) $\mathcal{P}_n$ is finite, and,

(iv) for every continuous $g : X \times Y \to Z$, there is a step function ψ_n on $\mathcal{P}_n$ such that

$$\sup_{(x,y) \in X \times Y} d(g(x,y), \psi_n(x,y)) \leq \omega(g; 2^{-n}).$$

Given $\mathcal{P}_{n-1}$, we fix a finite open over $\mathcal{C}$ of $X \times Y$ with $\operatorname{diam}(C) \leq 2^{-n}$, $C \in \mathcal{C}$, and then use the same definition of $\mathcal{P}_n$ as in the proof of part (a). (i)-(iii) can be verified as above. Again we choose $(x_Q, y_Q) \in Q$ for every $Q \in \mathcal{P}_n$. We define ψ_n by $\psi_n(Q) \equiv g(x_Q, y_Q)$. Every $Q \in \mathcal{P}_n$ is contained in some $C \in \mathcal{C}$ and thus satisfies $\operatorname{diam}(Q) \leq \operatorname{diam}(C) \leq 2^{-n}$. Now the proof of (iv) follows the same lines as that of claim (v) in the proof of Theorem 1. □

5 Subspaces of $\mathbb{R} \times \mathbb{R}$ to Which Applies Theorem 5.

We present a class of spaces $X, Y \subseteq \mathbb{R}$ satisfying the supposition of Theorem 5.

Theorem 6. *Let $X, Y \subseteq \mathbb{R}$ be topological subspaces without isolated points. Then there exists a chain $(\mathcal{R}_j)_{j=1}^{\infty}$ of partitions of $X \times Y$ into sets from $SOS(X,Y)$ such that $\lim_{j\to\infty} \sup_{R\in\mathcal{R}_j} \operatorname{diam}(R) = 0$.*

PROOF. *Step 1. A chain $(\mathcal{Q}_j)_{j=1}^{\infty}$ of open packings in $\mathbb{R}\times\mathbb{R}$ related to $X \times Y$.*

Every point of X is approached by other points from X. Hence it can belong to at most one of the sets $X_L = \{x \in X : (x-\varepsilon, x) \cap X = \emptyset$ for some $\varepsilon > 0\}$ (locally lower points of X) or $X^U = \{x \in X : (x, x+\varepsilon) \cap X = \emptyset$ for some $\varepsilon > 0\}$ (locally upper points of X). These sets are at most countable, since every interval $(x-\varepsilon, x)$ or $(x, x+\varepsilon)$ contains some rational number. We define Y_L and Y^U in the analogous way.

We shall construct families $\mathcal{Q}_j$, $j \geq 1$, such that

(i) $(\mathcal{Q}_j)_{j=1}^{\infty}$ is a chain of open packings in $\mathbb{R}\times\mathbb{R}$,

(ii) $\lim_{j\to\infty} \sup_{Q\in\mathcal{Q}_j} \operatorname{diam}(Q) = 0$,

(iii) for every $(x,y) \in \mathbb{R}^2$ and every $j \geq 1$, there exist $\delta > 0$ and unique sets $Q_1, Q_2 \in \mathcal{Q}_j$ such that

$$([x,x+\delta]\times[y,y+\delta])\setminus\{(x,y)\} \subseteq Q_1, \quad ([x-\delta,x]\times[y-\delta,y])\setminus\{(x,y)\} \subseteq Q_2,$$

(iv) $\left(\bigcup_{Q\in\bigcup_{j=1}^{\infty}\mathcal{Q}_j} \operatorname{bd}(Q)\right) \cap \left(\left(X_L \times Y^U\right) \cup \left(X^U \times Y_L\right)\right) = \emptyset.$

First we consider the packings $\mathcal{P}_j = \left\{2^{-j}P(l,m) : l, m \in \mathbb{Z}\right\}$, $j \geq 1$, based on the disjoint open parallelograms

$$P(l,m) = \{\lambda(2,-1) + \mu(-1,2) : l < \lambda < l+1,\, m < \mu < m+1\},$$

whose edges have the directions $(2,-1)$ and $(-1,2)$. Clearly, $(\mathcal{P}_j)_{j=1}^{\infty}$ satisfies (i) and (ii). Property (iii) applies, since all edges of the parallelograms are graphs of strictly decreasing functions. However, (iv) may fail. The final chain $(\mathcal{Q}_j)_{j=1}^{\infty}$ can be obtained by translating $(\mathcal{P}_j)_{j=1}^{\infty}$ with some appropriate vector (x_t, y_t), that is, $\mathcal{Q}_j = \{P + (x_t, y_t) : P \in \mathcal{P}_j\}$. Indeed, $\left(\bigcup_{P\in\bigcup_{j=1}^{\infty}\mathcal{P}_j} \operatorname{bd}(P)\right)$ is a countable union of straight lines l. Given l, the set of all vectors (x,y) such that $l + (x,y)$ misses the countable set $\left(\left(X_L \times Y^U\right) \cup \left(X^U \times Y_L\right)\right)$ is of the second category. Hence there exists an appropriate translation vector (x_t, y_t) such that resulting shifted packings $\mathcal{Q}_j$, $j \geq 1$, satisfy (iv).

Step 2. Construction of $(\mathcal{R}_j)_{j=1}^{\infty}$.

We use the packing $\mathcal{Q}_j = \{Q_i^{(j)} : i \in I_j\}$ for defining the respective partition $\mathcal{R}_j = \{R_i^{(j)} : i \in I_j\}$ of $X \times Y$ as follows. By (iii), for every $(x,y) \in X \times Y$, there exist $\delta > 0$ and unique $i_1, i_2 \in I_j$ such that

$$([x, x+\delta] \times [y, y+\delta]) \setminus \{(x,y)\} \subseteq Q_{i_1}^{(j)}, \quad ([x-\delta, x] \times [y-\delta, y]) \setminus \{(x,y)\} \subseteq Q_{i_2}^{(j)}. \tag{13}$$

If $x \notin X^U$ and $y \notin Y^U$, (x,y) is assigned to the set $R_{i_1}^{(j)}$ (*upper assignment*). If $x \in X^U$ or $y \in Y^U$ we assign (x,y) to $R_{i_2}^{(j)}$ (*lower assignment*).

This defines a partition $\mathcal{R}_j$ of $X \times Y$ where obviously

$$R_i^{(j)} \subseteq \mathrm{cl}\big(Q_i^{(j)}\big) \quad \text{and} \quad Q_i^{(j)} \cap (X \times Y) \subseteq \mathrm{int}\big(R_i^{(j)}\big). \tag{14}$$

Property (i) of $(\mathcal{Q}_j)_{j=1}^{\infty}$ yields the chain property of $(\mathcal{R}_j)_{j=1}^{\infty}$, because the choice of upper or lower assignment depends only on (x,y), but not on j. The claim $\lim_{j\to\infty} \sup_{R\in\mathcal{R}_j} \mathrm{diam}(R) = 0$ is a consequence of (ii).

It remains to prove that $R_i^{(j)} \in SOS(X,Y)$ for every $i \in I_j$. So assume that $(x,y) \in R_i^{(j)}$. By the definition of $SOS(X,Y)$, it suffices to show that, given $\varepsilon > 0$, there exist $\hat{x} \in X$ and $\hat{y} \in Y$ such that $|x - \hat{x}| < \varepsilon$, $|y - \hat{y}| < \varepsilon$, and $(x, \hat{y}), (\hat{x}, y) \in \mathrm{int}\big(R_i^{(j)}\big)$.

Case 1. $(x,y) \in \mathrm{int}\big(R_i^{(j)}\big)$. Putting $\hat{x} = x$ and $\hat{y} = y$ we are done.

Case 2. $(x,y) \notin \mathrm{int}\big(R_i^{(j)}\big)$. Then, by (14), $(x,y) \in \mathrm{bd}\big(Q_i^{(j)}\big)$. By (iv), $x \in X^U$ yields $y \notin Y_L$ and $y \in Y^U$ forces $x \notin X_L$. Thus only the following three subcases are possible.

Case 2.1. $x \notin X^U$ *and* $y \notin Y^U$. Using the notations from (13) we obtain $i = i_1$ according to the upper assignment. By the definitions of X^U and Y^U, there exist $\hat{x} \in X$ and $\hat{y} \in Y$ such that $x < \hat{x} < x + \min\{\varepsilon, \delta\}$ and $y < \hat{y} < y + \min\{\varepsilon, \delta\}$. Now (13) and (14) give our claim, namely

$$(x, \hat{y}), (\hat{x}, y) \in Q_{i_1}^{(j)} \cap (X \times Y) = Q_i^{(j)} \cap (X \times Y) \subseteq \mathrm{int}\big(R_i^{(j)}\big).$$

Case 2.2. $x \in X^U$ *and* $y \notin Y_L$. Now we have lower assignment, that is $i = i_2$. According to $x \in X^U$ and $y \notin Y_L$ there are $\hat{x} \in X$ and $\hat{y} \in Y$ such that $x - \min\{\varepsilon, \delta\} < \hat{x} < x$ and $y - \min\{\varepsilon, \delta\} < \hat{y} < y$. Again (13) and (14) yield

$$(x, \hat{y}), (\hat{x}, y) \in Q_{i_2}^{(j)} \cap (X \times Y) = Q_i^{(j)} \cap (X \times Y) \subseteq \mathrm{int}\big(R_i^{(j)}\big).$$

Case 2.3. $y \in Y^U$ *and* $x \notin X_L$. We argue as in Case 2.2.

This completes the proof of $R_i^{(j)} \in SOS(X,Y)$. □

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